Journal of
Mathematical
Inequalities
Volume 15, Number 4 (2021), 1341–1360 doi:10.7153/jmi-2021-15-91

GENERALIZATION AND REFINEMENTS OF THE JENSEN–MERCER INEQUALITY WITH APPLICATIONS

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(*Communicated by M. Niezgoda*)

Abstract. We give a generalization followed by refinements of Jensen-Mercer inequality in variety of ways. We also highlight its importance by stating plenty of applications. In this way our main results generalize many established results including Ky Fan's Inequality, Popoviciu's inequalities and Rado's inequalities etc.

1. Introduction and preliminaries

The core of mathematics is to generalize the concepts and results. Therefore, we would like to further generalize a variant of the Jensen's inequality which was first introduced by Mercer and then it was generalized by Niezgoda. The Jensen inequality for convex functions is one of the most celebrated inequalities in mathematics and statistics. It plays a paramount role in various branches of sciences. Many other renowned inequalities can be obtained from it. $E.\ g.$, the important Arithmetic-Geometric inequality or some general inequalities between means of order p and q, such as Minkowski's inequality and Hölder's inequality, are all consequences of Jensen's inequality for convex functions. There are given numerous variants, generalizations and refinements of Jensen's inequalities for reference see [2, 3, 6, 7, 8, 9, 11, 12, 14, 15, 16, 17, 28, 35, 36, 37, 38]. We also refer [5] and [31] for detailed discussion on Jensen's inequality and for some remarks on literature and history of the topic.

Throughout the article we assume that J is an interval in $\mathbb{R}$ and for real weights $w_1,\ldots,w_n$ we define the notation

$$W_i = \sum_{j=1}^{i} w_j,\ i \in \{1,\ldots,n\} \quad \text{and clearly} \quad W_n = \sum_{j=1}^{n} w_j.$$

Here we state some results from [31] (see also [24, 25, 34]). Let us start with Jensen's inequality.

Mathematics subject classification (2020): 26A51, 39B62, 26D15, 26D20, 26D99.
Keywords and phrases: Convex functions, Jensen's inequality, Mercer's inequality, refinements, index sets.
This research work is financially supported by Dean Science Research Grant, University of Karachi.

PROPOSITION 1. *Let* $\mathbf{x}$ *be a* n*-tuple such that* $x_i \in J$, $i \in \{1,\dots,n\}$ *and let* $\mathbf{w}$ *be a nonnegative* n*-tuple with* $W_n > 0$. *If* f *is convex function on* J, *then the following inequality is valid*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leqslant \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i). \tag{1}$$

Steffensen in 1919 [31, p. 57] presented a more general form of Jensen's inequality which we usually refer to as Jensen-Steffensen's inequality, which may be stated as:

PROPOSITION 2. *Let* $\mathbf{x}$ *be a real monotonic* n*-tuple such that* $x_i \in J$, $i \in \{1,\dots,n\}$ *and let* $\mathbf{w}$ *be a real* n*-tuple such that*

$$0 \leqslant W_i \leqslant W_n, \quad W_n > 0 \quad \textit{for} \quad i \in \{1,\dots,n\}. \tag{2}$$

If f *is convex function on* J, *then* (1) *is valid.*

The following inequality is usually known in literature as reverse-Jensen's inequality [31, p. 83].

PROPOSITION 3. *Let* $\mathbf{x}$ *be a* n*-tuple such that* $x_i \in J$, $i \in \{1,\dots,n\}$ *and let* $\mathbf{w}$ *be a real* n*-tuple with* $\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \in J$, *where* $w_1 > 0$, $w_i \leqslant 0$ *for* $i \in \{2,\dots,n\}$ *and* $W_n > 0$. *If* f *is convex function on* J, *then reverse inequality in* (1) *is valid.*

In article [23], A. Mercer proved the following variant of Jensen's inequality, which we wouldrefer to as the Jensen-Mercer's inequality (see also [22]).

PROPOSITION 4. *Let the assumptions of Proposition* 1 *be valid. Then following inequality is valid*

$$f\left(L+M-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leqslant f(L)+f(M)-\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i), \tag{3}$$

where

$$L = \min_{x_i \in J}\{x_i\} \quad \textit{and} \quad M = \max_{x_i \in J}\{x_i\}.$$

In [1] we can find the following variant of Jensen-Mercer's inequality.

PROPOSITION 5. *Let* $\mathbf{x}$ *be a monotonic nondecreasing* n*-tuple such that* $x_i \in J$, $i \in \{1,\dots,n\}$ *and let* $\mathbf{w}$ *be a real* n*-tuple such that conditions on weights given in* (2) *be valid. If* f *is convex function on* J, *then inequality* (3) *is valid.*

In [22] the following result has been proved:

PROPOSITION 6. *Under the assumptions of Proposition* 3 *the inequality* (3) *is valid.*

Now, let us state definition of majorization from [21] as follows. Let $\mathbf{x} = (x_1,\dots,x_m)$ and $\mathbf{y} = (y_1,\dots,y_m)$ denote two m-tuples and $x_{[1]} \geqslant \cdots \geqslant x_{[m]}$, $y_{[1]} \geqslant \cdots \geqslant y_{[m]}$ be their ordered components.

DEFINITION 1. For $\mathbf{x},\mathbf{y}\in\mathbb{R}^m$,

$$\mathbf{x}\prec\mathbf{y}\quad\text{if}\quad\begin{cases}\sum_{i=1}^{k}x_{[i]}\leqslant\sum_{i=1}^{k}y_{[i]}\ , & k\in\{1,\dots,m-1\},\\ \sum_{i=1}^{m}x_{[i]}=\sum_{i=1}^{m}y_{[i]}\ , \end{cases}$$

when $\mathbf{x}\prec\mathbf{y}$, $\mathbf{x}$ *is said to be majorized by* $\mathbf{y}$ or $\mathbf{y}$ *majorizes* $\mathbf{x}$.

This notion and notation of majorization was first introduced by Hardy et al. in [10]. We can find the well-known majorization theorem in the same book [10].

Now we are ready to state the following extension of (3) that was given by M. Niezgoda in [27] which we would refer to as Niezgoda's inequality (see [19, 20, 29, 30] for recent extensions of (3)).

PROPOSITION 7. *Suppose that* $\boldsymbol{a}$ *be an* m*-tuple such that* $a_i\in J$ *and* $\boldsymbol{X}=(\mathbf{x}_j)=(x_{ij})$ *is a* $n\times m$ *matrix such that* $x_{ij}\in J$ $\forall i\in\{1,\dots,n\}$ *and* $j\in\{1,\dots,m\}$. *Let* f *be continuous convex function on* J.

If $\boldsymbol{a}$ *majorizes each row of* $\boldsymbol{X}$, *that is,*

$$\mathbf{x}_{i.}=(x_{i1},\dots,x_{im})\prec(a_1,\dots,a_m)=\boldsymbol{a}\text{ for each } i\in\{1,\dots,n\},$$

then we have the inequality

$$f\left(\sum_{j=1}^{m}a_j-\sum_{j=1}^{m-1}\sum_{i=1}^{n}w_ix_{ij}\right)\leqslant\sum_{j=1}^{m}f(a_j)-\sum_{j=1}^{m-1}\sum_{i=1}^{n}w_if(x_{ij}),\tag{4}$$

where $\sum_{i=1}^{n}w_i=1$ *with* $w_i\geqslant0$.

The main theme of this work is to generalize the Jensen-Mercer inequality. Various refinements of generalized result is also discussed. The article is organized in the following manner. The first section is devoted to preliminaries and introduction. In second section we generalize the Jensen-Mercer inequality by considering real weights satisfying the assumptions of the Jensen-steffensen's inequality as stated in (2). In third section we use index set functions to give various refinements of result proved in the second section. The forth section is completely based on applications of our generalized results and some of its refinements.

Our article generalizes various results stated in [1, 13, 18, 22, 23, 25, 26, 27, 31, 32, 33].

2. Generalization of Jensen-Mercer inequality

THEOREM 1. *Suppose that* $\boldsymbol{a}$ *be an* m*-tuple such that* $a_j\in J$ *for* $j\in\{1,\dots,m\}$ *and* $\boldsymbol{X}=(\mathbf{x}_j)=(x_{ij})$ *is a* $n\times m$ *matrix such that* $x_{ij}\in J$ $\forall i\in\{1,\dots,n\}$, $j\in\{1,\dots,m\}$ *and each sequence* $\mathbf{x}_{i1},\mathbf{x}_{i2},\dots,\mathbf{x}_{im}$ *is nondecreasing. Let* $\mathbf{w}$ *be a* n*-tuple such that conditions on weights given in* (2) *are valid. Let* f *be continuous convex function on* J. *If*

$$\sum_{j=1}^{m}x_{ij}=\sum_{j=1}^{m}a_j\quad\forall i\in\{1,\dots,n\}\tag{5}$$

and

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_{ij}) \leqslant f(a_j) \quad \forall j \in \{1,\dots,m\}, \tag{6}$$

then we have the inequality

$$f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i x_{ij} - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i x_{ij}\right) \leqslant \sum_{j=1}^{m} f(a_j) - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i f(x_{ij}) - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i f(x_{ij}), \tag{7}$$

where $k \in \{1,\dots,m\}$.

Proof. Fix $k \in \{1,\dots,m\}$, using first Jensen-Steffensen's inequality and then using (6) we get,

$$\begin{aligned}
& f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i x_{ij} - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i x_{ij}\right) \\
&= f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} x_{ij} - \sum_{j=k+1}^{m} x_{ij}\right)\right) \\
&\leqslant \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} x_{ij} - \sum_{j=k+1}^{m} x_{ij}\right) \\
&= \frac{1}{W_n}\sum_{i=1}^{n} w_i f\left(x_{ik}\right) \\
&\leqslant \frac{1}{W_n}\sum_{i=1}^{n} w_i \left(\sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{k-1} f(x_{ij}) - \sum_{j=k+1}^{m} f(x_{ij})\right) \\
&= \sum_{j=1}^{m} f(a_j) - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i f(x_{ij}) - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i f(x_{ij}). \quad \square
\end{aligned}$$

Here we give a couple of remarks related to our first main result.

REMARK 1. (a) It worth mentioning that Theorem 1 is stated for real weights w_i's satisfying conditions of Jensen-Steffensen (2) and we relaxed the condition of majorization (as given in the statement of Proposition 7) at the expense of conditions (5) and (6).

(b) Our main inequality (7) has close connection with the inequality (3) of [18] (see Theorem 4 there) which was proved for positive weights and we assumed weights to be real and not necessarily all positive.

(b) If $W_n = 1$ and $k = m$ in the inequality (7), then the sum $\sum_{j=k+1}^{m}$ becomes zero; in this case the inequality coincides with the inequality (4) for real weights. Here we do not claim that this is true generalization of Neizgoda's inequality (4) but it captures the inequality with slightly different assumptions involving real weights instead of nonnegative weights.

(c) If in the inequality (7) we set $k = m = 2$, $a_1 = L$, $a_2 = M$ and $x_{i1} = x_i$ for $i \in \{1,\dots,n\}$, then this inequality reduces to the inequality (3) for real weights as stated in Proposition 5 and hence the part of results of Theorem 2 of [1] is a special case of Theorem 1. Further by imposing different conditions on weights we easily obtain Propositions 4 and 6.

(d) For further remarks see [18].

3. Index set functions and refinements of generalized Niezgoda's inequality

In start of this section we give some construction which we would use throughout this section: Let I be a finite nonempty set of positive integers. Let $\mathbf{w} = (w_i)$, $i \in I$ be a real sequence and let $(\mathbf{x}_j) = (x_{ij})$ be a sequence of vectors such that $x_{ij} \in J$ $\forall i \in I$, $j \in \{1,\dots,m\}$. Moreover we define $A_I(\mathbf{x}_j,\mathbf{w}) = \frac{1}{W_I}\sum_{i\in I} w_i x_{ij}$ where $W_I = \sum_{i\in I} w_i$. For the generalized Jensen-Mercer inequality (7), we define the index set function F as

$$F(I) = W_I\left[\sum_{j=1}^{m} f(a_j) - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i f(x_{ij}) - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i f(x_{ij}) \right.$$
$$\left. - f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}\right)\right] \quad (8)$$

where $\mathbf{a}$ is an m-tuple such that $a_j \in J$ for $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,m\}$.

Now, by using techniques of article [22] we proof some results here. Throughout this section $I_n = \{1,\dots,n\}$.

THEOREM 2. *Let $\mathbf{a}$ be an m-tuple such that $a_j \in J$ for $j \in \{1,\dots,m\}$, I and I' be nonempty sets such that $I \bigcup I' = I_n$ and $I \cap I' = \emptyset$. Let $(\mathbf{x}_j) = (x_{ij})$ be a sequence of vectors such that $x_{ij} \in J$ $\forall i \in I$, $j \in \{1,\dots,m\}$ and $\mathbf{w} = (w_i)$, $i \in I \cup I'$ such that $W_{I\cup I'} > 0$. Let $A_S(\mathbf{x}_j,\mathbf{w}) \in J$ $(S \in \{I, I', I\cup I'\})$. If $W_I > 0$ and $W_{I'} > 0$, then under the assumptions of Theorem 1*

$$F(I \cup I') \geqslant F(I) + F(I'). \quad (9)$$

If $W_I \cdot W_{I'} < 0$, then the inequality (9) is reversed.

Proof. Fix $k \in \{1 \dots m\}$. Since f is continuous convex and composition with an affine function, we get convex function g which we may define as:

$$g(\mathbf{t}_\alpha) = f\Big(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} t_j^{(\alpha)} - \sum_{j=k+1}^{m} t_j^{(\alpha)}\Big)$$

where $\mathbf{t}_\alpha = (t_1^{(\alpha)},\dots,t_m^{(\alpha)}) \in J^m$. Using the definition of convex function, for all $\mathbf{t}_1$, $\mathbf{t}_2 \in J^m$ and $\lambda_1, \lambda_2 > 0$, we have

$$g\left(\frac{\lambda_1\mathbf{t_1} + \lambda_2\mathbf{t_2}}{\lambda_1+\lambda_2}\right) \leqslant \frac{\lambda_1 g(\mathbf{t_1}) + \lambda_2 g(\mathbf{t_2})}{\lambda_1+\lambda_2}, \quad (10)$$

which gives

$$(\lambda_1+\lambda_2)f\left(\sum_{j=1}^{m}a_j-\sum_{j=1}^{k-1}\frac{\lambda_1 t_j^{(1)}+\lambda_2 t_j^{(2)}}{\lambda_1+\lambda_2}-\sum_{j=k+1}^{m}\frac{\lambda_1 t_j^{(1)}+\lambda_2 t_j^{(2)}}{\lambda_1+\lambda_2}\right)$$
$$\leqslant \lambda_1 f\Big(\sum_{j=1}^{m}a_j-\sum_{j=1}^{k-1}t_j^{(1)}-\sum_{j=k+1}^{m}t_j^{(1)}\Big)+\lambda_2 f\Big(\sum_{j=1}^{m}a_j-\sum_{j=1}^{k-1}t_j^{(2)}-\sum_{j=k+1}^{m}t_j^{(2)}\Big). \quad (11)$$

Now, by putting $\lambda_1=W_I,\ \lambda_2=W_{I'},\ t_j^{(1)}=A_I(\mathbf{x}_j,\mathbf{w})$ and $t_j^{(2)}=A_{I'}(\mathbf{x}_j,\mathbf{w})$ we have

$$W_{I\cup I'}f\left(\sum_{j=1}^{m}a_j-\sum_{j=1}^{k-1}\frac{W_IA_I(\mathbf{x}_j,\mathbf{w})+W_{I'}A_{I'}(\mathbf{x}_j,\mathbf{w})}{W_{I\cup I'}}-\sum_{j=k+1}^{m}\frac{W_IA_I(\mathbf{x}_j,\mathbf{w})+W_{I'}A_{I'}(\mathbf{x}_j,\mathbf{w})}{W_{I\cup I'}}\right)$$
$$\leqslant W_I f\left(\sum_{j=1}^{m}a_j-\sum_{j=1}^{k-1}A_I(\mathbf{x}_j,\mathbf{w})-\sum_{j=k+1}^{m}A_I(\mathbf{x}_j,\mathbf{w})\right)$$
$$+W_{I'}f\left(\sum_{j=1}^{m}a_j-\sum_{j=1}^{k-1}A_{I'}(\mathbf{x}_j,\mathbf{w})-\sum_{j=k+1}^{m}A_{I'}(\mathbf{x}_j,\mathbf{w})\right).$$

Now,

$$W_{I\cup I'}f\left(\sum_{j=1}^{m}a_j-\sum_{j=1}^{k-1}A_{I\cup I'}(\mathbf{x}_j,\mathbf{w})-\sum_{j=k+1}^{m}A_{I\cup I'}(\mathbf{x}_j,\mathbf{w})\right)$$
$$\leqslant W_I f\Big(\sum_{j=1}^{m}a_j-\sum_{j=1}^{k-1}A_I(\mathbf{x}_j,\mathbf{w})-\sum_{j=k+1}^{m}A_I(\mathbf{x}_j,\mathbf{w})\Big)$$
$$+W_{I'}f\Big(\sum_{j=1}^{m}a_j-\sum_{j=1}^{k-1}A_{I'}(\mathbf{x}_j,\mathbf{w})-\sum_{j=k+1}^{m}A_{I'}(\mathbf{x}_j,\mathbf{w})\Big).$$

Multiplying both sides of the last inequality by (-1), putting values of A_S and adding the following term on the both sides

$$W_{I\cup I'}\left[\sum_{j=1}^{m}f(a_j)-\frac{1}{W_{I\cup I'}}\sum_{j=1}^{k-1}\sum_{i\in I\cup I'}w_i f(x_{ij})-\frac{1}{W_{I\cup I'}}\sum_{j=k+1}^{m}\sum_{i\in I\cup I'}w_i f(x_{ij})\right]$$

we get

$$W_{I\cup I'}\left[\sum_{j=1}^{m}f(a_j)-\frac{1}{W_{I\cup I'}}\sum_{j=1}^{k-1}\sum_{i\in I\cup I'}w_i f(x_{ij})-\frac{1}{W_{I\cup I'}}\sum_{j=k+1}^{m}\sum_{i\in I\cup I'}w_i f(x_{ij})\right.$$
$$\left.-f\left(\sum_{j=1}^{m}a_j-\frac{1}{W_{I\cup I'}}\sum_{j=1}^{k-1}\sum_{i\in I\cup I'}w_i x_{ij}-\frac{1}{W_{I\cup I'}}\sum_{j=k+1}^{m}\sum_{i\in I\cup I'}w_i x_{ij}\right)\right]$$
$$\geqslant W_I\left[\sum_{j=1}^{m}f(a_j)-\frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I}w_i f(x_{ij})-\frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I}w_i f(x_{ij})\right.$$
$$\left.-f\left(\sum_{j=1}^{m}a_j-\frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I}w_i x_{ij}-\frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I}w_i x_{ij}\right)\right]$$

$$+W_{I'}\left[\sum_{j=1}^{m} f(a_j)-\frac{1}{W_{I'}}\sum_{j=1}^{k-1}\sum_{i\in J} w_i f(x_{ij})-\frac{1}{W_{I'}}\sum_{j=k+1}^{m}\sum_{i\in J} w_i f(x_{ij})\right.$$
$$\left.-f\left(\sum_{j=1}^{m} a_j-\frac{1}{W_{I'}}\sum_{j=1}^{k-1}\sum_{i\in J} w_i x_{ij}-\frac{1}{W_{I'}}\sum_{j=k+1}^{m}\sum_{i\in J} w_i x_{ij}\right)\right].$$

In index set function notation we finally get

$$F(I\cup I')\geqslant F(I)+F(I').$$

In case when $W_I.W_{I'}<0$, for instance $W_I>0$ and $W_{I'}<0$, we again let $\lambda_1=W_I$, $\lambda_2=W_{I'}$, $t_j^{(1)}=A_I(\mathbf{x}_j,\mathbf{w})$ and $t_j^{(2)}=A_{I'}(\mathbf{x}_j,\mathbf{w})$ and reversed inequality in (9) follows by using reverse Jensen's inequality for two variable case. □

COROLLARY 1. *Let* $\boldsymbol{a}$ *be an* m*-tuple such that* $a_j\in J$ *for* $j\in\{1,\dots,m\}$*. Let* I_t*,* $t\in\{1,\dots,l\}$ *be finite nonempty sets of positive integers such that* $I_s\cap I_t=\emptyset$ *for all* $s\neq t\in\{1,\dots,l\}$*. We further suppose that* $(\mathbf{x}_j)=(x_{ij})$ *be a real sequence of vectors such that* $x_{ij}\in J\ \forall\, i\in\bigcup_{t=1}^{l} I_t,\ j\in\{1,\dots,m\}$ *and let* $\mathbf{w}=(w_i)$*,* $i\in\bigcup_{t=1}^{l} I_t$ *such that* $W_{i\in\bigcup_{t=1}^{l} I_t}>0$ *and* $A_S(\mathbf{x}_j,\mathbf{w})\in J$ $(S\in\{I_1,\dots,I_t,\bigcup_{t=1}^{r} I_t\})$ $(r\in\{2,\dots,l\})$*. Then under the assumptions of Theorem 1 we have*

(a) If $W_{I_t}>0$ *for* $t\in\{1,\dots,l\}$*,*

$$F\left(\bigcup_{t=1}^{l} I_t\right)\geqslant\sum_{t=1}^{l} F(I_t). \tag{12}$$

(b) If $W_{I_1}>0$ *and* $W_{I_t}<0$ *for* $t\in\{2,\dots,l\}$*, then the inequality (12) is reversed.*

Proof. Proof follows from Theorem 2 by using induction. □

Following results give us refinements of Niezgoda's Inequality. For the rest of this section we assume $x_{ij}\in[a,b]\subseteq J\ \forall i$ and j.

COROLLARY 2. *Let* $\boldsymbol{a}$ *be an* m*-tuple such that* $a_j\in J$ *for* $j\in\{1,\dots,m\}$*. Let* $I_k=\{1,\dots,k\}$*,* $k\in\{1,\dots,n\}$*. We further suppose that* $(\mathbf{x}_j)=(x_{ij})$ *be a real sequence of vectors such that* $x_{ij}\in J\ \forall\, i\in I_n,\ j\in\{1,\dots,m\}$ *and if* $w_1>0$ *and* $w_i\geqslant 0$ *for* $i\in\{2,\dots,n\}$*, then under the assumptions of Theorem 1 we have*

$$F(I_n)\geqslant F(I_{n-1})\geqslant\cdots\geqslant F(I_2)\geqslant F(I_1)\geqslant 0. \tag{13}$$

If $w_i\leqslant 0$ *for* $i\in\{2,\dots,n\}$*,* $W_{I_n}>0$ *and* $A_{I_n}(\mathbf{x}_j,\mathbf{w})\in[a,b]\subseteq J$*, then*

$$0\leqslant F(I_n)\leqslant F(I_{n-1})\leqslant\cdots\leqslant F(I_2)\leqslant F(I_1). \tag{14}$$

Proof. Fix $k\in\{1,\dots,m\}$. Suppose that $w_i\geqslant 0$ for $i\in\{2,\dots,n\}$. From generalized Niezgoda's inequality (7) it follows that

$$F(\{t\})=w_t\left[\sum_{j=1}^{m} f(a_j)-\sum_{j=1}^{k-1} f(x_{tj})-\sum_{j=k+1}^{m} f(x_{tj})-f\left(\sum_{j=1}^{m} a_j-\sum_{j=1}^{k-1} x_{tj}-\sum_{j=k+1}^{m} x_{tj}\right)\right]$$
$$\geqslant 0$$

for $t \in I_n$. Now, by Theorem 2 we have

$$F(I_t) = F(I_{t-1} \cup \{t\}) \geqslant F(I_{t-1}) + F(\{t\}) \geqslant F(I_{t-1})$$

for all $t \in \{2,\ldots,n\}$.

For second part, we suppose that $w_i \leqslant 0$ for $i \in \{2,\ldots,n\}$ with $W_{I_n} > 0$ and $A_{I_n}(\mathbf{x}_j,\mathbf{w}) \in [a,b]$. Now we show that $A_{I_{n-1}}(\mathbf{x}_j,\mathbf{w}) \in [a,b]$ as follows.

Given that

$$a \leqslant A_{I_n}(\mathbf{x}_j,\mathbf{w}) \leqslant b$$

multiplying both sides by $W_{I_n} > 0$ and adding $-w_n x_{nj}$ we obtain

$$W_{I_n}a - w_n x_{nj} \leqslant \sum_{i \in I_n} w_i x_{ij} - w_n x_{nj} \leqslant W_{I_n}b - w_n x_{nj}$$

or we may write

$$W_{I_n}a - w_n x_{nj} \leqslant \sum_{i \in I_{n-1}} w_i x_{ij} \leqslant W_{I_n}b - w_n x_{nj}$$

Now multiplying both sides by $\frac{1}{W_{I_{n-1}}} > 0$ we get

$$\frac{1}{W_{I_{n-1}}}(W_{I_n}a - w_n x_{nj}) \leqslant A_{I_{n-1}}(\mathbf{x}_j,\mathbf{w}) \leqslant \frac{1}{W_{I_{n-1}}}(W_{I_n}b - w_n x_{nj}),$$

or we may write

$$a + \frac{w_n}{W_{I_{n-1}}}(a - x_{nj}) \leqslant A_{I_{n-1}}(\mathbf{x}_j,\mathbf{w}) \leqslant b + \frac{w_n}{W_{I_{n-1}}}(b - x_{nj}),$$

clearly

$$\frac{w_n}{W_{I_{n-1}}}(a - x_{nj}) \geqslant 0 \quad \text{and} \quad \frac{w_n}{W_{I_{n-1}}}(b - x_{nj}) \leqslant 0,$$

and hence we conclude that

$$a \leqslant A_{I_{n-1}}(\mathbf{x}_j,\mathbf{w}) \leqslant b.$$

By iteration we obtain $A_{I_t}(\mathbf{x}_j,\mathbf{w}) \in [a,b]$ for all $t \in \{2,\ldots,n\}$. Similarly as before we have $F(\{t\}) \leqslant 0$ for all $t \in \{2,\ldots,n\}$. Now, by reversed (9) we have

$$F(I_t) = F(I_{t-1} \cup \{t\}) \leqslant F(I_{t-1}) + F(\{t\}) \leqslant F(I_{t-1})$$

for all $t \in \{2,\ldots,n\}$ and finally by Theorem 1 $F(I_n) \geqslant 0$. □

COROLLARY 3. *Let all the assumptions of Corollary 2 be valid. If $w_i > 0$ for $i \in \{1,\ldots,n\}$, then*

$$F(I_n) \geqslant \max_{1 \leqslant s \leqslant t \leqslant n} \Bigg[(w_s + w_t) \Bigg[\sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{k-1} \frac{w_s f(x_{sj}) + w_t f(x_{tj})}{w_s + w_t} - \sum_{j=k+1}^{m} \frac{w_s f(x_{sj}) + w_t f(x_{tj})}{w_s + w_t} - f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} \frac{w_s x_{sj} + w_t x_{tj}}{w_s + w_t} - \sum_{j=k+1}^{m} \frac{w_s x_{sj} + w_t x_{tj}}{w_s + w_t} \right) \Bigg] \Bigg] \tag{15}$$

and

$$F(I_n) \geqslant \max_{1\leqslant t\leqslant n}\left[w_t\left[\sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{k-1} f(x_{tj}) - \sum_{j=k+1}^{m} f(x_{tj}) - f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} x_{tj} - \sum_{j=k+1}^{m} x_{tj}\right)\right]\right]. \quad (16)$$

If $w_i \leqslant 0$ *for* $i \in \{2,\ldots,n\}$ *with* $W_{I_n} > 0$ *and* $A_{I_n}(\mathbf{x}_j,\mathbf{w}) \in [a,b]$, *then*

$$F(I_n) \leqslant \min_{2\leqslant t\leqslant n}\left[(w_1+w_t)\left[\sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{k-1} \frac{w_1 f(x_{1j}) + W_T F(X_{TJ})}{W_1+W_T} - \sum_{j=k+1}^{m} \frac{w_1 f(x_{1j}) + w_t f(x_{tj})}{w_1+w_t} - f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} \frac{w_1x_{1j}+w_tx_{tj}}{w_1+w_t} - \sum_{j=k+1}^{m} \frac{w_1x_{1j}+w_tx_{tj}}{w_1+w_t}\right)\right]\right]. \quad (17)$$

Proof. Suppose that $w_i > 0$ for $i \in I_n$. As $F(I_n) \geqslant F(I_2)$ in (13), we may conclude that

$$F(I_n) \geqslant F(\{s,t\}) \quad (18)$$

for all $s \neq t \in I_n$, so the inequality (15) immediately follows. From (18) we have that $F(I_n) \geqslant F(\{t\})$ for all $t \in I_n$, so the inequality (16) is also proved. Inequality (17) can be proved in the similar way. □

REMARK 2. If in Theorem 2 we set $k = m = 2$, $a_1 = L$, $a_2 = M$ and $x_{i1} = x_i$ for $i \in \{1,\ldots,n\}$ and in its corollaries, then we obtain Theorem 2.1 and Corollaries 2.4, 2.5 and 2.6 of [22] as special case of our results.

Throughout this section we assume that $I \subseteq I_n$ unless stated otherwise. Now we give refinement of (7) as follows.

THEOREM 3. *Let all the assumptions of Theorem 1 be valid. Then the following refinement hold:*

$$f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_ix_{ij} - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_ix_{ij}\right) \leqslant D(\mathbf{w},\mathbf{X},f;I) \leqslant \sum_{j=1}^{m} f(a_j) - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i f(x_{ij}) - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i f(x_{ij}), \quad (19)$$

where $W_I = \sum_{i\in I} w_i$, $W_{\overline{I}} = \sum_{i\in\overline{I}} w_i$, $\overline{I} = I_n \backslash I$ *and* $k \in \{1,\ldots,m\}$, *and*

$$D(\mathbf{w},\mathbf{X},f;I) = \frac{W_I}{W_n} f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_ix_{ij} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_ix_{ij}\right) + \frac{W_{\overline{I}}}{W_n} f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_{\overline{I}}}\sum_{j=1}^{k-1}\sum_{i\in\overline{I}} w_ix_{ij} - \frac{1}{W_{\overline{I}}}\sum_{j=k+1}^{m}\sum_{i\in\overline{I}} w_ix_{ij}\right).$$

Proof. Fixing $k \in \{1,\dots,m\}$, and supposing that $w_i^* = \frac{w_i}{W_n}$ where $\sum_{i=1}^n w_i^* = 1$. Also $W_I^* = \sum_{i\in I} w_i^*$ By convexity of function f we have

$$
\begin{aligned}
& f\left(\sum_{j=1}^m a_j - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^n w_i x_{ij} - \frac{1}{W_n}\sum_{j=k+1}^m\sum_{i=1}^n w_i x_{ij}\right) \\
&= f\left(\sum_{j=1}^m a_j - \sum_{j=1}^{k-1}\sum_{i=1}^n w_i^* x_{ij} - \sum_{j=k+1}^m\sum_{i=1}^n w_i^* x_{ij}\right) \\
&= f\left(\sum_{i=1}^n w_i^*\left(\sum_{j=1}^m a_j - \sum_{j=1}^{k-1} x_{ij} - \sum_{j=k+1}^m x_{ij}\right)\right) \\
&= f\left(W_I^*\left(\frac{1}{W_I^*}\sum_{i\in I} w_i^*\left(\sum_{j=1}^m a_j - \sum_{j=1}^{k-1} x_{ij} - \sum_{j=k+1}^m x_{ij}\right)\right)\right. \\
&\quad \left. + W_{\bar I}^*\left(\frac{1}{W_{\bar I}^*}\sum_{i\in \bar I} w_i^*\left(\sum_{j=1}^m a_j - \sum_{j=1}^{k-1} x_{ij} - \sum_{j=k+1}^m x_{ij}\right)\right)\right) \\
&\leqslant W_I^* f\left(\sum_{j=1}^m a_j - \frac{1}{W_I^*}\sum_{j=1}^{k-1}\sum_{i\in I} w_i^* x_{ij} - \frac{1}{W_I^*}\sum_{j=k+1}^m\sum_{i\in I} w_i^* x_{ij}\right) \\
&\quad + W_{\bar I}^* f\left(\sum_{j=1}^m a_j - \frac{1}{W_{\bar I}^*}\sum_{j=1}^{k-1}\sum_{i\in I} w_i^* x_{ij} - \frac{1}{W_{\bar I}^*}\sum_{j=k+1}^m\sum_{i\in I} w_i^* x_{ij}\right) \\
&\leqslant \frac{W_I}{W_n} f\left(\sum_{j=1}^m a_j - \frac{1}{\frac{W_I}{W_n}}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{\frac{W_I}{W_n}}\sum_{j=k+1}^m\sum_{i\in I} w_i x_{ij}\right) \\
&\quad + \frac{W_{\bar I}}{W_n} f\left(\sum_{j=1}^m a_j - \frac{1}{\frac{W_{\bar I}}{W_n}}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{\frac{W_{\bar I}}{W_n}}\sum_{j=k+1}^m\sum_{i\in I} w_i x_{ij}\right) \\
&= \frac{W_I}{W_n} f\left(\sum_{j=1}^m a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{W_I}\sum_{j=k+1}^m\sum_{i\in I} w_i x_{ij}\right) \\
&\quad + \frac{W_{\bar I}}{W_n} f\left(\sum_{j=1}^m a_j - \frac{1}{W_{\bar I}}\sum_{j=1}^{k-1}\sum_{i\in \bar I} w_i x_{ij} - \frac{1}{W_{\bar I}}\sum_{j=k+1}^m\sum_{i\in \bar I} w_i x_{ij}\right) = D(\mathbf{w},\mathbf{X},f;I)
\end{aligned}
$$

for any I, which proves the first inequality in (19).

By the inequality (7) we also have

$$
\begin{aligned}
D(\mathbf{w},\mathbf{X},f;I) &= \frac{W_I}{W_n} f\left(\sum_{j=1}^m a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{W_I}\sum_{j=k+1}^m\sum_{i\in I} w_i x_{ij}\right) \\
&\quad + \frac{W_{\bar I}}{W_n} f\left(\sum_{j=1}^m a_j - \frac{1}{W_{\bar I}}\sum_{j=1}^{k-1}\sum_{i\in \bar I} w_i x_{ij} - \frac{1}{W_{\bar I}}\sum_{j=k+1}^m\sum_{i\in \bar I} w_i x_{ij}\right) \\
&\leqslant \frac{W_I}{W_n}\left(\frac{1}{W_n}\sum_{j=1}^m f(a_j) - \frac{1}{W_n}\frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i=1}^n w_i f(x_{ij}) - \frac{1}{W_n}\frac{1}{W_I}\sum_{j=k+1}^m\sum_{i=1}^n w_i f(x_{ij})\right)
\end{aligned}
$$

$$+\frac{W_{\bar{I}}}{W_n}\left(\sum_{j=1}^{m} f(a_j) - \frac{1}{W_{\bar{I}}}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i f(x_{ij}) - \frac{1}{W_{\bar{I}}}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i f(x_{ij})\right)$$
$$=\sum_{j=1}^{m} f(a_j) - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i f(x_{ij}) - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i f(x_{ij})$$

for any I, which proves the second inequality in (19). □

REMARK 3.

$$f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i x_{ij} - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i x_{ij}\right) \leqslant \min_I D(\mathbf{w}, \mathbf{X}, f; I)$$

$$\max_I D(\mathbf{w}, \mathbf{X}, f; I) \leqslant \sum_{j=1}^{m} f(a_j) - \frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i f(x_{ij}) - \frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i f(x_{ij}).$$

REMARK 4. Similar remarks as given in Remark 1 hold for Theorem 3 as well. Also some special cases of Theorem 3 can be found in [13].

For our next corollary we need the following definition.

DEFINITION 2. [21, p. 10] An $m \times m$ matrix $\mathbf{A} = (a_{jk})$ is said to be doubly stochastic, if $a_{jk} \geqslant 0$ and $\sum_{j=1}^{m} a_{jk} = \sum_{k=1}^{m} a_{jk} = 1$ for all $j, k \in \{1, \ldots, m\}$.

It is well known [21, p. 31] that if $\mathbf{A}$ is an $m \times m$ doubly stochastic matrix, then

$$\mathbf{aA} \prec \mathbf{a} \text{ for each real } m\text{-tuple } \mathbf{a} = (a_1, \ldots, a_m). \tag{20}$$

By applying Theorem 3 and (20), one obtains:

COROLLARY 4. *Let f be continuous convex function on J. Suppose that $\boldsymbol{a} = (a_1, \ldots, a_m) \in J^m$ for $j \in \{1, \ldots, m\}$ and $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are $m \times m$ doubly stochastic matrices. Set*

$$\mathbf{X} = (x_{ij}) = \begin{pmatrix} \boldsymbol{a}\mathbf{A}_1 \\ \vdots \\ \boldsymbol{a}\mathbf{A}_n \end{pmatrix}.$$

Then the inequality (19) is valid.

REMARK 5. Special cases of Corollary 4 can be found in [6] and [14].

REMARK 6. Analagous assertion can be formulated for concave functions using the fact that f is concave iff $-f$ is convex.

4. Applications

H: Let all the assumptions of Theorem 1 be valid. Further we let, for $\emptyset \neq I \subseteq I_n = \{1,\dots,n\}$, $x_{ij} \in [a,b] \subseteq J$ for $j \in \{1,\dots,m\}$ and $i \in I$, where $0<a<b$, formed with weights w_i, $i \in I$ satisfies conditions stated in (2) we define the arithmetic, geometric, harmonic means and power mean of order $r \in \mathbb{R}$ as A_I, G_I, H_I and $M_I^{[r]}$ respectively. While For $I = I_n$ we denote the arithmetic, geometric, harmonic and power means by A_n, G_n, H_n and $M_n^{[r]}$ respectively. For the various properties of these means and relations among them we refer the reader to [5] and [15].

$E.\ g.$, it is well known that

$$A_n \geqslant G_n \geqslant H_n, \tag{21}$$

$$\left(\frac{A_n}{G_n}\right)^{W_n} \geqslant \left(\frac{A_{n-1}}{G_{n-1}}\right)^{W_{n-1}} \geqslant \cdots \geqslant \left(\frac{A_1}{G_1}\right)^{W_1} \geqslant 1. \tag{22}$$

$$W_n(A_n - G_n) \geqslant W_{n-1}(A_{n-1} - G_{n-1}) \geqslant \cdots \geqslant W_1(A_1 - G_1) \geqslant 0. \tag{23}$$

Also we have renowned Ky Fan Inequality [4, p. 5] given by

$$\frac{A_n(\mathbf{x})}{A_n(\mathbf{1}-\mathbf{x})} \geqslant \frac{G_n(\mathbf{x})}{G_n(\mathbf{1}-\mathbf{x})},\ 0 < x_{ij} \leqslant \frac{1}{2}\ \forall\, i,\ j. \tag{24}$$

If we define

$$\begin{aligned}
\tilde{A}_I :&= \sum_{j=1}^{m} a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij} \\
&= \sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} A_I(\mathbf{x}_j,\mathbf{w}) - \sum_{j=k+1}^{m} A_I(\mathbf{x}_j,\mathbf{w})
\end{aligned}$$

$$\tilde{G}_I := \frac{\prod_{j=1}^{m} a_j}{\left(\prod_{j=1}^{k-1}\prod_{i\in I} x_{ij}^{w_i}\right)^{\frac{1}{W_I}} \left(\prod_{j=k+1}^{m}\prod_{i\in I} x_{ij}^{w_i}\right)^{\frac{1}{W_I}}}$$

$$\tilde{H}_I := \left(\sum_{j=1}^{m} a_j{}^{-1} - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij}^{-1} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}^{-1}\right)^{-1}$$

$$\tilde{M}_I^{[r]} := \begin{cases} \left(\sum_{j=1}^{m} a_j{}^{r} - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij}^{r} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}^{r}\right)^{\frac{1}{r}} & r \neq 0, \\ \tilde{G}_I & r = 0, \end{cases}$$

THEOREM 4. *Under the assumptions given in* **H**, *the following inequalities hold:*

$$\tilde{A}_n(\mathbf{x}) \geqslant \tilde{G}_n(\mathbf{x}) \tag{25}$$

$$\frac{\tilde{A}_n(\mathbf{x})}{\tilde{A}_n(\mathbf{1}-\mathbf{x})} \geqslant \frac{\tilde{G}_n(\mathbf{x})}{\tilde{G}_n(\mathbf{1}-\mathbf{x})} \quad \textit{provided that } 0 < x_{ij} \leqslant \frac{1}{2}\ \ \forall i,\ j. \tag{26}$$

Proof. Applying (7) to convex function $f(x) = -\ln x$, we obtain (25).

Applying (7) to convex function $f(x) = \ln\left(\frac{1-x}{x}\right)$ for $0 < x \leqslant \frac{1}{2}$, we obtain required inequality (26). $\square$

THEOREM 5. *Under the assumptions given in* $\mathbf{H}$, *the following inequalities hold:*

$$\left(\frac{\tilde{A}_n}{\tilde{G}_n}\right)^{W_n} \geqslant \left(\frac{\tilde{A}_{n-1}}{\tilde{G}_{n-1}}\right)^{W_{n-1}} \geqslant \cdots \geqslant \left(\frac{\tilde{A}_1}{\tilde{G}_1}\right)^{W_1} \geqslant 1. \tag{27}$$

$$W_n(\tilde{A}_n - \tilde{G}_n) \geqslant W_{n-1}(\tilde{A}_{n-1} - \tilde{G}_{n-1}) \geqslant \cdots \geqslant W_1(\tilde{A}_1 - \tilde{G}_1) \geqslant 0. \tag{28}$$

Proof. Applying (13) to convex function $f(x) = -\ln x$, we obtain

$$\ln\left(\frac{\tilde{A}_n}{\tilde{G}_n}\right)^{W_n} \geqslant \ln\left(\frac{\tilde{A}_{n-1}}{\tilde{G}_{n-1}}\right)^{W_{n-1}} \geqslant \cdots \geqslant \ln\left(\frac{\tilde{A}_1}{\tilde{G}_1}\right)^{W_1} \geqslant 0. \tag{29}$$

from which (27) follows. Applying (13) to convex function $f(x) = \exp x$ and replacing a_j and x_{ij} with $\ln(a_j)$ and $\ln(x_{ij})$ respectively, we obtain

$$W_n(\tilde{A}_n - \tilde{G}_n) \geqslant W_{n-1}(\tilde{A}_{n-1} - \tilde{G}_{n-1}) \geqslant \cdots \geqslant W_1(\tilde{A}_1 - \tilde{G}_1) \geqslant 0,$$

since in this case

$$\begin{aligned} F(I_t) = W_t \Bigg[&\sum_{j=1}^{m} a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij} \\ &- \exp\left(\sum_{j=1}^{m}\ln(a_j) - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i \ln(x_{ij}) - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i \ln(x_{ij})\right)\Bigg] \\ = W_t(\tilde{A}_t - \tilde{G}_t). \quad \square \end{aligned}$$

REMARK 7. If in Theorem 5 we simply put $w_i = 1\ \forall i \in I_n$, then we get the following results which are of Popoviciu- [32] and Rado- [33] types, respectively, (see also [26, p. 13] and [34, p. 194]).

COROLLARY 5. *Under the assumptions of Theorem 5, we have*

$$\left(\frac{\tilde{A}_n}{\tilde{G}_n}\right)^{n} \geqslant \left(\frac{\tilde{A}_{n-1}}{\tilde{G}_{n-1}}\right)^{n-1} \geqslant \cdots \geqslant \left(\frac{\tilde{A}_1}{\tilde{G}_1}\right)^{1} \geqslant 1.$$

$$n(\tilde{A}_n - \tilde{G}_n) \geqslant (n-1)(\tilde{A}_{n-1} - \tilde{G}_{n-1}) \geqslant \cdots \geqslant 1\cdot(\tilde{A}_1 - \tilde{G}_1 \geqslant 0$$

COROLLARY 6. *Under the assumptions of Theorem 5, we have*

$$\left(\frac{\tilde{G}_n}{\tilde{H}_n}\right)^{W_n} \geqslant \left(\frac{\tilde{G}_{n-1}}{\tilde{H}_{n-1}}\right)^{W_{n-1}} \geqslant \cdots \geqslant \left(\frac{\tilde{G}_1}{\tilde{H}_1}\right)^{W_1} \geqslant 1.$$

$$W_n\left(\frac{1}{\tilde{H}_n} - \frac{1}{\tilde{G}_n}\right) \geqslant W_{n-1}\left(\frac{1}{\tilde{H}_{n-1}} - \frac{1}{\tilde{G}_{n-1}}\right) \geqslant \cdots \geqslant W_1\left(\frac{1}{\tilde{H}_1} - \frac{1}{\tilde{G}_1}\right) \geqslant 0.$$

Proof. Follows from Theorem 5 by the substitutions $a_j \to \frac{1}{a_j}$ and $x_{ij} \to \frac{1}{x_{ij}}$. $\square$

THEOREM 6. *For $r \leqslant 1$ and under the assumptions given in* **H**, *the following series of inequalities hold:*

$$W_n(\tilde{A}_n - \tilde{M}_n^{[r]}) \geqslant W_{n-1}(\tilde{A}_{n-1} - \tilde{M}_{n-1}^{[r]}) \geqslant \cdots \geqslant W_1(\tilde{A}_1 - \tilde{M}_1^{[r]}) \geqslant 0 \tag{30}$$

For $r \geqslant 1$, inequalities in (30) are reversed.

Proof. For $r \leqslant 1$, use (13) for convex function $f(x) = x^{\frac{1}{r}}$ and replacing a_j and x_{ij} with a_j^r and x_{ij}^r respectively we obtain (30), since in this case

$$F(I_t) = W_t\left[\sum_{j=1}^{m} a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij} - \left(\sum_{j=1}^{m} a_j^r - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij}^r - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}^r\right)^{1/r}\right] = W_t(\tilde{A}_t - \tilde{M}_t^{[r]}).$$

If $r \geqslant 1$, then function $f(x) = x^{\frac{1}{r}}$ is concave, so inequalities in (30) are reversed. □

By simply taking $r = -1$ we get the following corollary.

COROLLARY 7. *Under the assumptions of Theorem 6, we have*

$$W_n(\tilde{A}_n - \tilde{H}_n) \geqslant W_{n-1}(\tilde{A}_{n-1} - \tilde{H}_{n-1}) \geqslant \cdots \geqslant W_1(\tilde{A}_1 - \tilde{H}_1) \geqslant 0.$$

REMARK 8. It is easy to see that, (28) is also direct consequences of Theorem 6.

THEOREM 7. *Under the assumptions given in* **H**, *let $r, s \in \mathbb{R}$, $r \leqslant s$. If $s > 0$, then*

$$W_n\left(\left(\tilde{M}_n^{[s]}\right)^s - \left(\tilde{M}_n^{[r]}\right)^s\right) \geqslant W_{n-1}\left(\left(\tilde{M}_{n-1}^{[s]}\right)^s - \left(\tilde{M}_{n-1}^{[r]}\right)^s\right) \geqslant \cdots \geqslant W_1\left(\left(\tilde{M}_1^{[s]}\right)^s - \left(\tilde{M}_1^{[r]}\right)^s\right) \geqslant 0. \tag{31}$$

If $s < 0$, then inequalities in (31) are reversed.

Proof. For $s > 0$, use (13) for convex function $f(x) = x^{\frac{s}{r}}$ and replacing a_j and x_{ij} with a_j^r and x_{ij}^r respectively we obtain (31), since in this case

$$\begin{aligned} F(I_t) = W_t\Bigg[&\sum_{j=1}^{m} a_j^s - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij}^s - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}^s \\ &- \left(\sum_{j=1}^{m} a_j^r - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij}^r - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}^r\right)^{s/r}\Bigg] \\ = W_t&\left(\left(\tilde{M}_t^{[s]}\right)^s - \left(\tilde{M}_t^{[r]}\right)^s\right). \end{aligned}$$

If $s < 0$, then function $f(x) = x^{\frac{s}{r}}$ is concave, so inequalities in (31) are reversed. □

THEOREM 8. *Under the assumptions given in* $\mathbf{H}$, *the following inequalities hold:*

$$(i)\ \tilde{G}_n \leqslant \tilde{A}_I^{\frac{W_I}{W_n}} \tilde{A}_{\bar{I}}^{\frac{W_{\bar{I}}}{W_n}} \leqslant \tilde{A}_n \tag{32}$$

$$(ii)\ \tilde{G}_n \leqslant \frac{W_I}{W_n}\tilde{G}_I + \frac{W_{\bar{I}}}{W_n}\tilde{G}_{\bar{I}} \leqslant \tilde{A}_n \tag{33}$$

Proof. (i) Applying Theorem 3 to convex function $f(x) = -\ln x$, we obtain

$$-\ln\tilde{A}_n \leqslant -\frac{W_I}{W_n}\ln\tilde{A}_I - \frac{W_{\bar{I}}}{W_n}\ln\tilde{A}_{\bar{I}} \leqslant -\ln\tilde{G}_n.$$

After some simplifications we obtain our required result.

(ii) Applying Theorem 3 to convex function $f(x) = \exp x$ and replacing a_j and x_{ij} with $\ln a_j$ and $\ln x_{ij}$ respectively we get what we wanted. □

COROLLARY 8. *Under the assumptions of Theorem 8, we have*

$$(i)\ \tilde{G}_n \leqslant \min_I \tilde{A}_I^{\frac{W_I}{W_n}} \tilde{A}_{\bar{I}}^{\frac{W_{\bar{I}}}{W_n}} \quad \text{and } \tilde{A}_n \geqslant \max_I \tilde{A}_I^{\frac{W_I}{W_n}} \tilde{A}_{\bar{I}}^{\frac{W_{\bar{I}}}{W_n}}. \tag{34}$$

$$(ii)\ \tilde{G}_n \leqslant \min_I \left[\frac{W_I}{W_n}\tilde{G}_I + \frac{W_{\bar{I}}}{W_n}\tilde{G}_{\bar{I}}\right] \text{ and } \tilde{A}_n \geqslant \max_I \left[\frac{W_I}{W_n}\tilde{G}_I + \frac{W_{\bar{I}}}{W_n}\tilde{G}_{\bar{I}}\right]. \tag{35}$$

Proof. Inequalities (34) and (35) follow from (32) and (33) respectively by using Remark 3. □

Following particular cases of Theorem 8 are of interest which follows from Theorem 8 and Corollary 8 respectively by the substitutions $a_j \to \frac{1}{a_j}$ and $x_{ij} \to \frac{1}{x_{ij}}$.

COROLLARY 9. *Under the assumptions of Theorem 8, we have*

$$(i)\ \frac{1}{\tilde{G}_n} \leqslant \frac{1}{\tilde{H}_I^{\frac{W_I}{W_n}} \tilde{H}_{\bar{I}}^{\frac{W_{\bar{I}}}{W_n}}} \leqslant \frac{1}{\tilde{H}_n}.$$

$$(ii)\ \frac{1}{\tilde{G}_n} \leqslant \left[\frac{W_I}{W_n\tilde{G}_I} + \frac{W_{\bar{I}}}{W_n\tilde{G}_{\bar{I}}}\right] \leqslant \frac{1}{\tilde{H}_n}$$

COROLLARY 10. *Under the assumptions of Theorem 8, we have*

$$(i)\ \frac{1}{\tilde{G}_n} \leqslant \min_I \frac{1}{\tilde{H}_I^{\frac{W_I}{W_n}} \tilde{H}_{\bar{I}}^{\frac{W_{\bar{I}}}{W_n}}} \quad \text{and} \quad \frac{1}{\tilde{H}_n} \geqslant \max_I \frac{1}{\tilde{H}_I^{\frac{W_I}{W_n}} \tilde{H}_{\bar{I}}^{\frac{W_{\bar{I}}}{W_n}}}.$$

$$(ii)\ \frac{1}{\tilde{G}_n} \leqslant \min_I \left[\frac{W_I}{W_n\tilde{G}_I} + \frac{W_{\bar{I}}}{W_n\tilde{G}_{\bar{I}}}\right] \quad \text{and} \quad \frac{1}{\tilde{H}_n} \geqslant \max_I \left[\frac{W_I}{W_n\tilde{G}_I} + \frac{W_{\bar{I}}}{W_n\tilde{G}_{\bar{I}}}\right].$$

Here we have another important result with some corollaries.

THEOREM 9. *Under the assumptions given in* $\mathbf{H}$ *and for* $r \leqslant 1$, *we have the following inequalities*

$$\tilde{M}_n^{[r]} \leqslant \frac{W_I}{W_n}\tilde{M}_I^{[r]} + \frac{W_{\bar{I}}}{W_n}\tilde{M}_{\bar{I}}^{[r]} \leqslant \tilde{A}_n \tag{36}$$

For $r \geqslant 1$, *inequalities in* (36) *are reversed.*

Proof. For $r \leqslant 1$, $r \neq 0$, use Theorem 3 for convex function $f(x) = x^{\frac{1}{r}}$ and replacing a_j and x_{ij} with a_j^r and x_{ij}^r respectively and for $r = 0$ use Theorem 3 for convex function $f(x) = \exp x$, replacing a_j and x_{ij} with $\ln a_j$ and $\ln x_{ij}$ respectively, we obtain (36).

If $r \geqslant 1$, then function $f(x) = x^{\frac{1}{r}}$ is concave, so inequalities in (36) are reversed. □

By using previous result and Remark 3 we obtain the following result.

COROLLARY 11. *Let all the assumptions of Theorem 9 be valid. Then for $r \leqslant 1$, we have following inequalities*

$$\tilde{M}_n^{[r]} \leqslant \min_I \left[\frac{W_I}{W_n}\tilde{M}_I^{[r]} + \frac{W_{\bar{I}}}{W_n}\tilde{M}_{\bar{I}}^{[r]}\right], \quad \tilde{A}_n \geqslant \max_I \left[\frac{W_I}{W_n}\tilde{M}_I^{[r]} + \frac{W_{\bar{I}}}{W_n}\tilde{M}_{\bar{I}}^{[r]}\right]. \tag{37}$$

For $r \geqslant 1$, inequalities in (37) are reversed.

Here we have some consequences of last two results.

COROLLARY 12. *Under the assumptions of Theorem 9, we have*

$$\tilde{H}_n \leqslant \frac{W_I}{W_n}\tilde{H}_I + \frac{W_{\bar{I}}}{W_n}\tilde{H}_{\bar{I}} \leqslant \tilde{A}_n.$$

COROLLARY 13. *Under the assumptions of Theorem 9, we have*

$$\tilde{H}_n \leqslant \min_I \left[\frac{W_I}{W_n}\tilde{H}_I + \frac{W_{\bar{I}}}{W_n}\tilde{H}_{\bar{I}}\right], \quad \tilde{A}_n \geqslant \max_I \left[\frac{W_I}{W_n}\tilde{H}_I + \frac{W_{\bar{I}}}{W_n}\tilde{H}_{\bar{I}}\right].$$

REMARK 9. It is easy to see that, (33) is also direct consequence of Theorem 9.

THEOREM 10. *Under the assumptions given in* **H** *let $r, s \in \mathbb{R}$, $r \leqslant s$.*
(i) If $s \geqslant 0$, then

$$\left(\tilde{M}_n^{[r]}\right)^s \leqslant \frac{W_I}{W_n}\left(\tilde{M}_I^{[r]}\right)^s + \frac{W_{\bar{I}}}{W_n}\left(\tilde{M}_{\bar{I}}^{[r]}\right)^s \leqslant \left(\tilde{M}_n^{[s]}\right)^s. \tag{38}$$

(ii) If $s < 0$, then inequalities in (38) are reversed.

Proof. Let $s \geqslant 0$. Using Theorem 3 to convex function $f(x) = x^{\frac{s}{r}}$ and replacing a_j and x_{ij} with a_j^r and x_{ij}^r respectively, we obtain (38).

If $s < 0$, then function $f(x) = x^{\frac{s}{r}}$ is concave so inequalities in (38) are reversed. □

Following result follows from previous theorem and Remark 3.

COROLLARY 14. *Let all the assumptions of Theorem 10 be valid and let $r, s \in \mathbb{R}$, $r \leqslant s$.*
(i) If $s \geqslant 0$, then

$$\left(\tilde{M}_n^{[r]}\right)^s \leqslant \min_I \left[\frac{W_I}{W_n}\left(\tilde{M}_I^{[r]}\right)^s + \frac{W_{\bar{I}}}{W_n}\left(\tilde{M}_{\bar{I}}^{[r]}\right)^s\right], \tag{39}$$

$$\left(\tilde{M}_n^{[s]}\right)^s \geqslant \max_I \left[\frac{W_I}{W_n}\left(\tilde{M}_I^{[r]}\right)^s + \frac{W_{\bar{I}}}{W_n}\left(\tilde{M}_{\bar{I}}^{[r]}\right)^s\right]. \tag{40}$$

(ii) If $s < 0$, then inequalities in (39) and (40) are reversed.

Let ϕ be continuous and strictly monotonic function on J. Then for a given n-tuple $\mathbf{x}=(x_1,\dots,x_n)\in J^n$ and real n-tuple $\mathbf{w}=(w_1,\dots,w_n)$ with $W_n\neq 0$, the value

$$M_{\phi}^{[n]}=\phi^{-1}\left(\frac{1}{W_n}\sum_{i=1}^{n}w_i\phi(x_i)\right)$$

is well defined and is called *quasi − arithmetic mean* of $\mathbf{x}$ with weight $\mathbf{w}$ (see e. g. [5, p. 215]). If we define, under the assumptions of Theorem 1,

$$\tilde{M}_{\phi}^{[n]}=\phi^{-1}\left(\sum_{j=1}^{m}\phi(a_j)-\frac{1}{W_n}\sum_{j=1}^{k-1}\sum_{i=1}^{n}w_i\phi(x_{ij})-\frac{1}{W_n}\sum_{j=k+1}^{m}\sum_{i=1}^{n}w_i\phi(x_{ij})\right). \tag{41}$$

then we have the following results.

THEOREM 11. *Let ϕ and ψ be two continuous and strictly monotonic functions on J. If $\psi\circ\phi^{-1}$ is convex on J, then under the assumptions given in* $\mathbf{H}$, *the following series of inequalities hold:*

$$\begin{aligned}W_n\left(\psi\left(\tilde{M}_{\psi}^{[n]}\right)-\psi\left(\tilde{M}_{\phi}^{[n]}\right)\right)&\geqslant W_{n-1}\left(\psi\left(\tilde{M}_{\psi}^{[n-1]}\right)-\psi\left(\tilde{M}_{\phi}^{[n-1]}\right)\right)\\&\geqslant\cdots\geqslant W_1\left(\psi\left(\tilde{M}_{\psi}^{[1]}\right)-\psi\left(\tilde{M}_{\phi}^{[1]}\right)\right)\geqslant 0\end{aligned}\tag{42}$$

If $\psi\circ\phi^{-1}$ is concave on J, then inequalities in (42) are reversed.

Proof. Applying (13) to convex function $f=\psi\circ\phi^{-1}$ and replacing a_j and x_{ij} with $\phi(a_j)$ and $\phi(x_{ij})$ respectively we obtain (42), since in this case

$$\begin{aligned}F(I_t)=W_t&\left[\sum_{j=1}^{m}\psi(a_j)-\frac{1}{W_{I_t}}\sum_{j=1}^{k-1}\sum_{i\in I_t}w_i\psi(x_{ij})-\frac{1}{W_{I_t}}\sum_{j=k+1}^{m}\sum_{i\in I_t}w_i\psi(x_{ij})\right].\\&-(\psi\circ\phi^{-1})\left[\sum_{j=1}^{m}\phi(a_j)-\frac{1}{W_{I_t}}\sum_{j=1}^{k-1}\sum_{i\in I_t}w_i\phi(x_{ij})-\frac{1}{W_{I_t}}\sum_{j=k+1}^{m}\sum_{i\in I_t}w_i\phi(x_{ij})\right]\\&=W_t\left(\psi\left(\tilde{M}_{\psi}^{[t]}\right)-\psi\left(\tilde{M}_{\phi}^{[t]}\right)\right).\end{aligned}$$ □

REMARK 10. Theorem 5, 6 and 7 follow from Theorem 11, by choosing adequate functions ϕ, ψ and appropriate substitutions.

COROLLARY 15. *Under the assumptions of Theorem 11, we have*

$$\begin{aligned}&W_n\left(\psi\left(\tilde{M}_{\psi}^{[n]}\right)-\psi\left(\tilde{M}_{\phi}^{[n]}\right)\right)\\&\geqslant\max_{1\leqslant s\leqslant t\leqslant n}\left[(w_s+w_t)\left[\sum_{j=1}^{m}\psi(a_j)-\sum_{j=1}^{k-1}\frac{w_s\psi(x_{sj})+w_t\psi(x_{tj})}{w_s+w_t}\right.\right.\\&\qquad\left.-\sum_{j=k+1}^{m}\frac{w_s\psi(x_{sj})+w_t\psi(x_{tj})}{w_s+w_t}\right.\\&\left.\left.-(\psi\circ\phi^{-1})\left(\sum_{j=1}^{m}\phi(a_j)-\sum_{j=1}^{k-1}\frac{w_s\phi(x_{sj})+w_t\phi(x_{tj})}{w_s+w_t}-\sum_{j=k+1}^{m}\frac{w_s\phi(x_{sj})+w_t\phi(x_{tj})}{w_s+w_t}\right)\right]\right]\end{aligned}\tag{43}$$

and

$$W_n\left(\psi\left(\tilde{M}_{\psi}^{[n]}\right)-\psi\left(\tilde{M}_{\phi}^{[n]}\right)\right)\geqslant\max_{1\leqslant t\leqslant n}\left[w_t\left[\sum_{j=1}^{m}\psi(a_j)-\sum_{j=1}^{k-1}\psi(x_{tj})-\sum_{j=k+1}^{m}\psi(x_{tj})\right.\right.$$
$$\left.\left.-(\psi\circ\phi^{-1})\left(\sum_{j=1}^{m}\phi(a_j)-\sum_{j=1}^{k-1}\phi(x_{tj})-\sum_{j=k+1}^{m}\phi(x_{tj})\right)\right]\right] \quad (44)$$

If $\psi\circ\phi^{-1}$ is concave on J, then inequalities in (43) and (44) with maximum replaced with minimum, are reversed.

THEOREM 12. *Let ϕ and ψ be two continuous and strictly monotonic functions on J. If $\psi\circ\phi^{-1}$ is convex on J, then under the assumptions given in $\mathbf{H}$, the following series of inequalities hold:*

$$\psi\left(\tilde{M}_{\phi}^{[n]}\right)\leqslant\frac{W_I}{W_n}\psi\left(\tilde{M}_{\phi}^{[I]}\right)+\frac{W_{\bar{I}}}{W_n}\psi\left(\tilde{M}_{\phi}^{[\bar{I}]}\right)\leqslant\psi\left(\tilde{M}_{\psi}^{[n]}\right), \quad (45)$$

where $\tilde{M}_{\phi}^{[I]}=\phi^{-1}\left(\sum_{j=1}^{m}\phi(a_j)-\frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I}w_i\phi(x_{ij})-\frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I}w_i\phi(x_{ij})\right)$.

Proof. Applying Theorem 3 to convex function $f=\psi\circ\phi^{-1}$ and replacing a_j and x_{ij} with $\phi(a_j)$ and $\phi(x_{ij})$ respectively, we obtain (45). □

Remark 3 gives us the following result as a special case of previous result.

COROLLARY 16. *Under the assumptions of Theorem 12, we have*

$$\psi\left(\tilde{M}_{\phi}^{[n]}\right)\leqslant\min_{I}\left[\frac{W_I}{W_n}\psi\left(\tilde{M}_{\phi}^{[I]}\right)+\frac{W_{\bar{I}}}{W_n}\psi\left(\tilde{M}_{\phi}^{[\bar{I}]}\right)\right],$$
$$\psi\left(\tilde{M}_{\psi}^{[n]}\right)\geqslant\max_{I}\left[\frac{W_I}{W_n}\psi\left(\tilde{M}_{\phi}^{[I]}\right)+\frac{W_{\bar{I}}}{W_n}\psi\left(\tilde{M}_{\phi}^{[\bar{I}]}\right)\right].$$

REMARK 11. (a) Theorem 8, 9 and 10 follow from Theorem 12, by choosing adequate functions ϕ, ψ and appropriate substitutions.

(b) In all theorems reverse inequalities hold for concave functions.

(c) By imposing different conditions on k and weights w_i's we can obtain many special cases of our results of this section in articles [13, 18, 22].

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(Received April 23, 2019)

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