Journal of Mathematical Inequalities
Volume 15, Number 4 (2021), 1447–1458 doi:10.7153/jmi-2021-15-99

SHARP ESTIMATES FOR m–LINEAR p–ADIC HARDY AND HARDY–LITTLEWOOD–PÓLYA OPERATORS ON p–ADIC CENTRAL MORREY SPACES

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(*Communicated by V. Dmitrievič Stepanov*)

Abstract. In this paper, we consider the mapping property of both m-linear p-adic Hardy operator $\mathscr{H}_m^p$ and Hardy-Littlewood-Pólya operators T_m^p on p-adic central Morrey-type spaces $B^{p_1,\lambda_1}(\mathbb{Q}_p^n)\times\cdots\times B^{p_m,\lambda_m}(\mathbb{Q}_p^n)$. The obtained bounds turn to be sharp when $\lambda_1 p_1=\cdots=\lambda_m p_m$.

1. Introduction

In recent years, the applications of p-adic analysis in the fields of quantum mechanics, probability theory, dynamics and other mathematical physics have attracted widespread attentions [13, 14].

For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers, which is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p=0$; if any nonzero rational number x is represented as $x=p^{\gamma}(m/n)$, where m, n are integers which are indivisible by p, and γ is an integer, then $|x|_p=p^{-\gamma}$. It's easy to show that the norm satisfies the following properties:

$$|xy|_p=|x|_p|y|_p,\qquad |x+y|_p\leqslant\max\{|x|_p,|y|_p\}.$$

From the standard p-adic analysis, we see that any nonzero p-adic number $x\in\mathbb{Q}_p$ can be uniquely represented in the canonical series $x=p^{\gamma}\sum_{j=0}^{\infty}a_jp^j$, where $\gamma\in\mathbb{Z}$, $a_j\in\{0,1,\cdots,p-1\}$ and $a_0\neq 0$, so we have $|x|_p=p^{-\gamma}$.

Let $\mathbb{Q}_p^*=\mathbb{Q}_p\setminus\{0\}$ and $\mathbb{Z}_p=\{x\in\mathbb{Q}_p:|x|_p\leqslant 1\}$.

The space $\mathbb{Q}_p^n$, which is called n-dimensional p-adic field, consists of points $x=(x_1,x_2,\cdots,x_n)$, where $x_j\in\mathbb{Q}_p$, $j=1,2,\cdots,n$. The p-adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p:=\max_{1\leqslant j\leqslant n}|x_j|_p, x\in\mathbb{Q}_p^n.$$

Mathematics subject classification (2020): 26A33, 42B25, 42B35.
Keywords and phrases: Hardy operator, Hardy-Littlewood-Pólya operator, p-adic fields, p-adic central Morrey spaces, sharp bound.

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Paper JMI-15-99

Denote by

$$B_\gamma(a)=\{x\in\mathbb{Q}_p^n : |x-a|_p\leqslant p^\gamma\}$$

the ball with center at $a\in\mathbb{Q}_p^n$ and radius p^γ, and

$$S_\gamma(a):=\{x\in\mathbb{Q}_p^n : |x-a|_p=p^\gamma\}=B_\gamma(a)\setminus B_{\gamma-1}(a)$$

the sphere with center at $a\in\mathbb{Q}_p^n$ and radius p^γ.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists a Haar measure $\mathrm{d}x$ on $\mathbb{Q}_p^n$, which is unique up to positive constant multiple and is translation invariant. We normalize the measure $\mathrm{d}x$ by the equality

$$\int_{B_0(0)}\mathrm{d}x=|B_0(0)|_H=1,$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^n$. By simple calculation, we can obtain that

$$|B_\gamma(a)|_H=p^{\gamma n},\quad |S_\gamma(a)|_H=p^{\gamma n}(1-p^{-n}),$$

for any $a\in\mathbb{Q}_p^n$. For more information about the p-adic field, we refer readers to [12] and [14].

The famous Hardy's integral inequality yields that

$$\|Hf\|_{L^q(\mathbb{R}^+)}\leqslant\frac{q}{q-1}\|f\|_{L^q(\mathbb{R}^+)},$$

where $1<q<\infty$ and the classical Hardy operator H is defined by

$$Hf(x):=\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t,$$

in which f is a nonnegative integrable function on $\mathbb{R}^+$, and the constant $\frac{q}{q-1}$ is the best possible (see [8]). That is,

$$\|H\|_{L^q(\mathbb{R}^+)\to L^q(\mathbb{R}^+)}=\frac{q}{q-1}.$$

In [5], the n-dimensional Hardy operator was introduced by Faris. For any non-negative locally integrable function f on $\mathbb{R}^n$,

$$\mathscr{H}f(x):=\frac{1}{\Omega_n|x|^n}\int_{|y|<|x|}f(y)\,\mathrm{d}y,\quad x\in\mathbb{R}^n\setminus\{0\},$$

where Ω_n is the volume of the unit ball in $\mathbb{R}^n$. The norm of $\mathscr{H}$ on $L^q(\mathbb{R}^n)$ obtained by Christ and Grafakos is

$$\|\mathscr{H}\|_{L^q(\mathbb{R}^n)\to L^q(\mathbb{R}^n)}=\frac{q}{q-1},$$

which is equal to that of the classical Hardy operator (see [4]). In [7], Fu et al. introduced the m-linear Hardy operator, which is defined by

$$\mathscr{H}^m(f_1,\cdots,f_m)(x) = \frac{1}{\Omega_{mn}|x|^{mn}} \int_{|(y_1,\cdots,y_m)|<|x|} f_1(y_1)\cdots f_m(y_m)\,\mathrm{d}y_1\cdots\mathrm{d}y_m,$$

where $x \in \mathbb{R}^n \setminus \{0\}$ and $f_1,\cdots,f_m$ are nonnegative locally integrable functions on $\mathbb{R}^n$. Similarly, in [15], Fu et al. defined the m-linear p-adic Hardy operator as follows:

DEFINITION 1. Let m be a positive integer and $f_1,\cdots,f_m$ be nonnegative locally integrable functions on $\mathbb{Q}_p^n$. The m-linear p-adic Hardy operator is defined by

$$\mathscr{H}_m^p(f_1,\cdots,f_m)(x) = \frac{1}{|x|_p^{mn}} \int_{|(y_1,\cdots,y_m)|_p\leqslant|x|_p} f_1(y_1)\cdots f_m(y_m)\,\mathrm{d}y_1\cdots\mathrm{d}y_m, \qquad (1)$$

where $x \in \mathbb{Q}_p^n \setminus \{0\}$.

In [15], Fu et al. obtained the sharp estimate of the p-adic Hardy operator on Lebesgue spaces with power weights. Recently, there are amounts of paper dealing with sharp strong and weak estimates of the p-adic Hardy type operators on various function spaces, see for example [2, 9, 11, 10, 6]. Inspired by [15], we will consider the sharp estimate of the m-linear p-adic Hardy operator on the product of p-adic central Morrey spaces.

For $k \in \mathbb{Z}$, we denote the ball and the sphere in the n-dimensional p-adic field by $B_k = \{x \in \mathbb{Q}_p^n : |x|_p \leqslant p^k\}$ and $S_k = \{x \in \mathbb{Q}_p^n : |x|_p = p^k\}$. The p-adic central Morrey space is defined as follows.

DEFINITION 2. Let $1 \leqslant q < \infty$ and $-1/q \leqslant \lambda < 0$. The p-adic central Morrey space $B^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$\|f\|_{B^{q,\lambda}(\mathbb{Q}_p^n)} := \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(x)|^q\,\mathrm{d}x \right)^{1/q} < \infty.$$

It is clear that $B^{q,-1/q}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$. If $1 \leqslant q_1 \leqslant q_2 < \infty$, by the Hölder's inequality.

$$B^{q_1,\lambda}(\mathbb{Q}_p^n) \subset B^{q_2,\lambda}(\mathbb{Q}_p^n)$$

holds for $-1/q_2 \leqslant \lambda < 0$.

Noting that the definition is very similar to that of central Morrey spaces on $\mathbb{R}^n$, which was introduced in [7]. For more about Morrey spaces, we refer the reader to [1, 16] and the references therein.

The Hardy-Littlewood-Pólya operator, which is the sum of the Hardy operator and its dual operator, was defined in [3] by

$$Tf(x) = \int_0^\infty \frac{f(y)}{\max(x,y)}\,\mathrm{d}y.$$

The authors [3] also obtained the norm of the Hardy-Littlewood-Pólya operator on $L^q(\mathbb{R}^+)$:

$$\|Tf(x)\|_{L^q(R^+)\to L^q(R^+)} = \frac{q^2}{q-1},$$

where $1 < q < \infty$.

On p-adic field, Fu et al. [15] also defined the m-linear p-adic Hardy-Littlewood-Pólya operator.

DEFINITION 3. Let m be an integer and $f_1,\cdots,f_m$ be nonnegative locally integrable functions on $\mathbb{Q}_p$. The m-linear p-adic Hardy-Littlewood-Pólya operator is defined by

$$T_m^p(f_1,\cdots,f_m)(x) = \int_{\mathbb{Q}_p}\cdots\int_{\mathbb{Q}_p} \frac{f_1(y_1)\cdots f_m(y_m)}{[\max(|x|_p,|y_1|_p,\cdots,|y_m|_p)]^m}\,\mathrm{d}y_1\cdots\mathrm{d}y_m.$$

In this paper, we are also interested in the mapping property of T_m^p on central Morrey spaces, which can be seen as a generalization of the results in [15].

In Section 2, we will furnish the estimate of the m-linear p-adic Hardy operator and the m-linear p-adic Hardy-Littlewood-Pólya operator on the product of p-adic central Morrey spaces, and then obtain the sharp bound for the particular case $\lambda_1 p_1 = \cdots = \lambda_m p_m$.

2. Main results

This section contains two main theorems and their proofs. Our first result below discovers the upper bound for the m-linear p-adic Hardy operator on p-adic central Morrey-type spaces.

THEOREM 1. *Let $m\in\mathbb{Z}^+$, $f_i \in B^{q_i,\lambda_i}(\mathbb{Q}_p^n)$, $1\leqslant q_i<\infty$, $-1/q_i \leqslant \lambda_i < 0$, $i = 1,\cdots,m$, $1\leqslant q<\infty$, $1/q=\sum_{i=1}^m 1/q_i$, $\lambda = \sum_{i=1}^m \lambda_i$ (It is easy to see that $-1/q\leqslant\lambda<0$). Then*

$$\|\mathscr{H}_m^p(f_1,\cdots,f_m)\|_{B^{q,\lambda}(\mathbb{Q}_p^n)} \leqslant C_{\mathscr{H}}\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\cdots\|f_m\|_{B^{q_m,\lambda_m}(\mathbb{Q}_p^n)}, \quad (2)$$

where

$$C_{\mathscr{H}} = \frac{(1-p^{-n})^m}{\prod_{i=1}^m (1-p^{-n(1+\lambda_i)})}.$$

Moreover, if $\lambda_1 p_1 = \cdots = \lambda_m p_m$, the constant $C_{\mathscr{H}}$ is the best possible.

Proof. The proof of the case when $m=1$ is similar to and even simpler than that of the case when $m\geqslant 2$, so for simplicity, we will only discuss the case $m\geqslant 2$.

(I) When $m=2$.
Set
$$g_i(x)=\frac{1}{1-p^{-n}}\int_{|\xi_i|_p=1} f_i(|x|_p^{-1}\xi_i)\,\mathrm{d}\xi_i,\quad x\in\mathbb{Q}_p^n,\quad i=1,2.$$
On the one hand, we have
$$\begin{aligned}
&\mathscr{H}_2^p(g_1,g_2)(x)\\
=&\frac{1}{|x|_p^{2n}}\int_{|(y_1,y_2)|_p\leqslant|x|_p} g_1(y_1)g_2(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2\\
=&\frac{1}{(1-p^{-n})^2}\frac{1}{|x|_p^{2n}}\int_{|(y_1,y_2)|_p\leqslant|x|_p}\prod_{i=1}^{2}\left(\int_{|\xi_i|_p=1} f_i(|y_i|_p^{-1}\xi_i)\,\mathrm{d}\xi_i\right)\mathrm{d}y_1\,\mathrm{d}y_2\\
=&\frac{1}{(1-p^{-n})^2}\frac{1}{|x|_p^{2n}}\int_{|(y_1,y_2)|_p\leqslant|x|_p}\prod_{i=1}^{2}\left(\int_{|z_i|_p=|y_i|_p} f_i(z_i)|y_i|_p^{-n}\,\mathrm{d}z_i\right)\mathrm{d}y_1\,\mathrm{d}y_2\\
=&\frac{1}{(1-p^{-n})^2}\frac{1}{|x|_p^{2n}}\int_{|(z_1,z_2)|_p\leqslant|x|_p}\prod_{i=1}^{2}\left(\int_{|y_i|_p=|z_i|_p} |y_i|_p^{-n}\,\mathrm{d}y_i\right)f_1(z_1)f_2(z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\\
=&\frac{1}{|x|_p^{2n}}\int_{|(z_1,z_2)|_p\leqslant|x|_p} f_1(z_1)f_2(z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\\
=&\mathscr{H}_2^p(f_1,f_2)(x).
\end{aligned}$$
On the other hand, by the Hölder's inequality, we obtain
$$\begin{aligned}
&\|g_i\|_{B^{q_i,\lambda_i}(\mathbb{Q}_p^n)}\\
=&\sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma|_H^{1+\lambda_iq_i}}\int_{B_\gamma}\left|\frac{1}{1-p^{-n}}\int_{|\xi_i|_p=1} f_i(|x|_p^{-1}\xi_i)\,\mathrm{d}\xi_i\right|^{q_i}\mathrm{d}x\right)^{1/q_i}\\
\leqslant&\sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma|_H^{1+\lambda_iq_i}}\left(\frac{1}{1-p^{-n}}\right)^{q_i}\int_{B_\gamma}\left(\int_{|\xi_i|_p=1}|f_i(|x|_p^{-1}\xi_i)|\,\mathrm{d}\xi_i\right)^{q_i}\mathrm{d}x\right)^{1/q_i}\\
\leqslant&\sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma|_H^{1+\lambda_iq_i}}\left(\frac{1}{1-p^{-n}}\right)^{q_i}\right.\\
&\left.\times\int_{B_\gamma}\left(\int_{|\xi_i|_p=1}|f_i(|x|_p^{-1}\xi_i)|^{q_i}\,\mathrm{d}\xi_i\right)\left(\int_{|\xi_i|_p=1}\mathrm{d}\xi_i\right)^{q_i-1}\mathrm{d}x\right)^{1/q_i}\\
=&\sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma|_H^{1+\lambda_iq_i}}\frac{1}{1-p^{-n}}\int_{B_\gamma}\left(\int_{|z_i|_p=|x|_p}|f_i(z_i)|^{q_i}\,\mathrm{d}z_i\right)|x|_p^{-n}\,\mathrm{d}x\right)^{1/q_i}\\
=&\sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma|_H^{1+\lambda_iq_i}}\frac{1}{1-p^{-n}}\int_{B_\gamma}\left(\int_{|x|_p=|z_i|_p}|x|_p^{-n}\,\mathrm{d}x\right)|f_i(z_i)|^{q_i}\,\mathrm{d}z_i\right)^{1/q_i}\\
=&\|f_i\|_{B^{q_i,\lambda_i}(\mathbb{Q}_p^n)},\quad i=1,2.
\end{aligned}$$

The above two formulas yield that

$$\frac{\|\mathscr{H}_2^p(f_1,f_2)\|_{B^{q,\lambda}(\mathbb{Q}_p^n)}}{\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)}} \leqslant \frac{\|\mathscr{H}_2^p(g_1,g_2)\|_{B^{q,\lambda}(\mathbb{Q}_p^n)}}{\|g_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|g_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)}}.$$

So the operator $\mathscr{H}_2^p$ and its restriction to the functions g satisfying $g(x)=g(|x|_p^{-1})$ have the same operator norm on $B^{q,\lambda}(\mathbb{Q}_p^n)$. In the following, we may assume that $f_i\in B^{q_i,\lambda_i}(\mathbb{Q}_p^n)$, $i=1,2$, which satisfy that $f_i(x)=f_i(|x|_p^{-1})$, $i=1,2$.

By changing of variables again, we get

$$\begin{aligned}
&\|\mathscr{H}_2^p(f_1,f_2)\|_{B^{q,\lambda}(\mathbb{Q}_p^n)}\\
&=\sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}\left|\frac{1}{|x|_p^{2n}}\int_{|(z_1,z_2)|_p\leqslant|x|_p}f_1(z_1)f_2(z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\right|^q\mathrm{d}x\right)^{1/q}\\
&=\sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}\left|\int_{|(z_1,z_2)|_p\leqslant1}f_1(|x|_p^{-1}z_1)f_2(|x|_p^{-1}z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\right|^q\mathrm{d}x\right)^{1/q}\\
&=\sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}\left|\int_{|(z_1,z_2)|_p\leqslant1}f_1(|z_1|_p^{-1}x)f_2(|z_2|_p^{-1}x)\,\mathrm{d}z_1\,\mathrm{d}z_2\right|^q\mathrm{d}x\right)^{1/q}.
\end{aligned}$$

Then using the Minkowski's integral inequality and the Hölder's inequality with $q/q_1+q/q_2=1$, we get

$$\begin{aligned}
&\|\mathscr{H}_2^p(f_1,f_2)\|_{B^{q,\lambda}(\mathbb{Q}_p^n)}\\
&\leqslant\sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}\left|\int_{|(z_1,z_2)|_p\leqslant1}f_1(|z_1|_p^{-1}x)f_2(|z_2|_p^{-1}x)\,\mathrm{d}z_1\,\mathrm{d}z_2\right|^q\mathrm{d}x\right)^{1/q}\\
&\leqslant\sup_{\gamma\in\mathbb{Z}}\frac{1}{|B_\gamma|_H^{1/q+\lambda}}\int_{|(z_1,z_2)|_p\leqslant1}\left(\int_{B_\gamma}\left|f_1(|z_1|_p^{-1}x)f_2(|z_2|_p^{-1}x)\right|^q\mathrm{d}x\right)^{1/q}\mathrm{d}z_1\,\mathrm{d}z_2\\
&\leqslant\sup_{\gamma\in\mathbb{Z}}\frac{1}{|B_\gamma|_H^{1/q+\lambda}}\int_{|(z_1,z_2)|_p\leqslant1}\prod_{i=1}^2\left(\int_{B_\gamma}\left|f_i(|z_i|_p^{-1}x)\right|^{q_i}\mathrm{d}x\right)^{1/q_i}\mathrm{d}z_1\,\mathrm{d}z_2\\
&=\sup_{\gamma\in\mathbb{Z}}\int_{|(z_1,z_2)|_p\leqslant1}\prod_{i=1}^2\left(\frac{1}{|B_{\gamma\log_p|z_i|_p}|_H^{1+\lambda_iq_i}}\int_{B_{\gamma\log_p|z_i|_p}}|f_i(x)|^{q_i}\,\mathrm{d}x\right)^{1/q_i}\\
&\qquad\times|z_1|_p^{n\lambda_1}|z_2|_p^{n\lambda_2}\,\mathrm{d}z_1\,\mathrm{d}z_2\\
&\leqslant\int_{|(z_1,z_2)|_p\leqslant1}|z_1|_p^{n\lambda_1}|z_2|_p^{n\lambda_2}\,\mathrm{d}z_1\,\mathrm{d}z_2\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p^n)}.
\end{aligned}\tag{3}$$

It is easy to see that $\int_{|(z_1,z_2)|_p\leqslant1}|z_1|_p^{n\lambda_1}|z_2|_p^{n\lambda_2}\,\mathrm{d}z_1\,\mathrm{d}z_2$ is a finite constant, and we will prove that it's equal to $C_{\mathscr{H}}$ soon.

(II) When $m \geqslant 3$,

The proof in this case is similar to that of the previous case. We derive from formula (3) that

$$\begin{aligned}
&\|\mathscr{H}_m^p(f_1,\cdots,f_m)\|_{B^{q,\lambda}(\mathbb{Q}_p^n)}\\
&\leqslant \int_{|(z_1,\cdots,z_m)|_p\leqslant 1} |z_1|_p^{n\lambda_1}\cdots|z_m|_p^{n\lambda_m}\,\mathrm{d}z_1\cdots\mathrm{d}z_m \|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\cdots\|f_m\|_{B^{q_m,\lambda_m}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Let

$$\begin{aligned}
D_1 &= \left\{(z_1,\cdots,z_m)\in\mathbb{Q}_p^n\times\cdots\times\mathbb{Q}_p^n : |z_1|_p\leqslant 1, |z_k|_p\leqslant|z_1|_p, 1<k\leqslant m\right\};\\
D_i &= \{(z_1,\cdots,z_m)\in\mathbb{Q}_p^n\times\cdots\times\mathbb{Q}_p^n :\\
&\qquad |z_i|_p\leqslant 1, |z_j|_p<|z_i|_p, |z_k|_p\leqslant|z_i|_p, 1\leqslant j<i<k\leqslant m\};\\
D_m &= \left\{(z_1,\cdots,z_m)\in\mathbb{Q}_p^n\times\cdots\times\mathbb{Q}_p^n : |z_m|_p\leqslant 1, |z_j|_p<|z_m|_p, 1\leqslant j\leqslant m\right\}.
\end{aligned}$$

It is obvious that

$$\bigcup_{j=1}^m D_j = \left\{(z_1,\cdots,z_m)\in\mathbb{Q}_p^n\times\cdots\times\mathbb{Q}_p^n : |(z_1,\cdots,z_m)|_p\leqslant 1\right\},$$

and $D_i\bigcap D_j=\varnothing$.

Let

$$I_j := \int_{D_j} |z_1|_p^{n\lambda_1}\cdots|z_m|_p^{n\lambda_m}\,\mathrm{d}z_1\cdots\mathrm{d}z_m.$$

Now, we begin to calculate each $I_j, j=1,\cdots,m$.

$$\begin{aligned}
I_1 &= \int_{D_1} |z_1|_p^{n\lambda_1}\cdots|z_m|_p^{n\lambda_m}\,\mathrm{d}z_1\cdots\mathrm{d}z_m\\
&= \int_{|z_1|_p\leqslant 1} |z_1|_p^{n\lambda_1}\prod_{k=2}^m\left(\int_{|z_k|_p\leqslant|z_1|_p}|z_k|_p^{n\lambda_k}\,\mathrm{d}z_k\right)\mathrm{d}z_1\\
&= \int_{|z_1|_p\leqslant 1} |z_1|_p^{n\lambda_1}\prod_{k=2}^m\left(\frac{1-p^{-n}}{1-p^{-n(1+\lambda_k)}}|z_1|_p^{n(1+\lambda_k)}\right)\mathrm{d}z_1\\
&= \frac{(1-p^{-n})^{m-1}}{\prod\limits_{1<k\leqslant m}(1-p^{-n(1+\lambda_k)})}\int_{|z_1|_p\leqslant 1}|z_1|_p^{n(m-1+\lambda)}\,\mathrm{d}z_1\\
&= \frac{(1-p^{-n})^m}{1-p^{-n(m+\lambda)}}\cdot\frac{1}{\prod\limits_{1<k\leqslant m}(1-p^{-n(1+\lambda_k)})}.
\end{aligned}$$

Similarly, we can obtain

$$I_j = \frac{(1-p^{-n})^m}{1-p^{-n(m+\lambda)}}\cdot\frac{1}{\prod\limits_{1\leqslant k\leqslant m, k\neq j}(1-p^{-n(1+\lambda_k)})\prod\limits_{k=1}^{j-1}p^{n(1+\lambda_k)}},\quad j=2,\cdots,m.$$

Then, we have

$$\begin{aligned}&\int_{|(z_1,\cdots,z_m)|_p\leqslant 1}|z_1|_p^{n\lambda_1}\cdots|z_m|_p^{n\lambda_m}\,\mathrm{d}z_1\cdots\mathrm{d}z_m\\&=\sum_{i=1}^{m}I_i=\frac{(1-p^{-n})^m}{\prod\limits_{k=1}^{m}(1-p^{-n(1+\lambda_k)})}\\&=C_{\mathscr{H}},\end{aligned}$$

so (2) is proved.

Finally, when $\lambda_1 q_1=\cdots=\lambda_m q_m$, we can get $\lambda q=\lambda_1 q_1=\cdots=\lambda_m q_m$. We set $f_i(x)=|x|_p^{n\lambda_i}$. It is easy to see that

$$\mathscr{H}_m^p(f_1,\cdots,f_m)(x)=C_{\mathscr{H}}|x|_p^{n\lambda},$$

and then

$$\|\mathscr{H}_m^p(f_1,\cdots,f_m)\|_{B^{q,\lambda}(\mathbb{Q}_p^n)}=C_{\mathscr{H}}\left(\frac{1-p^{-n}}{1-p^{-n(1+\lambda q)}}\right)^{1/q}.$$

By direct computation, there holds

$$\|f_i\|_{B^{q_i,\lambda_i}(\mathbb{Q}_p^n)}=\left(\frac{1-p^{-n}}{1-p^{-n(1+\lambda_i q_i)}}\right)^{1/q_i},$$

so

$$\frac{\|\mathscr{H}_m^p(f_1,\cdots,f_m)\|_{B^{q,\lambda}(\mathbb{Q}_p^n)}}{\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\cdots\|f_m\|_{B^{q_m,\lambda_m}(\mathbb{Q}_p^n)}}=C_{\mathscr{H}}.$$

We are done. □

Our second result is mainly about the estimate of T_m^p on p-adic central Morrey-type spaces.

THEOREM 2. *Let* $m\in\mathbb{Z}^+,\ f_i\in B^{q_i,\lambda_i}(\mathbb{Q}_p),\ 1\leqslant q_i<\infty,\ -1/q_i\leqslant\lambda_i<0,\ i=1,\cdots,m,\ 1\leqslant q<\infty,\ 1/q=\sum_{i=1}^m 1/q_i,\ \lambda=\sum_{i=1}^m\lambda_i$ *(also* $-1/q\leqslant\lambda<0$*). Then*

$$\|T_m^p(f_1,\cdots,f_m)\|_{B^{q,\lambda}(\mathbb{Q}_p)}\leqslant C_T\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p)}\cdots\|f_m\|_{B^{q_m,\lambda_m}(\mathbb{Q}_p)},\tag{4}$$

where

$$C_T=\frac{(1-p^{-1})^m(1-p^{-m})}{(1-p^{\lambda})\prod\limits_{i=1}^{m}(1-p^{-(\lambda_i+1)})}.$$

Moreover, if $\lambda_1 p_1=\cdots=\lambda_m p_m$*, the constant* C_T *is the best possible.*

Proof. We will only discuss the case $m \geqslant 2$, since $m = 1$ is much simpler.

(I) When $m = 2$.

By the change of variables $y_i = xz_i$, $i = 1,2$, we have

$$\begin{aligned}
&\|T_2^p(f_1,f_2)\|_{B^{q,\lambda}(\mathbb{Q}_p)}\\
&= \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma|_H^{1/q+\lambda}} \left(\int_{B_\gamma} \left| \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \frac{f_1(y_1)f_2(y_2)}{[\max(|x|_p,|y_1|_p,|y_2|_p)]^2} \mathrm{d}y_1\,\mathrm{d}y_2 \right|^q \mathrm{d}x \right)^{1/q}\\
&\leqslant \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma|_H^{1/q+\lambda}} \left(\int_{B_\gamma} \left(\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \frac{|f_1(y_1)f_2(y_2)|}{[\max(|x|_p,|y_1|_p,|y_2|_p)]^2} \mathrm{d}y_1\,\mathrm{d}y_2 \right)^q \mathrm{d}x \right)^{1/q}\\
&= \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma|_H^{1/q+\lambda}} \left(\int_{B_\gamma} \left(\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \frac{|f_1(xz_1)f_2(xz_2)|}{[\max(1,|z_1|_p,|z_2|_p)]^2} \mathrm{d}z_1\,\mathrm{d}z_2 \right)^q \mathrm{d}x \right)^{1/q}.
\end{aligned}$$

Then using the Minkowski's integral inequality and the Hölder's inequality with $q/q_1 + q/q_2 = 1$, we get

$$\begin{aligned}
&\|T_2^p(f_1,f_2)\|_{B^{q,\lambda}(\mathbb{Q}_p)}\\
&\leqslant \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma|_H^{1/q+\lambda}} \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \left(\int_{B_\gamma} |f_1(xz_1)f_2(xz_2)|^q \mathrm{d}x \right)^{1/q} \frac{1}{[\max(1,|z_1|_p,|z_2|_p)]^2} \mathrm{d}z_1\,\mathrm{d}z_2\\
&\leqslant \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma|_H^{1/q+\lambda}} \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \prod_{i=1}^{2} \left(\int_{B_\gamma} |f_i(xz_i)|^{q_i} \mathrm{d}x \right)^{1/q_i} \frac{1}{[\max(1,|z_1|_p,|z_2|_p)]^2} \mathrm{d}z_1\,\mathrm{d}z_2\\
&= \sup_{\gamma\in\mathbb{Z}} \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \prod_{i=1}^{2} \left(\frac{1}{|B_{\gamma \log_p |z_i|_p}|_H^{1+\lambda_i q_i}} \int_{B_{\gamma \log_p |z_i|_p}} |f_i(x)|^{q_i} \mathrm{d}x \right)^{1/q_i}\\
&\quad \times \frac{|z_1|^{\lambda_1}|z_2|^{\lambda_2}}{[\max(1,|z_1|_p,|z_2|_p)]^2} \mathrm{d}z_1\,\mathrm{d}z_2\\
&\leqslant \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \frac{|z_1|^{\lambda_1}|z_2|^{\lambda_2}}{[\max(1,|z_1|_p,|z_2|_p)]^2} \mathrm{d}z_1\,\mathrm{d}z_2 \|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p)} \|f_2\|_{B^{q_2,\lambda_2}(\mathbb{Q}_p)}. \qquad (5)
\end{aligned}$$

It is easy to see that $\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \frac{|z_1|^{\lambda_1}|z_2|^{\lambda_2}}{[\max(1,|z_1|_p,|z_2|_p)]^2} \mathrm{d}z_1\,\mathrm{d}z_2$ is a finite constant, and we will prove that it's equal to C_T later.

(II) When $m \geqslant 3$,

The proof in this case is similar to that of the previous case. Like the formula (5), we can get

$$\begin{aligned}
&\|T_m^p(f_1,\cdots,f_m)\|_{B^{q,\lambda}(\mathbb{Q}_p)}\\
&\leqslant \int_{\mathbb{Q}_p}\cdots\int_{\mathbb{Q}_p} \frac{|z_1|^{\lambda_1}\cdots|z_m|^{\lambda_m}}{[\max(1,|z_1|_p,\cdots,|z_m|_p)]^m} \mathrm{d}z_1\cdots\mathrm{d}z_m \|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p)} \cdots \|f_m\|_{B^{q_m,\lambda_m}(\mathbb{Q}_p)}.
\end{aligned}$$

Let

$$\begin{aligned}E_0 =& \left\{(z_1,\cdots,z_m)\in\mathbb{Q}_p\times\cdots\times\mathbb{Q}_p : |z_k|_p\leqslant 1, 1\leqslant k\leqslant m\right\};\\ E_1 =& \left\{(z_1,\cdots,z_m)\in\mathbb{Q}_p\times\cdots\times\mathbb{Q}_p : |z_1|_p> 1, |z_k|_p\leqslant |z_1|_p, 1< k\leqslant m\right\};\\ E_i =& \{(z_1,\cdots,z_m)\in\mathbb{Q}_p\times\cdots\times\mathbb{Q}_p :\\ & |z_i|_p>1, |z_j|_p<|z_i|_p, |z_k|_p\leqslant |z_i|_p, 1\leqslant j<i<k\leqslant m\};\\ E_m =& \left\{(z_1,\cdots,z_m)\in\mathbb{Q}_p\times\cdots\times\mathbb{Q}_p : |z_m|_p> 1, |z_j|_p< |z_m|_p, 1\leqslant j\leqslant m\right\}.\end{aligned}$$

It is obvious that

$$\bigcup_{j=0}^{m} E_j=\mathbb{Q}_p\times\cdots\times\mathbb{Q}_p,$$

and $E_i\bigcap E_j=\varnothing$. Let

$$J_j := \int_{E_j}\frac{|z_1|^{\lambda_1}\cdots|z_m|^{\lambda_m}}{[\max(1,|z_1|_p,\cdots,|z_m|_p)]^m}\,\mathrm{d}z_1\cdots\mathrm{d}z_m.$$

Now, we begin to calculate $J_j,\ j=0,\cdots,m$.

$$J_0=\prod_{i=1}^{m}\int_{|z_i|_p\leqslant 1}|z_i|_p^{\lambda_i}\,\mathrm{d}z_i=\frac{(1-p^{-1})^m}{\prod_{i=1}^{m}(1-p^{-(\lambda_i+1)})}.$$

$$\begin{aligned}J_1 &= \int_{|z_1|_p>1}|z_1|_p^{\lambda_1-m}\prod_{k=2}^{m}\left(\int_{|z_k|_p\leqslant|z_1|_p}|z_k|_p^{\lambda_k}\,\mathrm{d}z_k\right)\mathrm{d}z_1\\ &= \frac{(1-p^{-1})^{m-1}}{\prod_{i=2}^{m}(1-p^{-(\lambda_i+1)})}\int_{|z_1|_p>1}|z_1|_p^{\lambda-1}\,\mathrm{d}z_1\\ &= \frac{(1-p^{-1})^m p^{\lambda}}{\prod_{i=2}^{m}(1-p^{-(\lambda_i+1)})(1-p^{\lambda})}.\end{aligned}$$

Similarly, we can deduce that

$$J_j=\frac{(1-p^{-1})^m p^{\lambda}}{\prod_{1\leqslant i\leqslant m, i\neq j}(1-p^{-(\lambda_i+1)})(1-p^{\lambda})\prod_{k=1}^{j-1}p^{\lambda_k+1}},\qquad j=2,\cdots,m.$$

Then, it yields that

$$\begin{aligned}&\int_{\mathbb{Q}_p}\cdots\int_{\mathbb{Q}_p}\frac{|z_1|^{\lambda_1}\cdots|z_m|^{\lambda_m}}{[\max(1,|z_1|_p,\cdots,|z_m|_p)]^m}\,\mathrm{d}z_1\cdots\mathrm{d}z_m\\ =&\sum_{j=0}^{m}J_j=\frac{(1-p^{-1})^m(1-p^{-m})}{(1-p^{\lambda})\prod_{i=1}^{m}(1-p^{-(\lambda_i+1)})}=C_T,\end{aligned}$$

so (4) is proved.

Finally, when $\lambda_1 q_1 = \cdots = \lambda_m q_m$, there holds $\lambda q = \lambda_1 q_1 = \cdots = \lambda_m q_m$. We set $f_i(x) = |x|_p^{\lambda_i}$. It is easy to see that

$$T_m^p(f_1, \cdots, f_m)(x) = C_T |x|_p^{\lambda},$$

which implies

$$\|T_m^p(f_1, \cdots, f_m)\|_{B^{q,\lambda}(\mathbb{Q}_p)} = C_T \left(\frac{1 - p^{-1}}{1 - p^{-(1+\lambda q)}} \right)^{1/q}.$$

It is not hard to calculate that

$$\|f_i\|_{B^{q_i,\lambda_i}(\mathbb{Q}_p^n)} = \left(\frac{1 - p^{-1}}{1 - p^{-(1+\lambda_i q_i)}} \right)^{1/q_i},$$

so

$$\frac{\|T_m^p(f_1, \cdots, f_m)\|_{B^{q,\lambda}(\mathbb{Q}_p)}}{\|f_1\|_{B^{q_1,\lambda_1}(\mathbb{Q}_p)} \cdots \|f_m\|_{B^{q_m,\lambda_m}(\mathbb{Q}_p)}} = C_T.$$

Thus, we complete the proof. □

Acknowledgements. This work was supported by National Natural Science Foundation of China (Grant Nos. 11871452), Natural Science Foundation of Henan Province (Grant Nos. 202300410338) and the Nanhu Scholar Program for Young Scholars of Xinyang Normal University.

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(Received August 12, 2020)

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