FEJER INEQUALITY FOR s−CONVEX FUNCTIONS IN THE FOURTH SENSE

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Abstract. In this paper, Fejer inequality for s-convex functions in the fourth sense is established. Some integral inequalities related to Fejer inequalities are presented through the functions whose first derivative is s-convex in the fourth sense. Also some application examples of these inequalities are presented. These applications of Fejer inequality and other obtained integrals result in the bound functions for Gauss error function, incomplete gamma function and Fresnel integrals.

1. Introduction

Convexity is one of the desired properties for a function to have, which attracts special interest in many branches of mathematics, e.g. geometry, analysis, optimization. The convexity of a function f is stated as follows:

Let A be a convex set in a vector space and let $f : A \to \mathbb{R}$. f is said to be convex on A if

$$f(\lambda x + \mu y) \leqslant \lambda f(x) + \mu f(y) \tag{1}$$

for $x, y \in A$ and $\lambda, \mu > 0$ with $\lambda + \mu = 1$. As its definition is based on an inequality, some of their characteristics are also expressed by some inequalities such as Jensen inequality and they give rise the emergence of many new inequalities such as Hermite-Hadamard inequality. Hermite-Hadamard inequality for a convex function f defined on $[a,b]$ is given as follows:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}.$$

This inequality has both geometrical and algebraical interpretation, which asserts that average integral value of such a function interpolates between average of the images of endpoints and the image of the average value of end points. Geometrical interpretation can be thought in terms of the area under f on the interval. About 23 years after Hermite's first introduction of this inequality in 1883 [11], Fejer in [10] gave the generalization of this inequality by multiplication with a symmetrical weight function as follows:

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Let $f : [a,b] \to \mathbb{R}$ be a integrable and convex function, $w : [a,b] \to \mathbb{R}$ be a non-negative, integrable, symmetric function with respect to $x = \frac{a+b}{2}$. Then

$$f\left(\frac{a+b}{2}\right) \int_a^b w(x)dx \leqslant \frac{1}{b-a} \int_a^b f(x)w(x)dx \leqslant \frac{f(a)+f(b)}{2} \int_a^b w(x)dx.$$

While innovative ideas on the concepts of convexity result in the novel convexity types , several extensions, generalizations for Hermite-Hadamard, Fejer inequalities and the versions of them for novel convexity types have been established by many researchers [1, 2, 3, 5, 6, 7, 8, 14, 15, 17, 18, 21, 22]. One of them is given for s-convex functions. s-convex functions have various types. First and second sense s-convex functions were defined by Orlicz and Breckner and used in modular spaces [4, 16]. Then Hermite-Hadamard, Fejer inequalities and similar type inequalities are obtained for s-convex functions in the second sense [19, 23]. Recently, the definitions of s-convex functions in third and fourth sense have been given [9, 13] and Hermite-Hadamard inequality for s-convex functions in the third sense has been intoduced [20].

In this paper, we establish Fejer inequality for s-convex functions in the fourth sense. Then, we present some inequalities for the functions whose derivative is s-convex in the fourth sense. Also, we exemplify an application of these inequalities, which determines bound functions for Gauss error function, the difference between two incomplete gamma functions and Fresnel integral.

2. Preliminaries

By changing the condition on the parameters λ, μ in (1), s-convex functions are obtained in different types.

DEFINITION 1. [9] Let $U \subseteq \mathbb{R}^n$ be a convex set and let $s \in (0,1]$ and $f : U \to \mathbb{R}$. f is said to be s-convex function in the fourth sense if

$$f(\lambda x + \mu y) \leqslant \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y)$$

for all $x, y \in U$ and $\lambda, \mu \in [0,1]$ such that $\lambda + \mu = 1$.

The class of s-convex functions in the fourth sense is denoted by K_s^4.

EXAMPLE 1. Let $0 < s < 1$ and $a, b, c \in \mathbb{R}$ with $b < 0$ and $a, c \leqslant 0$. The function

$$f(x) = \begin{cases} a, & \text{if } x = 0 \\ bx^{\frac{1}{s}} + c, & \text{if } x > 0 \end{cases}$$

is s-convex function in the fourth sense on $(0, \infty)$. Adding extra condition $a = c$, we can say f is s-convex function in the fourth sense on $[0, \infty)$.

Suppose $x,y \in (0,\infty)$. Then we have $\lambda x + \mu y > 0$ with $\lambda + \mu = 1$ and

$$
\begin{aligned}
f(\lambda x + \beta y) &= b(\lambda x + \mu y)^{\frac{1}{s}} + c \\
&\leqslant b\left(\lambda^{\frac{1}{s}} x^{\frac{1}{s}} + \mu^{\frac{1}{s}} y^{\frac{1}{s}}\right) + c \\
&= b\left(\lambda^{\frac{1}{s}} x^{\frac{1}{s}} + \mu^{\frac{1}{s}} y^{\frac{1}{s}}\right) + c(\lambda + \mu) \\
&\leqslant b\left(\lambda^{\frac{1}{s}} x^{\frac{1}{s}} + \mu^{\frac{1}{s}} y^{\frac{1}{s}}\right) + c\left(\lambda^{\frac{1}{s}} + \mu^{\frac{1}{s}}\right) \\
&= \lambda^{\frac{1}{s}}\left(bx^{\frac{1}{s}} + c\right) + \mu^{\frac{1}{s}}\left(by^{\frac{1}{s}} + c\right) \\
&= \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y).
\end{aligned}
$$

For $x,y \in [0,\infty)$, we have to check only the cases $y > x = 0$ and $x = y = 0$. Let $y > x = 0$. Then

$$
\begin{aligned}
f(\lambda 0 + \mu y) = f(\mu y) &= b\mu^{\frac{1}{s}} y^{\frac{1}{s}} + c = b\mu^{\frac{1}{s}} y^{\frac{1}{s}} + c(\lambda + \mu) \\
&\leqslant b\mu^{\frac{1}{s}} y^{\frac{1}{s}} + c\left(\lambda^{\frac{1}{s}} + \mu^{\frac{1}{s}}\right) = \lambda^{\frac{1}{s}} c + \mu^{\frac{1}{s}}\left(by^{\frac{1}{s}} + c\right) \\
&= \lambda^{\frac{1}{s}} c + \mu^{\frac{1}{s}} f(y) = \lambda^{\frac{1}{s}} a + \mu^{\frac{1}{s}} f(y) = \lambda^{\frac{1}{s}} f(0) + \mu^{\frac{1}{s}} f(y).
\end{aligned}
$$

Let $y = x = 0$. Then

$$
f(\lambda 0 + \mu 0) = a \leqslant a\left(\lambda^{\frac{1}{s}} + \mu^{\frac{1}{s}}\right) = \lambda^{\frac{1}{s}} f(0) + \mu^{\frac{1}{s}} f(0).
$$

The following theorem states that one can obtain s-convex function in the fourth sense on $[0,1]$ from an integrable s-convex function in fourth sense on $[a,b]$:

THEOREM 1. [13] *Let f be an integrable and s-convex function in the fourth sense on $[a,b]$ and G be defined on $[0,1]$ as follows*

$$
G(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx. \tag{2}
$$

Then G is s-convex function in the fourth sense on $[0,1]$.

One of the main properties of s-convex function in the fourth sense we use in this paper is nonpositivity:

THEOREM 2. [9] *Let $f: \mathbb{R} \to \mathbb{R}$. If f is s-convex in the fourth sense, then $f \leqslant 0$.*

From now on, when we say s-convex function even without indication of fourth sense, it will be accepted as in the fourth sense throughout the rest of the paper.

3. Main results

The following theorem gives Fejer inequality for s-convex function:

THEOREM 3. *Let $f, w : \mathbb{R} \to \mathbb{R}$ be integrable functions and let $f \in K_s^4$ and $a, b \in \mathbb{R}$ with $a < b$. Let w be nonnegative and symmetric with respect to $x = \frac{a+b}{2}$, i.e. $w(x) = w(a+b-x)$. Then the following inequality holds:*

$$\frac{2^{\frac{1}{s}-1}}{b-a} f\left(\frac{a+b}{2}\right) \int_a^b w(x)dx \leqslant \frac{1}{b-a} \int_a^b f(x)w(x)dx$$

$$\leqslant \frac{1}{(b-a)^{1+\frac{1}{s}}} \frac{f(a)+f(b)}{2} \int_a^b \left[(b-x)^{\frac{1}{s}} + (x-a)^{\frac{1}{s}}\right] w(x)dx.$$

Proof. We can write from s-convexity of f the following inequality:

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2^{1/s}}.$$

The substitutions $x = ta + (1-t)b$ and $y = tb + (1-t)a$ yield to

$$f\left(\frac{a+b}{2}\right) \leqslant 2^{-\frac{1}{s}}(f(ta+(1-t)b)+f(tb+(1-t)a)).$$

Using $w(ta+(1-t)b) = w(tb+(1-t)a)$, we have

$$f\left(\frac{a+b}{2}\right) w(ta+(1-t)b) \leqslant 2^{-\frac{1}{s}}(f(ta+(1-t)b)+f(tb+(1-t)\alpha))w(ta+(1-t)b).$$

By integrating both sides, we obtain

$$f\left(\frac{a+b}{2}\right) \int_0^1 w(ta+(1-t)b) \leqslant 2^{-\frac{1}{s}} \int_0^1 f(ta+(1-t)b)w(ta+(1-t)b)dt$$

$$+ 2^{-\frac{1}{s}} \int_0^1 f(tb+(1-t)\alpha)w(tb+(1-t)a)dt$$

$$= 2^{-\frac{1}{s}} \left(\int_a^b f(x)w(x)\frac{dx}{b-a} + \int_a^b f(x)w(x)\frac{dx}{b-a} \right)$$

$$= \frac{2^{1-\frac{1}{s}}}{b-a} \int_a^b f(x)w(x)dx.$$

Since

$$f\left(\frac{a+b}{2}\right)\int_0^1 w(ta+(1-t)b) = \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx,$$

as a result, we get

$$\frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leqslant \frac{2^{1-\frac{1}{s}}}{b-a}\int_a^b f(x)w(x)dx. \qquad (3)$$

On the other hand, using s-convexity of f, we have

$$f(ta+(1-t)b) \leqslant t^{\frac{1}{s}}f(a)+(1-t)^{\frac{1}{s}}f(b),$$
$$f(tb+(1-t)a) \leqslant t^{\frac{1}{s}}f(b)+(1-t)^{\frac{1}{s}}f(a).$$

Summing above inequalities side by side, we have

$$f(ta+(1-t)b)+f(tb+(1-t)a) \leqslant \left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)(f(a)+f(b)).$$

Then multiplying each side by $w(ta+(1-t)b)$, we have

$$f(ta+(1-t)b)w(ta+(1-t)b)+f(tb+(1-t)a)w(ta+(1-t)b)$$
$$\leqslant \left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)(f(a)+f(b))w(ta+(1-t)b).$$

Integration with respect to t on $[0,1]$ gives the following inequality:

$$\int_0^1 f(ta+(1-t)b)w(ta+(1-t)b)dt + \int_0^1 f(tb+(1-t)a)w(tb+(1-t)a)dt$$

$$\leqslant \int_0^1 \left(t^{\frac{1}{s}}+(1-t)^{\frac{1}{s}}\right)(f(a)+f(b))w(ta+(1-t)b)dt.$$

The substitutions $x=ta+(1-t)b$, $x=tb+(1-t)a$ in each integrals in the inequality, transform them into

$$\frac{1}{b-a}\int_a^b f(x)w(x)dx \leqslant \frac{f(a)+f(b)}{2(b-a)^{1+\frac{1}{s}}}\int_a^b \left[(b-x)^{\frac{1}{s}}+(x-a)^{\frac{1}{s}}\right]w(x)dx. \qquad (4)$$

Combining (3) and (4), we obtain

$$\frac{2^{\frac{1}{s}-1}}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leqslant \frac{1}{b-a}\int_a^b f(x)w(x)dx$$

$$\leqslant \frac{1}{(b-a)^{1+\frac{1}{s}}}\frac{f(a)+f(b)}{2}\int_a^b \left[(b-x)^{\frac{1}{s}}+(x-a)^{\frac{1}{s}}\right]w(x)dx. \quad \square$$

Let us consider the function G defined in Theorem 1. Accepting $f = G$ in Theorem 3, we get another Fejer inequality for s-convex functions on $[0,1]$.

COROLLARY 1. *Let $f,w : \mathbb{R} \to \mathbb{R}$ be integrable functions and let $f \in K_s^4$, $a,b \in (0,1]$. Let w be nonnegative and symmetric with respect to $x = \frac{a+b}{2}$. Then the following inequality holds:*

$$\frac{2^{\frac{1}{s}}}{(a+b)} \int_{(\frac{a+1}{2})^2 - (\frac{b-1}{2})^2}^{(\frac{b+1}{2})^2 - (\frac{a-1}{2})^2} f(x)\,dx \int_a^b w(x)\,dx \leqslant \int_a^b \int_{\frac{a+b}{2} - \frac{x}{2}(b-a)}^{\frac{a+b}{2} + \frac{x}{2}(b-a)} \frac{w(x)f(u)}{x}\,du\,dx$$

$$\leqslant \frac{1}{2(b-a)^{\frac{1}{s}}} \left(a \int_{\frac{a+b}{2} - a\frac{b-a}{2}}^{\frac{a+b}{2} + a\frac{b-a}{2}} f(x)\,dx + b \int_{\frac{a+b}{2} - b\frac{b-a}{2}}^{\frac{a+b}{2} + b\frac{b-a}{2}} f(x)\,dx \right)$$

$$\times \int_a^b \left[(b-x)^{\frac{1}{s}} + (x-a)^{\frac{1}{s}} \right] w(x)\,dx.$$

Let us give Fejer type inequalities by using the functions whose first derivatives are s-convex function. We need to prove some integral identities as lemmas.

LEMMA 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $w : [a,b] \to [0,\infty)$ be integrable function on $[a,b]$ with $a < b$. If $f' \in L[a,b]$, i.e. f' is integrable on $[a,b]$, then the following equality holds:*

$$\frac{f(b)}{2(b-a)} \int_a^b w(x)\,dx - \frac{1}{2(b-a)} \int_a^b f(x)w(x)\,dx \tag{5}$$

$$= \frac{(b-a)}{2} \int_0^1 \left(\int_t^1 w(as + (1-s)b)\,ds \right) f'(ta + (1-t)b)\,dt.$$

Proof. Let us apply partial integration to $\int_a^b f(x)w(x)\,dx$. Assume $u(x) = f(x)$ and $w(x)\,dx = d(v(x))$. Then

$$v(x) = \int_a^x w(z)\,dz \text{ and } d(u(x)) = f'(x)\,dx$$

so

$$\int_a^b f(x)w(x)\,dx = f(b) \int_a^b w(z)\,dz - \int_a^b \left(\int_a^x w(z)\,dz \right) f'(x)\,dx. \tag{6}$$

Putting (6) in the left side of (5), we have

$$\frac{f(b)}{2(b-a)} \int_a^b w(x)dx - \frac{1}{2(b-a)} \int_a^b f(x)w(x)dx = \frac{1}{2(b-a)} \int_a^b \left(\int_a^x w(z)dz \right) f'(x)dx.$$

The substitution $z = as + (1-s)b$ gives that

$$\int_a^x w(z)dz = (b-a) \int_{\frac{x-b}{a-b}}^1 w(as+(1-s)b)ds.$$

Then the substitution $x = at + (1-t)b$ yields to the desired result:

$$\frac{1}{2(b-a)} \int_a^b \left(\int_a^x w(z)dz \right) f'(x)dx$$

$$= \frac{(b-a)}{2} \int_0^1 \left(\int_t^1 w(as+(1-s)b)ds \right) f'(ta+(1-t)b)dt. \quad \square$$

THEOREM 4. *Let $a,b \in \mathbb{R}$ with $a < b$, $f : [a,b] \to \mathbb{R}$ be differentiable and $w : [a,b] \to [0,\infty)$. Suppose that $f' \in L[a,b]$, $w \in C[a,b]$, i.e. w is continuous on $[a,b]$, and $f' \in K_s^4$. Then the following inequality holds:*

$$\frac{f(b)}{2(b-a)} \int_a^b w(x)dx - \frac{1}{2(b-a)} \int_a^b f(x)w(x)dx$$

$$\leqslant \frac{b-a}{2(2s+1)} \min_{x \in [a,b]} w(x) \left(\frac{s^2}{s+1} f'(a) + sf'(b) \right).$$

Proof. Using Lemma 1, Theorem 2, s-convexity of f' and the continuity of w, we have

$$\frac{f(b)}{2(b-a)} \int_a^b w(x)dx - \frac{1}{2(b-a)} \int_a^b f(x)w(x)dx$$

$$= \frac{b-a}{2} \int_0^1 \left(\int_t^1 w(as+(1-s)b)ds \right) f'(ta+(1-t)b)dt$$

$$\leqslant \frac{b-a}{2} \min_{x \in [a,b]} w(x) \int_0^1 (1-t) f'(ta+(1-t)b)dt$$

$$\leqslant \frac{b-a}{2} \min_{x \in [a,b]} w(x) \int_0^1 (1-t) \left(t^{\frac{1}{s}} f'(a) + (1-t)^{\frac{1}{s}} f'(b) \right) dt$$

$$\leqslant \frac{b-a}{2} \min_{x\in[a,b]} w(x) \int_0^1 ((1-t)t^{\frac{1}{s}} f'(a) + (1-t)^{\frac{1}{s}+1} f'(b)) dt$$

$$\leqslant \frac{b-a}{2(2s+1)} \min_{x\in[a,b]} w(x) \left(\frac{s^2}{s+1} f'(a) + s f'(b) \right). \quad \square$$

LEMMA 2. *Let $a,b \in \mathbb{R}$ with $a < b$, $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping and $w : [a,b] \to [0,\infty)$. If $f', w \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)}{2(b-a)} \int_a^b w(x)dx - \frac{1}{2(b-a)} \int_a^b f(x)w(x)dx$$

$$= -\frac{(b-a)}{2} \int_0^1 \left(\int_0^t w(as + (1-s)b)ds \right) f'(ta + (1-t)b)dt.$$

Proof. The same operations in the proof of Lemma 1 result in the desired equality. $\square$

THEOREM 5. *Let $a,b \in \mathbb{R}$ with $a < b$, $f : [a,b] \to \mathbb{R}$ be a differentiable function and $w : [a,b] \to [0,\infty)$. Suppose that $f' \in L[a,b]$ and $w \in C[a,b]$, $f' \in K_s^4$. Then, the following inequality holds:*

$$-\frac{f(a)}{2(b-a)} \int_a^b w(x)dx + \frac{1}{2(b-a)} \int_a^b f(x)w(x)dx$$

$$\leqslant \frac{(b-a)s}{2(2s+1)} \min_{x\in[a,b]} w(x) \left(f'(a) + \frac{s}{s+1} f'(b) \right).$$

Proof. By making use of Lemma 2, Theorem 2, s-convexity of f' and continuity of w, we get

$$-\frac{f(a)}{2(b-a)} \int_a^b w(x)dx + \frac{1}{2(b-a)} \int_a^b f(x)w(x)dx$$

$$= \frac{b-a}{2} \int_0^1 \left(\int_0^t w(as + (1-s)b)ds \right) f'(ta + (1-t)b)dt$$

$$\leqslant \frac{b-a}{2} \min_{x\in[a,b]} w(x) \int_0^1 t f'(ta + (1-t)b)dt$$

$$\leqslant \frac{b-a}{2} \min_{x\in[a,b]} w(x) \int_0^1 t \left(t^{\frac{1}{s}} f'(a) + (1-t)^{\frac{1}{s}} f'(b) \right) dt$$

$$= \frac{b-a}{2} \min_{x\in[a,b]} w(x) \int_0^1 (t^{\frac{1}{s}+1} f'(a) + t(1-t)^{\frac{1}{s}} f'(b)) dt$$

$$= \frac{(b-a)s}{2(2s+1)} \min_{x\in[a,b]} w(x) \left(f'(a) + \frac{s}{s+1} f'(b) \right). \quad \square$$

By combining Theorem 4 and Theorem 5, we have the following result.

COROLLARY 2. *Let* $a,b \in \mathbb{R}$ *with* $a < b$, $f : [a,b] \to \mathbb{R}$ *be differentiable and* $w :$ $[a,b] \to (0,\infty)$. *Suppose that* $w \in C[a,b]$, $f' \in K_s^4$, $f(a) \neq f(b)$. *Then, the following inequality holds:*

$$\int_a^b w(x)dx \leqslant \min_{x\in[0,1]} w(x) \frac{(b-a)^2}{2} \frac{s(s^2+s+1)}{(2s+1)(s+1)} \frac{f'(a)-f'(b)}{f(a)-f(b)}.$$

The following lemma presents the similar inequalities above in terms of midpoint of interval.

LEMMA 3. *Let* $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function and* $w : [a,b] \to$ $[0,\infty)$ *with* $a < b$. *If* $w, f' \in L[a,b]$, *then the following equality holds:*

$$\frac{1}{b-a} \int_{\frac{a+b}{2}}^b f(x)w(x)dx - \frac{1}{b-a} f\left(\frac{a+b}{2} \right) \int_{\frac{a+b}{2}}^b w(x)dx$$

$$= (b-a) \int_0^{1/2} \left(\int_0^t w(as + (1-s)b)ds \right) f'(ta + (1-t)b)dt. \quad (7)$$

Proof. Let us apply partial integration to $\int_{\frac{a+b}{2}}^b f(x)w(x)dx$. Assume $u(x) = f(x)$ and $w(x)dx = d(v(x))$. Then we obtain

$$v(x) = \int_{\frac{a+b}{2}}^x w(z)dz \text{ and } d(u(x)) = f'(x)dx.$$

We have

$$\int_{\frac{a+b}{2}}^b f(x)w(x)dx = f(b) \int_{\frac{a+b}{2}}^b w(z)dz - \int_{\frac{a+b}{2}}^b \left(\int_{\frac{a+b}{2}}^x w(z)dz \right) f'(x)dx. \quad (8)$$

By putting (8) in the left side of (7), we get

$$\frac{1}{b-a}\left(f(b)-f\left(\frac{a+b}{2}\right)\right)\int_{\frac{a+b}{2}}^{b}w(x)dx-\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}\left(\int_{\frac{a+b}{2}}^{x}w(z)dz\right)f'(x)dx$$

$$=\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}f'(x)dx\int_{\frac{a+b}{2}}^{b}w(x)dx-\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}\left(\int_{\frac{a+b}{2}}^{x}w(z)dz\right)f'(x)dx$$

$$=\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}\left(\int_{\frac{a+b}{2}}^{b}w(z)dz\right)f'(x)dx-\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}\left(\int_{\frac{a+b}{2}}^{x}w(z)dz\right)f'(x)dx$$

$$=\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}\left(\int_{x}^{b}w(z)dz\right)f'(x)dx.$$

In the last integral, the substitutions $z=as+(1-s)b$ and $x=at+(1-t)b$, respectively, give the right side of (7). □

THEOREM 6. *Let $a,b\in\mathbb{R}$ with $a<b$, f be a differentiable function on $[a,b]$ and $w:[a,b]\to[0,\infty)$. Suppose that $f'\in L[a,b]$, $w\in C[a,b]$ and $f'\in K_s^4$. Then the following inequality holds:*

$$\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}f(x)w(x)dx-\frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}w(x)dx$$

$$\leqslant(b-a)\frac{s}{2^{2+\frac{1}{s}}(2s+1)}\min_{x\in[\frac{a+b}{2},b]}w(x)\left(f'(a)+\frac{2^{2+\frac{1}{s}}s-3s-1}{s+1}f'(b)\right).$$

Proof. Using Lemma 3, Theorem 2, s-convexity of f' and continuity of w, we have

$$\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}f(x)w(x)dx-\frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}w(x)dx$$

$$\leqslant(b-a)\int_{0}^{1/2}\left(\int_{0}^{t}\min_{x\in[\frac{a+b}{2},b]}w(x)ds\right)f'(ta+(1-t)b)dt$$

$$=(b-a)\min_{x\in[\frac{a+b}{2},b]}w(x)\int_{0}^{1/2}tf'(ta+(1-t)b)dt$$

$$\leqslant (b-a) \min_{x\in[\frac{a+b}{2},b]} w(x) \int_0^{1/2} t\left(t^{\frac{1}{s}}f'(a)+(1-t)^{\frac{1}{s}}f'(b))\right)dt$$

$$\leqslant (b-a) \min_{x\in[\frac{a+b}{2},b]} w(x) \left(\frac{s}{2^{2+\frac{1}{s}}(2s+1)}f'(a)+\frac{s((2^{2+\frac{1}{s}}-3)s-1)}{2^{2+\frac{1}{s}}(s+1)(2s+1)}f'(b)\right)$$

$$\leqslant (b-a)\frac{s}{2^{2+\frac{1}{s}}(2s+1)} \min_{x\in[\frac{a+b}{2},b]} w(x)\left(f'(a)+\frac{2^{2+\frac{1}{s}}s-3s-1)}{s+1}f'(b)\right). \quad \square$$

LEMMA 4. *Let* $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ *be differentiable function and* $w:[a,b]\to[0,\infty)$ *with* $a<b$. *If* $w,f'\in L[a,b]$, *then the following equality holds:*

$$\frac{1}{b-a}\int_a^{\frac{a+b}{2}} f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}} w(x)dx$$

$$= -(b-a)\int_{1/2}^1\left(\int_t^1 w(as+(1-s)b)ds\right)f'(ta+(1-t)b)dt.$$

Proof. The same successive operations in the proof of Lemma 3 result in the desired equality. $\square$

THEOREM 7. *Let* $a,b\in\mathbb{R}$ *with* $a<b$ *and* $f:[a,b]\to\mathbb{R}$ *be differentiable function and* $w:[a,b]\to[0,\infty)$. *Suppose that* $f'\in L[a,b]$, $w\in C[a,b]$ *and* $f'\in K_s^4$. *Then the following inequality holds:*

$$-\frac{1}{b-a}\int_a^{\frac{a+b}{2}} f(x)w(x)dx + \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}} w(x)dx$$

$$\leqslant (b-a)\frac{s}{2^{2+\frac{1}{s}}(2s+1)} \min_{x\in[a,\frac{a+b}{2}]} w(x)\left(\frac{2^{2+\frac{1}{s}}s-3s-1}{s+1}f'(a)+f'(b)\right).$$

Proof. Using Lemma 4, Theorem 2, s-convexity of f' and continuity of w, we have

$$-\frac{1}{b-a}\int_a^{\frac{a+b}{2}} f(x)w(x)dx + \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}} w(x)dx$$

$$\leqslant (b-a)\int_{1/2}^1\left(\int_t^1 w(as+(1-s)b)ds\right)f'(ta+(1-t)b)dt$$

$$\leqslant (b-a) \min_{x\in[a,\frac{a+b}{2}]} w(x) \int_{1/2}^{1} (1-t)f'(ta+(1-t)b)dt$$

$$\leqslant (b-a) \min_{x\in[a,\frac{a+b}{2}]} w(x) \int_{1/2}^{1} (1-t)\left(t^{\frac{1}{s}}f'(a)+(1-t)^{\frac{1}{s}}f'(b)\right)dt$$

$$= (b-a) \min_{x\in[a,\frac{a+b}{2}]} w(x) \left(\frac{s((2^{2+\frac{1}{s}}-3)s-1)}{2^{2+\frac{1}{s}}(s+1)(2s+1)}f'(a)+\frac{s}{2^{2+\frac{1}{s}}(2s+1)}f'(b) \right)$$

$$= (b-a)\frac{s}{2^{2+\frac{1}{s}}(2s+1)} \min_{x\in[a,\frac{a+b}{2}]} w(x) \left(\frac{2^{2+\frac{1}{s}}s-3s-1}{s+1}f'(a)+f'(b) \right). \quad \square$$

Combining Theorem 6 and Theorem 7 with symmetry condition on w, we have the following result:

COROLLARY 3. *Let $a,b \in \mathbb{R}$ with $a < b$, $f : [a,b] \to \mathbb{R}$ be differentiable and $w : [a,b] \to [0,\infty)$. Suppose that $w \in C[a,b]$, $f' \in K_s^4$ and w is symmetric on $[a,b]$. Then the following inequality holds:*

$$\frac{1}{b-a} \int_{\frac{a+b}{2}}^{b} f(x)w(x)dx - \frac{1}{b-a} \int_{a}^{\frac{a+b}{2}} f(x)w(x)dx$$

$$\leqslant (b-a)\frac{s^2(1-2^{-1-\frac{1}{s}})}{(2s+1)(s+1)} \min_{x\in[a,b]} w(x) \left(f'(a)+f'(b)\right).$$

4. Applications

The inequalities obtained above can be used in getting bound for some integrals which has not elemantary antiderivative. Depending on the choice of f and w, one can likely obtain sharper results. The following proposition is given application of Theorem 3 which indicates Fejer inequality for s-convex functions.

PROPOSITION 1. *Let x be positive real number. If $0 < x < \frac{1}{\sqrt{2}}$, then*

$$\frac{2x}{\sqrt{\pi}(1-2x^2)e^{x^2}} \leqslant \operatorname{erf}(x),$$

where erf is Gauss error function, i.e.

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt, \ x \in \mathbb{R}.$$

If $x > \frac{1}{\sqrt{2}}$, the inequality is reversed.

Proof. From Example 1, $f(t) = -t^2$ is $\frac{1}{2}$-convex function in the fourth sense. Consider $w(t) = e^{-(t-\frac{z}{2})^2}$ for fixed positive number z. It is clear that w is symmetric to $t = \frac{z}{2}$ on $[0,z]$. Let us apply the left side of the inequality in Theorem 3 for these functions on $[0,z]$. We have

$$\frac{z}{2}\sqrt{\pi}\,\mathrm{erf}\left(\frac{z}{2}\right) \geqslant \frac{1}{4z}\left(\sqrt{\pi}(z^2+2)\,\mathrm{erf}\left(\frac{z}{2}\right) - 2ze^{-\frac{z^2}{4}}\right).$$

Solving the inequality for $\mathrm{erf}(\frac{z}{2})$, then replacement of $x = \frac{z}{2}$ yields to

$$\sqrt{\pi}(1-2x^2)\,\mathrm{erf}(x) \leqslant 2xe^{-x^2}.$$

This gives the desired inequality. □

Using Theorem 4 and Theorem 6, we present the following proposition which allows us to bound the difference of two incomplete gamma functions in terms of exponential functions.

PROPOSITION 2. Let $\alpha > 3$, $x > 0$. Then

$$\left(\frac{1}{\alpha}\left(\frac{1}{2}-2^{-\alpha}\right)-2^{-\alpha}\right)x^{\alpha-1}e^{-x} \leqslant \Gamma\left(\alpha,\frac{x}{2}\right) - \Gamma(\alpha,x)$$

$$\leqslant x^{\alpha-1}e^{-\frac{x}{2}}\left(e^{\frac{x}{2}}-2+\frac{1}{\alpha}\left(1-2^{2-\alpha}\right)\right),$$

where $\Gamma(\alpha,x)$ denotes the incomplete gamma function, i.e.

$$\Gamma(\alpha,x) = \int_x^\infty t^{\alpha-1}e^{-t}dt,\ \alpha > 0,\ x > 0.$$

Proof. It is clear from Example 1 that the derivative of $f(x) = -\frac{s}{s+1}x^{1+\frac{1}{s}}$ is s-convex function. Let us apply Theorem 4 and 6 for f defined on $[\frac{x}{2},x]$ and w defined by $w(x) = e^{-x}$ on and $[0,x]$.

Using Theorem 4, we have the following inequality

$$\Gamma\left(2+\frac{1}{s},\frac{x}{2}\right) - \Gamma\left(2+\frac{1}{s},x\right) \leqslant -\frac{(s+1)x^{2+\frac{1}{s}}}{4e^x(2s+1)}\left(1+\frac{s}{2^{\frac{1}{s}}(s+1)}\right) + x^{1+\frac{1}{s}}\left(e^{-\frac{x}{2}}-e^{-x}\right).$$

Using Theorem 6, we obtain

$$\frac{\left(2^{2+\frac{1}{s}}s - 3s - 1\right)x^{1+\frac{1}{s}}e^{-x}}{2^{2+\frac{1}{s}}(2s+1)} \leqslant \Gamma\left(2+\frac{1}{s},\frac{x}{2}\right) - \Gamma\left(2+\frac{1}{s},x\right).$$

Taking $2 + \frac{1}{s} = \alpha$ inequalities above and then combining them, we obtain

$$\left(\frac{1}{\alpha}\left(\frac{1}{2} - 2^{-\alpha}\right) - 2^{-\alpha}\right) x^{\alpha-1} e^{-x} \leqslant \Gamma\left(\alpha, \frac{x}{2}\right) - \Gamma(\alpha, x)$$

$$\leqslant x^{\alpha-1} e^{-\frac{x}{2}}\left(e^{\frac{x}{2}} - 2 + \frac{1}{\alpha}\left(1 - 2^{2-\alpha}\right)\right). \quad \square$$

The following proposition is given as an application example of Theorem 5 and Theorem 7, which yields to some bound functions for Fresnel integral for sinus. In a similar way, one can get some bound functions for Fresnel integral for cosinus.

PROPOSITION 3. Let $x > 0$. Then

$$\Phi(x) \leqslant S(x) \leqslant \Psi(x)$$

where $S(x)$ is Fresnel integral for sinus, i.e.

$$S(x) = \int_0^x \sin(t^2)\,dt$$

and

$$\Psi(x) = \frac{4x}{3}\left(\frac{\pi^2}{20}x^4 - \frac{\pi}{4}x^2\cos\left(\frac{\pi}{2}x^2\right) + \frac{3}{4}\sin\left(\frac{\pi}{2}x^2\right)\right)$$

and

$$\Phi(x) = \frac{4x}{\pi^2 x^4 + 3}\left(-\frac{\pi^2}{5}x^4 - \frac{\pi}{4}x^2\cos\left(\frac{\pi}{2}x^2\right) + \frac{3}{4}\sin\left(\frac{\pi}{2}x^2\right)\right).$$

Proof. Consider $s = \frac{1}{3}$, $f(t) = -\frac{t^{\frac{1}{4}}}{4}$, $w(t) = \sin(t^2)$ on $[0,z]$ in Theorem 5. Using $-1 \leqslant \min_{t\in[0,z]} w(t)$ for all z, we have

$$-\frac{3\sin z^2}{32} + \frac{z^2\cos z^2}{16} + \frac{3\sqrt{2\pi}}{64z}S\left(z\sqrt{\frac{2}{\pi}}\right) \leqslant \frac{z^4}{40}.$$

Thus we have

$$S\left(z\sqrt{\frac{2}{\pi}}\right) \leqslant \frac{8z}{3\sqrt{2\pi}}\left(\frac{z^4}{5} + \frac{3\sin z^2}{4} - \frac{z^2\cos z^2}{2}\right). \tag{9}$$

Let us consider the same value s and functions f, w on $[0, 2z]$ in Theorem 7. Using $-1 \leqslant \min_{t\in[0,z]} w(t)$ for all z, we have

$$\frac{3\sin z^2}{32} - \frac{z^2\cos z^2}{16} - \frac{3\sqrt{2\pi}}{64z}S\left(z\sqrt{\frac{2}{\pi}}\right) - \frac{z^3\sqrt{2\pi}}{16}S\left(z\sqrt{\frac{2}{\pi}}\right) \leqslant \frac{z^4}{10}.$$

So

$$\frac{32z}{\sqrt{2\pi}(4z^4+3)}\left(-\frac{z^4}{5}+\frac{3\sin z^2}{16}-\frac{z^2\cos z^2}{8}\right)\leqslant S\left(z\sqrt{\frac{2}{\pi}}\right). \qquad (10)$$

Combining (9) and (10) and then making the replacement $x=z\sqrt{\frac{2}{\pi}}$, we have

$$S(x)\leqslant\frac{4x}{3}\left(\frac{x^4\pi^2}{20}+\frac{3}{4}\sin\left(\frac{\pi}{2}x^2\right)-\frac{\pi}{4}x^2\cos\left(\frac{\pi}{2}x^2\right)\right)$$

and

$$\frac{4x}{\pi^2x^4+3}\left(-\frac{\pi^2}{5}x^4+\frac{3}{4}\sin\left(\frac{\pi}{2}x^2\right)-\frac{\pi}{4}x^2\cos\left(\frac{\pi}{2}x^2\right)\right)\leqslant S(x). \quad \square$$

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