Journal of Mathematical Inequalities
Volume 16, Number 1 (2022), 35–50 doi:10.7153/jmi-2022-16-03

A HILBERT–TYPE INEQUALITY IN THE WHOLE PLANE WITH THE CONSTANT FACTOR RELATED TO SOME SPECIAL CONSTANTS

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(*Communicated by M. Krnić*)

Abstract. In this work, a new kernel function defined in the whole plane, including both the homogeneous and the non-homogeneous cases, and involving multiple parameters is constructed. By the method of weight coefficient and using some techniques of real analysis, a new Hilbert-type inequality with the newly constructed kernel function, as well as its equivalent Hardy-type inequality are established. The constant factors of the obtained inequalities are proved to be the best possible. Furthermore, assuming special values to the parameters, some interesting and special Hilbert-type inequalities with the constant factors involving some special constants, such as the Euler number, Bernoulli number and the Catalan constant are presented at the end of the paper.

1. Introduction

In 1908, German mathematician D. Hilbert put forward the famous Hilbert's double series inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\|a\|_2\|b\|_2, \tag{1.1}$$

where $a = \{a_m\}_{m=1}^{\infty} \in l^2$, $b = \{b_n\}_{n=1}^{\infty} \in l^2$, and the constant factor π is the best possible. In 1911, Schur proved the integral analogy of (1.1):

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \pi\|f\|_2\|g\|_2, \tag{1.2}$$

where f,g are two nonnegative real-valued functions, and $f,g \in L^2(\mathbb{R}^+)$. By introducing a pair of conjugate parameters p and q satisfying $p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, inequalities

Mathematics subject classification (2020): 26D15, 41A17.
Keywords and phrases: Hilbert-type inequality, special constant, Bernoulli number, Euler number, Catalan constant.
This work was supported by scientific research project of Zhejiang Provincial Department of Education (Y202148139), and the incubation foundation of Zhejiang Institute of Mechanical and Electrical Engineering (A-0271-21-206).
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(1.1) and (1.2) can be extended to more general forms:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\frac{\pi}{p}}\|a\|_p\|b\|_p, \tag{1.3}$$

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin\frac{\pi}{p}}\|f\|_p\|g\|_q, \tag{1.4}$$

where the constant factor $\frac{\pi}{\sin\frac{\pi}{p}}$ is the best possible. (1.1)–(1.3) can be referred to the classical book [4] by Hardy et al. In addition, Some analogical forms of (1.3) are also presented in [4], such as

$$\int_0^{\infty}\int_0^{\infty}\frac{\log\frac{x}{y}}{x-y}f(x)g(y)\mathrm{d}x\mathrm{d}y < \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2\|f\|_p\|g\|_q. \tag{1.5}$$

Inequalities (1.1) and (1.2) are usually called Hilbert inequality while Such inequalities as (1.3)–(1.5) are usually named as Hilbert-type inequalities. Although (1.1)–(1.5) have been put forward for more than 100 years, their parameter extensions, coefficient refinements and high-dimension generalizations have attracted considerable attention, especially in the last 20 years (see [2, 3, 8, 9, 10, 11, 21, 22, 23, 24, 25, 26, 29, 30]). Meanwhile, by continuous construction of new kernel functions, considering the discrete, half-discrete and integral cases, and studying various forms of parameter extensions, researchers have already established a large number of new Hilbert-type inequalities [1, 5, 6, 12, 13, 14, 15, 16, 17, 18, 19, 27, 31].

It should be mentioned that the integral Hilbert-type inequalities are usually constructed in the first quadrant. However, considering the particularity of some newly constructed kernel functions, researchers sometimes extend the integral interval to the whole plane (see [7, 15, 16, 18, 20, 28]). The idea of this paper is mainly inspired by the above literature, and we will establish the following three inequalities in the whole plane with $\mu(x)=|x|^{p-1}$ and $\nu(y)=|y|^{q-1}$:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(|\log|xy||)^{2k-1}}{|1\pm xy|}\min\{1,|xy|\}^2 f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{B_k}{2k}(2\pi)^{2k}\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{1.6}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|1\pm xy|(|\log|xy||)^{2k}}{1+(xy)^2}\min\{1,|xy|\}f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{E_k}{2^{2k}}\pi^{2k+1}\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{1.7}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|1\pm xy||\log|xy||}{1+(xy)^2}\min\{1,|xy|\}f(x)g(y)\mathrm{d}x\mathrm{d}y < 4L_0\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{1.8}$$

where B_k $(k\in\mathbb{N}^+)$, E_k $(k\in\mathbb{N})$ and L_0 are the Bernoulli number, Euler number and the Catalan constant, respectively. More generally, we will construct a new kernel function which includes both the homogeneous and the non-homogeneous cases, and involves multiple parameters. By the method of weight coefficient and using some techniques of real analysis, we will establish a more general Hilbert-type inequality in the whole plane. Detailed lemmas will be presented in the second section.

2. Some Lemmas

LEMMA 2.1. *Let* $\tau_1, \tau_2 \in \{-1,1\}$ *and* $\delta \in \{0,1\}$. *Let* α, β *be such that* $\beta < \alpha < 2m-2n+1$, *where* $m,n \in \mathbb{N}$, *and* $n \neq 0$ *for* $\delta = 0$, $\tau_2 = -1$. *Suppose that* $\gamma \geqslant 0$ *for* $\delta = 0$, $\tau_2 = -1$ *or* $\delta = 1$, $\tau_1 = 1$, *and* $\gamma > 0$ *for* δ, τ_1, τ_2 *taking other values. Define*

$$K_\delta(z) := \frac{|\,1+\tau_2 z^{2n+\delta}\,|\,(|\log|\,z\,||)^\gamma}{|\,1+\tau_1 z^{2m+1+\delta}\,|\,(\min\{1,|\,z\,|\})^\beta},\ z \neq 0, \pm 1, \tag{2.1}$$

$$\begin{aligned} C_0(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2) :=& 2\sum_{j=0}^{\infty}\left\{\frac{1}{[(4m+2)j+\alpha-\beta]^{\gamma+1}}+\frac{\tau_2}{[(4m+2)j+2n+\alpha-\beta]^{\gamma+1}}\right.\\ &+\frac{\tau_2}{[(4m+2)j+2m-\alpha+1]^{\gamma+1}}\\ &\left.+\frac{1}{[(4m+2)j+2m-2n-\alpha+1]^{\gamma+1}}\right\}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} C_1(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2) :=& 2\sum_{j=0}^{\infty}\left\{\frac{(-\tau_1)^j}{[(2m+2)j+\alpha-\beta]^{\gamma+1}}\right.\\ &\left.+\frac{(-\tau_1)^j}{[(2m+2)j+2m-2n-\alpha+1]^{\gamma+1}}\right\}. \end{aligned} \tag{2.3}$$

Then

$$\int_{-\infty}^{\infty} K_\delta(z)\,|\,z\,|^{\alpha-1}\,\mathrm{d}z = \Gamma(\gamma+1)C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2), \tag{2.4}$$

where $\Gamma(s) = \int_0^\infty u^{s-1}e^{-u}\mathrm{d}u\,(s>0)$ *is the* Γ*-function* [32], *and* $\Gamma(s) = (s-1)!$ *for* $s \in \mathbb{N}^+$.

Proof. We first consider the case of $\delta = 0$, then it can be obtained that

$$\begin{aligned} &\int_{-\infty}^{\infty} K_0(z)\,|\,z\,|^{\alpha-1}\,\mathrm{d}z = \int_0^\infty [K_0(z)+K_0(-z)]\,|\,z\,|^{\alpha-1}\,\mathrm{d}z\\ =&\int_0^\infty \left[\frac{|\,1+\tau_2 z^{2n}\,|}{|\,1+\tau_1 z^{2m+1}\,|}+\frac{|\,1+\tau_2 z^{2n}\,|}{|\,1-\tau_1 z^{2m+1}\,|}\right]\frac{z^{\alpha-1}\,|\log z\,|^\gamma}{(\min\{1,z\})^\beta}\mathrm{d}z\\ =&2\int_0^1 \frac{(z^{\alpha-\beta-1}+\tau_2 z^{2n+\alpha-\beta-1})\,|\log z\,|^\gamma}{1-z^{4m+2}}\mathrm{d}z \end{aligned}$$

$$
\begin{aligned}
&+2\int_1^{\infty}\frac{(\tau_2 z^{2m+\alpha}+z^{2m+2n+\alpha})\mid\log z\mid^{\gamma}}{z^{4m+2}-1}\mathrm{d}z\\
&=2\int_0^1\frac{(z^{\alpha-\beta-1}+\tau_2 z^{2n+\alpha-\beta-1}+\tau_2 z^{2m-\alpha}+z^{2m-2n-\alpha})\mid\log z\mid^{\gamma}}{1-z^{4m+2}}\mathrm{d}z\\
&=:2(J_1+\tau_2 J_2+\tau_2 J_3+J_4).
\end{aligned}\tag{2.5}
$$

In view that $\frac{1}{1-z^{4m+2}}=\sum_{j=0}^{\infty}z^{(4m+2)j}$ for $0<z<1$, it follows from Lebesgue term-by-term integration theorem that

$$
J_1=\sum_{j=0}^{\infty}\int_0^1 z^{(4m+2)j+\alpha-\beta-1}\mid\log z\mid^{\gamma}\mathrm{d}z.\tag{2.6}
$$

Set $z=e^{\frac{-t}{(4m+2)j+\alpha-\beta}}$ in (2.6), then we get

$$
J_1=\sum_{j=0}^{\infty}\frac{\int_0^{\infty}e^{-t}t^{\gamma}\mathrm{d}t}{[(4m+2)j+\alpha-\beta]^{\gamma+1}}=\sum_{j=0}^{\infty}\frac{\Gamma(\gamma+1)}{[(4m+2)j+\alpha-\beta]^{\gamma+1}}.\tag{2.7}
$$

In the same way, we have

$$
J_2=\sum_{j=0}^{\infty}\frac{\Gamma(\gamma+1)}{[(4m+2)j+2n+\alpha-\beta]^{\gamma+1}};\tag{2.8}
$$

$$
J_3=\sum_{j=0}^{\infty}\frac{\Gamma(\gamma+1)}{[(4m+2)j+2m-\alpha+1]^{\gamma+1}};\tag{2.9}
$$

$$
J_4=\sum_{j=0}^{\infty}\frac{\Gamma(\gamma+1)}{[(4m+2)j+2m-2n-\alpha+1]^{\gamma+1}}.\tag{2.10}
$$

Applying (2.7)–(2.10) to (2.5), and in view of (2.2), we get

$$
\int_{-\infty}^{\infty}K_0(z)\mid z\mid^{\alpha-1}\mathrm{d}z=\Gamma(\gamma+1)C_0(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2).\tag{2.11}
$$

Similarly, for $\delta=1$, we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty}K_1(z)\mid z\mid^{\alpha-1}\mathrm{d}z=\int_0^{\infty}[K_1(z)+K_1(-z)]\mid z\mid^{\alpha-1}\mathrm{d}z\\
&=\int_0^{\infty}\left[\frac{\mid 1+\tau_2 z^{2n+1}\mid}{\mid 1+\tau_1 z^{2m+2}\mid}+\frac{\mid 1-\tau_2 z^{2n+1}\mid}{\mid 1+\tau_1 z^{2m+2}\mid}\right]\frac{z^{\alpha-1}\mid\log z\mid^{\gamma}}{(\min\{1,z\})^{\beta}}\mathrm{d}z\\
&=2\int_0^1\frac{z^{\alpha-\beta-1}\mid\log z\mid^{\gamma}}{1+\tau_1 z^{2m+2}}\mathrm{d}z+2\int_1^{\infty}\frac{z^{2n+\alpha}\mid\log z\mid^{\gamma}}{\tau_1+z^{2m+2}}\mathrm{d}z\\
&=2\int_0^1\frac{z^{\alpha-\beta-1}+z^{2m-2n-\alpha}}{1+\tau_1 z^{2m+2}}\mid\log z\mid^{\gamma}\mathrm{d}z\\
&=2\sum_{j=0}^{\infty}\left\{\frac{(-\tau_1)^j\Gamma(\gamma+1)}{[(2m+2)j+\alpha-\beta]^{\gamma+1}}+\frac{(-\tau_1)^j\Gamma(\gamma+1)}{[(2m+2)j+2m-2n-\alpha+1]^{\gamma+1}}\right\}\\
&=\Gamma(\gamma+1)C_1(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2).
\end{aligned}\tag{2.12}
$$

Combining (2.11) and (2.12), we have (2.3). Lemma 2.1 is proved. □

LEMMA 2.2. *Let* $\tau_1,\tau_2 \in \{-1,1\}$ *and* $\delta \in \{0,1\}$. *Let* α, β *be such that* $\beta < \alpha < 2m-2n+1$, *where* $m,n \in \mathbb{N}$, *and* $n \neq 0$ *for* $\delta = 0$, $\tau_2 = -1$. *Suppose that* $\gamma \geqslant 0$ *for* $\delta = 0$, $\tau_2 = -1$ *or* $\delta = 1$, $\tau_1 = 1$, *and* $\gamma > 0$ *for* δ,τ_1,τ_2 *taking other values. Let* $\lambda_1,\lambda_2 \in \{\pm 1, \pm 3, \pm\frac{1}{3}, \pm 5, \pm\frac{1}{5}, \cdots\}$, $p > 1$ *and* $\frac{1}{p}+\frac{1}{q} = 1$. *Suppose that* $\mu(x) = |x|^{p(1-\lambda_1\alpha)-1}$, $\nu(y) = |y|^{q(1-\lambda_2\alpha)-1}$. *For an arbitrary natural number* s *which is large enough, set*

$$\hat{f}(x) := \begin{cases} |x|^{\lambda_1\alpha-1-\frac{2\lambda_1}{sp}}, & x \in E_1 \\ 0 & x \in \mathbb{R}\setminus E_1 \end{cases},$$

$$\hat{g}(y) := \begin{cases} |y|^{\lambda_2\alpha-1+\frac{2\lambda_2}{sq}}, & y \in E_2 \\ 0 & y \in \mathbb{R}\setminus E_2 \end{cases},$$

where $E_1 = \left\{x : |x|^{\operatorname{sgn}\lambda_1} > 1\right\}$ *and* $E_2 = \left\{y : |y|^{\operatorname{sgn}\lambda_2} < 1\right\}$. *Then*

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right)\hat{f}(x)\hat{g}(y)\mathrm{d}x\mathrm{d}y \\ &= \frac{s}{|\lambda_1\lambda_2|}\left[\int_{[-1,1]} K_\delta(z)\,|z|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} K_\delta(z)\,|z|^{\alpha-1-\frac{2}{sp}}\,\mathrm{d}z\right]. \qquad (2.13) \end{aligned}$$

Proof. Let

$$E_1^+ = \{x : x \in E_1, x > 0\}, \qquad E_1^- = \{x : x \in E_1, x < 0\};$$

$$E_2^+ = \{y : y \in E_2, y > 0\}, \qquad E_2^- = \{y : y \in E_2, y < 0\}.$$

Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right)\hat{f}(x)\hat{g}(y)\mathrm{d}x\mathrm{d}y := L_1+L_2+L_3+L_4, \qquad (2.14)$$

where

$$\begin{aligned} L_1 &= \int_{E_1^-} |x|^{\lambda_1\alpha-1-\frac{2\lambda_1}{sp}} \int_{E_2^-} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right)|y|^{\lambda_2\alpha-1+\frac{2\lambda_2}{sq}}\,\mathrm{d}y\mathrm{d}x, \\ L_2 &= \int_{E_1^-} |x|^{\lambda_1\alpha-1-\frac{2\lambda_1}{sp}} \int_{E_2^+} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right)|y|^{\lambda_2\alpha-1+\frac{2\lambda_2}{sq}}\,\mathrm{d}y\mathrm{d}x, \\ L_3 &= \int_{E_1^+} |x|^{\lambda_1\alpha-1-\frac{2\lambda_1}{sp}} \int_{E_2^-} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right)|y|^{\lambda_2\alpha-1+\frac{2\lambda_2}{sq}}\,\mathrm{d}y\mathrm{d}x, \\ L_4 &= \int_{E_1^+} |x|^{\lambda_1\alpha-1-\frac{2\lambda_1}{sp}} \int_{E_2^+} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right)|y|^{\lambda_2\alpha-1+\frac{2\lambda_2}{sq}}\,\mathrm{d}y\mathrm{d}x. \end{aligned}$$

We first consider the case $\lambda_1,\lambda_2<0$. That is $\lambda_1,\lambda_2\in\{-1,-3,-\frac{1}{3},-5,-\frac{1}{5},\cdots\}$. Setting $x^{\lambda_1}y^{\lambda_2}=z$, and using Fubini's theorem, we have

$$
\begin{aligned}
L_1 &= \int_{-1}^{0} |\,x\,|^{\lambda_1\alpha-1-\frac{2\lambda_1}{sp}} \int_{-\infty}^{-1} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right) |\,y\,|^{\lambda_2\alpha-1+\frac{2\lambda_2}{sq}}\,\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{|\,\lambda_2\,|}\int_{-1}^{0} |\,x\,|^{-1-\frac{2\lambda_1}{s}} \int_{0}^{-x^{\lambda_1}} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z\mathrm{d}x \\
&= \frac{1}{|\,\lambda_2\,|}\int_{0}^{1} x^{-1-\frac{2\lambda_1}{s}} \left[\int_{0}^{1} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z + \int_{1}^{x^{\lambda_1}} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z\right]\mathrm{d}x \\
&= \frac{s}{2\,|\,\lambda_1\lambda_2\,|}\int_{0}^{1} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z \\
&\quad + \frac{1}{|\,\lambda_2\,|}\int_{1}^{\infty} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}} \int_{0}^{z^{\frac{1}{\lambda_1}}} x^{-1-\frac{2\lambda_1}{s}}\,\mathrm{d}x\mathrm{d}z \\
&= \frac{s}{2\,|\,\lambda_1\lambda_2\,|}\left[\int_{0}^{1} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z + \int_{1}^{\infty} K_\delta(z)\,|\,z\,|^{\alpha-1-\frac{2}{sp}}\,\mathrm{d}z\right]; \qquad (2.15)
\end{aligned}
$$

and

$$
\begin{aligned}
L_2 &= \int_{-1}^{0} |\,x\,|^{\lambda_1\alpha-1-\frac{2\lambda_1}{sp}} \int_{1}^{\infty} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right) |\,y\,|^{\lambda_2\alpha-1+\frac{2\lambda_2}{sq}}\,\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{|\,\lambda_2\,|}\int_{-1}^{0} |\,x\,|^{-1-\frac{2\lambda_1}{s}} \int_{x^{\lambda_1}}^{0} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z\mathrm{d}x \\
&= \frac{1}{|\,\lambda_2\,|}\int_{0}^{1} x^{-1-\frac{2\lambda_1}{s}} \left[\int_{-1}^{0} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z + \int_{-x^{\lambda_1}}^{-1} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z\right]\mathrm{d}x \\
&= \frac{s}{2\,|\,\lambda_1\lambda_2\,|}\int_{-1}^{0} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z \\
&\quad + \frac{1}{|\,\lambda_2\,|}\int_{-\infty}^{-1} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}} \int_{0}^{-z^{\frac{1}{\lambda_1}}} x^{-1-\frac{2\lambda_1}{s}}\,\mathrm{d}x\mathrm{d}z \\
&= \frac{s}{2\,|\,\lambda_1\lambda_2\,|}\left[\int_{-1}^{0} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z + \int_{-\infty}^{-1} K_\delta(z)\,|\,z\,|^{\alpha-1-\frac{2}{sp}}\,\mathrm{d}z\right]. \qquad (2.16)
\end{aligned}
$$

Similarly, it can also be obtained that

$$
L_3 = \frac{s}{2\,|\,\lambda_1\lambda_2\,|}\left[\int_{-1}^{0} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z + \int_{-\infty}^{-1} K_\delta(z)\,|\,z\,|^{\alpha-1-\frac{2}{sp}}\,\mathrm{d}z\right]; \qquad (2.17)
$$

$$
L_4 = \frac{s}{2\,|\,\lambda_1\lambda_2\,|}\left[\int_{0}^{1} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z + \int_{1}^{\infty} K_\delta(z)\,|\,z\,|^{\alpha-1-\frac{2}{sp}}\,\mathrm{d}z\right]. \qquad (2.18)
$$

Plugging (2.15)–(2.18) back into (2.14), we obtain (2.13) for $\lambda_1,\lambda_2<0$. Similarly, (2.13) can also be obtained for $\lambda_1\lambda_2<0$ or $\lambda_1,\lambda_2>0$. Lemma 2.2 is proved. □

LEMMA 2.3. *Let* $|z|<1$, $\psi(t)=\tan t$. *Then*

$$\psi^{(2k)}\left(\frac{z\pi}{2}\right)=\frac{2^{2k+1}(2k)!}{\pi^{2k+1}}\sum_{j=0}^{\infty}\left[\frac{1}{(2j+1-z)^{2k+1}}-\frac{1}{(2j+1+z)^{2k+1}}\right],\ k\in\mathbb{N}, \tag{2.19}$$

$$\psi^{(2k-1)}\left(\frac{z\pi}{2}\right)=\frac{2^{2k}(2k-1)!}{\pi^{2k}}\sum_{j=0}^{\infty}\left[\frac{1}{(2j+1-z)^{2k}}+\frac{1}{(2j+1+z)^{2k}}\right],\ k\in\mathbb{N}^{+}. \tag{2.20}$$

Proof. $\psi(t)=\tan t$ can be expressed as a rational fraction [32]:

$$\psi(t)=2\sum_{j=0}^{\infty}\left[\frac{1}{(2j+1)\pi-2t}-\frac{1}{(2j+1)\pi+2t}\right]. \tag{2.21}$$

Find the $(2k)$th derivative of (2.21), then we obtain

$$\psi^{(2k)}(t)=2^{2k+1}(2k)!\sum_{j=0}^{\infty}\left\{\frac{1}{[(2j+1)\pi-2t]^{2k+1}}-\frac{1}{[(2j+1)\pi+2t]^{2k+1}}\right\}. \tag{2.22}$$

Setting $t=\frac{z\pi}{2}$ in (2.22), we get (2.19). Finding the first derivative of (2.22), letting $t=\frac{z\pi}{2}$ and replacing $k+1$ with k, we can arrive at (2.20). Lemma 2.3 is proved. □

LEMMA 2.4. *Let* $|z|<1$, $\phi(t)=\sec t$. *Then*

$$\phi^{(2k)}\left(\frac{z\pi}{2}\right)=\frac{2^{2k+1}(2k)!}{\pi^{2k+1}}\sum_{j=0}^{\infty}\left[\frac{(-1)^j}{(2j+1-z)^{2k+1}}+\frac{(-1)^j}{(2j+1+z)^{2k+1}}\right],\ k\in\mathbb{N}. \tag{2.23}$$

Proof. Taking the $(2k)$th derivative of the following equality

$$2\phi(t)=\psi\left(\frac{\pi}{4}+\frac{t}{2}\right)+\psi\left(\frac{\pi}{4}-\frac{t}{2}\right), \tag{2.24}$$

we get

$$2^{2k+1}\phi^{(2k)}(t)=\psi^{(2k)}\left(\frac{\pi}{4}+\frac{t}{2}\right)+\psi^{(2k)}\left(\frac{\pi}{4}-\frac{t}{2}\right). \tag{2.25}$$

Setting $t=\frac{z\pi}{2}$ in (2.25), and using (2.19), we can get

$$\begin{aligned}\phi^{(2k)}\left(\frac{z\pi}{2}\right)&=\frac{2^{2k+1}(2k)!}{\pi^{2k+1}}\left\{\sum_{j=0}^{\infty}\left[\frac{1}{(4j+1-z)^{2k+1}}-\frac{1}{(4j+3+z)^{2k+1}}\right]\right.\\&\qquad\left.+\sum_{j=0}^{\infty}\left[\frac{1}{(4j+1+z)^{2k+1}}-\frac{1}{(4j+3-z)^{2k+1}}\right]\right\}\\&=\frac{2^{2k+1}(2k)!}{\pi^{2k+1}}\sum_{j=0}^{\infty}\left[\frac{(-1)^j}{(2j+1-z)^{2k+1}}+\frac{(-1)^j}{(2j+1+z)^{2k+1}}\right].\end{aligned}$$

Lemma 2.4 is proved. □

3. Main results

THEOREM 3.1. *Let $\tau_1, \tau_2 \in \{-1,1\}$ and $\delta \in \{0,1\}$. Let α, β be such that $\beta < \alpha < 2m-2n+1$, where $m,n \in \mathbb{N}$, and $n \neq 0$ for $\delta = 0$, $\tau_2 = -1$. Suppose that $\gamma \geqslant 0$ for $\delta = 0$, $\tau_2 = -1$ or $\delta = 1$, $\tau_1 = 1$, and $\gamma > 0$ for δ, τ_1, τ_2 taking other values. Let $\lambda_1, \lambda_2 \in \{\pm 1, \pm 3, \pm\frac{1}{3}, \pm 5, \pm\frac{1}{5}, \cdots\}$, $p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$. Suppose that $\mu(x) = |x|^{p(1-\lambda_1\alpha)-1}$, $\nu(y) = |y|^{q(1-\lambda_2\alpha)-1}$, and $f(x),\ g(y) \geqslant 0$ with $f(x) \in L^p_\mu(\mathbb{R})$, $g(y) \in L^q_\nu(\mathbb{R})$. $K_\delta(z)$ and $C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2)$ are defined via Lemma 2.1. Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right) f(x)g(y)\mathrm{d}x\mathrm{d}y \tag{3.1}$$
$$< |\lambda_1|^{-\frac{1}{q}} |\lambda_2|^{-\frac{1}{p}}\, \Gamma(\gamma+1) C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2) \|f\|_{p,\mu} \|g\|_{q,\nu},$$

where the constant factor $|\lambda_1|^{-\frac{1}{q}} |\lambda_2|^{-\frac{1}{p}}\, \Gamma(\gamma+1) C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2)$ is the best possible.

Proof. By Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right) f(x)g(y)\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \left[K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right)\right]^{\frac{1}{p}} |y|^{\frac{\lambda_2\alpha-1}{p}} |x|^{\frac{1-\lambda_1\alpha}{q}} f(x) \right\} \\
&\qquad \times \left\{ \left[K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right)\right]^{\frac{1}{q}} |x|^{\frac{\lambda_1\alpha-1}{q}} |y|^{\frac{1-\lambda_2\alpha}{p}} g(y) \right\} \mathrm{d}x\mathrm{d}y \\
&\leqslant \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right) |y|^{\lambda_2\alpha-1} |x|^{\frac{p(1-\lambda_1\alpha)}{q}} f^p(x)\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{p}} \\
&\qquad \times \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right) |x|^{\lambda_1\alpha-1} |y|^{\frac{q(1-\lambda_2\alpha)}{p}} g^q(y)\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{q}} \\
&= \left[\int_{-\infty}^{\infty} \omega(x) |x|^{\frac{p(1-\lambda_1\alpha)}{q}} f^p(x)\mathrm{d}x\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} \varpi(y) |y|^{\frac{q(1-\lambda_2\alpha)}{p}} g^q(y)\mathrm{d}y\right]^{\frac{1}{q}},
\end{aligned} \tag{3.2}$$

where

$$\omega(x) = \int_{-\infty}^{\infty} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right) |y|^{\lambda_2\alpha-1}\, \mathrm{d}y,$$

and

$$\varpi(y) = \int_{-\infty}^{\infty} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right) |x|^{\lambda_1\alpha-1}\, \mathrm{d}x.$$

For $x < 0$, $\lambda_2 < 0$, namely, $\lambda_2 \in \{-1,-3,-\frac{1}{3},-5,-\frac{1}{5},\cdots\}$, setting $x^{\lambda_1}y^{\lambda_2} = z$, and using (2.4), we can get

$$\begin{aligned}\omega(x) &= \int_{-\infty}^{0} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right) | y |^{\lambda_2\alpha-1} \mathrm{d}y + \int_{0}^{\infty} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right) | y |^{\lambda_2\alpha-1} \mathrm{d}y \\ &= | \lambda_2 |^{-1} | x |^{-\lambda_1\alpha} \left[\int_0^\infty K_\delta(z) | z |^{\alpha-1} \mathrm{d}z + \int_{-\infty}^{0} K_\delta(z) | z |^{\alpha-1} \mathrm{d}z\right] \\ &= | \lambda_2 |^{-1} \Gamma(\gamma+1) C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2) | x |^{-\lambda_1\alpha}. \end{aligned} \tag{3.3}$$

Similarly, it can be proved that (3.3) holds true for other cases: $x>0$, $\lambda_2>0$; $x>0$, $\lambda_2<0$ and $x<0$, $\lambda_2>0$. In addition, we can also prove that

$$\begin{aligned}\varpi(y) &= | \lambda_1 |^{-1} | y |^{-\lambda_2\alpha} \int_{-\infty}^{\infty} K_\delta(z) | z |^{\alpha-1} \mathrm{d}z \\ &= | \lambda_1 |^{-1} \Gamma(\gamma+1) C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2) | y |^{-\lambda_2\alpha}, \quad (y\neq 0). \end{aligned} \tag{3.4}$$

Applying (3.3) and (3.4) to (3.2), it follows that

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\delta(x,y) f(x) g(y) \mathrm{d}x\mathrm{d}y \\ &\leqslant | \lambda_1 |^{-\frac{1}{q}} | \lambda_2 |^{-\frac{1}{p}} \Gamma(\gamma+1) C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2) \|f\|_{p,\mu} \|g\|_{q,\nu}. \end{aligned} \tag{3.5}$$

Next, we will prove that the equality in (3.5) doesn't hold. In fact, suppose that (3.5) takes the form of equality, then it can be deduced that

$$M_1 | y |^{\lambda_2\alpha-1} | x |^{\frac{p(1-\lambda_1\alpha)}{q}} f^p(x) = M_2 | x |^{\lambda_1\alpha-1} | y |^{\frac{q(1-\lambda_2\alpha)}{p}} g^q(y) \tag{3.6}$$

a. e. in $\mathbb{R}\times\mathbb{R}$, where the constants M_1 and M_2 are not both equal to zero. Simplify (3.6), then we get

$$M_1 | x |^{p(1-\lambda_1\alpha)} f^p(x) = M_2 | y |^{q(1-\lambda_2\alpha)} g^q(y).$$

Hence, there exists a constant C such that $M_1 | x |^{p(1-\lambda_1\alpha)} f^p(x) = C$ and $M_2 | y |^{q(1-\lambda_2\alpha)} g^q(y) = C$ a. e. in $\mathbb{R}$. Assume that $M_1 \neq 0$, then we have $| x |^{p(1-\lambda_1\alpha)-1} f^p(x) = \frac{C}{M_1|x|}$ a.e. in $\mathbb{R}$, which contradicts the condition $f(x)\in L^p_\mu(\mathbb{R})$. Therefore, (3.5) doesn't take the form of equality, and we get (3.1).

In order to complete the proof of Theorem 3.1, it suffices to prove that $| \lambda_1 |^{-\frac{1}{q}} | \lambda_2 |^{-\frac{1}{p}} \Gamma(\gamma+1) C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2)$ is the best possible. Assuming that there exists a constant $k (k>0)$ such that $k < | \lambda_1 |^{-\frac{1}{q}} | \lambda_2 |^{-\frac{1}{p}} \Gamma(\gamma+1) C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2)$, and (3.1) still holds true if $| \lambda_1 |^{-\frac{1}{q}} | \lambda_2 |^{-\frac{1}{p}} \Gamma(\gamma+1) C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2)$ is replaced with k. That is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right) f(x) g(y) \mathrm{d}x\mathrm{d}y < k \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{3.7}$$

Let $f(x)=\hat{f}(x)$ and $g(y)=\hat{g}(y)$, where $\hat{f}(x)$ and $\hat{g}(y)$ are defined in Lemma 2.2. Then

$$\begin{aligned}
&\int_{[-1,1]} K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z+\int_{\mathbb{R}\setminus[-1,1]} K_\delta(z)\,|\,z\,|^{\alpha-1-\frac{2}{sp}}\,\mathrm{d}z\\
&\quad<\frac{k\,|\,\lambda_1\lambda_2\,|}{s}\left(\int_{E_1}|\,x\,|^{-\frac{2\lambda_1}{s}-1}\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{E_2}|\,y\,|^{\frac{2\lambda_2}{s}-1}\,\mathrm{d}y\right)^{\frac{1}{q}}\\
&\quad=k\,|\,\lambda_1\,|^{\frac{1}{q}}|\,\lambda_2\,|^{\frac{1}{p}}\,. \qquad (3.8)
\end{aligned}$$

By the use of Fatou's lemma, it follows that

$$\begin{aligned}
&\Gamma(\gamma+1)C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2)\\
&=\int_{-\infty}^{\infty}K_\delta(z)\,|\,z\,|^{\alpha-1}\,\mathrm{d}z\\
&=\int_{[-1,1]}\varliminf_{s\to\infty}K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z+\int_{\mathbb{R}\setminus[-1,1]}\varliminf_{s\to\infty}K_\delta(z)\,|\,z\,|^{\alpha-1-\frac{2}{sp}}\,\mathrm{d}z\\
&\leqslant\varliminf_{s\to\infty}\left[\int_{[-1,1]}K_\delta(z)\,|\,z\,|^{\alpha-1+\frac{2}{sq}}\,\mathrm{d}z+\int_{\mathbb{R}\setminus[-1,1]}K_\delta(z)\,|\,z\,|^{\alpha-1-\frac{2}{sp}}\,\mathrm{d}z\right]\\
&<k\,|\,\lambda_1\,|^{\frac{1}{q}}|\,\lambda_2\,|^{\frac{1}{p}}\,. \qquad (3.9)
\end{aligned}$$

Obviously, (3.9) contradicts the assumption, and therefore the constant factor in (3.1) is the best possible. Theorem 3.1 is proved. □

THEOREM 3.2. *Let* $\tau_1,\tau_2\in\{-1,1\}$ *and* $\delta\in\{0,1\}$. *Let* α, β *be such that* $\beta<\alpha<2m-2n+1$, *where* $m,n\in\mathbb{N}$, *and* $n\neq 0$ *for* $\delta=0,\tau_2=-1$. *Suppose that* $\gamma\geqslant 0$ *for* $\delta=0$, $\tau_2=-1$ *or* $\delta=1$, $\tau_1=1$, *and* $\gamma>0$ *for* δ,τ_1,τ_2 *taking other values. Let* $\lambda_1,\lambda_2\in\{\pm1,\pm3,\pm\frac{1}{3},\pm5,\pm\frac{1}{5},\cdots\}$, $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Suppose that* $\mu(x)=|\,x\,|^{p(1-\lambda_1\alpha)-1}$, $\nu(y)=|\,y\,|^{q(1-\lambda_2\alpha)-1}$, *and* $f(x)>0$ *with* $f(x)\in L^p_\mu(\mathbb{R})$. $K_\delta(z)$ *and* $C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2)$ *are defined via Lemma 2.1. Then*

$$\begin{aligned}
&\int_{-\infty}^{\infty}|\,y\,|^{p\lambda_2\alpha-1}\left[\int_{-\infty}^{\infty}K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right)f(x)\mathrm{d}x\right]^p\mathrm{d}y\\
&\qquad<\left[|\,\lambda_1\,|^{-\frac{1}{q}}|\,\lambda_2\,|^{-\frac{1}{p}}\,\Gamma(\gamma+1)C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2)\|f\|_{p,\mu}\right]^p, \qquad (3.10)
\end{aligned}$$

where $|\,\lambda_1\,|^{-\frac{1}{q}}|\,\lambda_2\,|^{-\frac{1}{p}}\,\Gamma(\gamma+1)C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2)$ *is the best possible, and* (3.10) *is equivalent to* (3.1).

Proof. Consider $G(y):=|\,y\,|^{p\lambda_2\alpha-1}\left[\int_{-\infty}^{\infty}K_\delta(x^{\lambda_1}y^{\lambda_2})f(x)\mathrm{d}x\right]^{p-1}$, then it follows from Theorem 3.1 that

$$
\begin{aligned}
(\|G\|_{q,v})^q &= \int_{-\infty}^{\infty} |\, y\,|^{q(1-\lambda_2\alpha)-1}\, G^q(y)\mathrm{d}y \\
&= \int_{-\infty}^{\infty} |\, y\,|^{p\lambda_2\alpha-1} \left[\int_{-\infty}^{\infty} K_\delta(x^{\lambda_1}y^{\lambda_2})f(x)\mathrm{d}x\right]^p \mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\delta(x^{\lambda_1}y^{\lambda_2})f(x)G(y)\mathrm{d}x\mathrm{d}y \\
&\leqslant |\, \lambda_1\,|^{-\frac{1}{q}} |\, \lambda_2\,|^{-\frac{1}{p}}\, \Gamma(\gamma+1)C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2)\|f\|_{p,\mu}\|G\|_{q,v}.
\end{aligned} \tag{3.11}
$$

Observing that $f(x)\in L^p_\mu(\mathbb{R})$, we can deduce from (3.11) that

$$
\begin{aligned}
0 &< \int_{-\infty}^{\infty} |\, y\,|^{p\lambda_2\alpha-1} \left[\int_{-\infty}^{\infty} K_\delta(x^{\lambda_1}y^{\lambda_2})f(x)\mathrm{d}x\right]^p \mathrm{d}y \\
&= (\|G\|_{q,v})^q = \left(\|G\|_{q,v}^{q-1}\right)^p \\
&\leqslant \left[|\, \lambda_1\,|^{-\frac{1}{q}} |\, \lambda_2\,|^{-\frac{1}{p}}\, \Gamma(\gamma+1)C_\delta(m,n,\alpha,\beta,\gamma,\tau_1,\tau_2)\|f\|_{p,\mu}\right]^p < \infty.
\end{aligned} \tag{3.12}
$$

It can be obtained from (3.12) that $G(y)\in L^q_v(\mathbb{R})$, and $f(x)$, $G(y)$ meet the conditions of Theorem 3.1. Therefore both (3.11) and (3.12) are strict inequalities and (3.10) is proved.

Alternatively, suppose that (3.10) holds true, it follows from Hölder's inequality that

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right) f(x)g(y)\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \left[|\, y\,|^{-\left(1-\lambda_2\alpha-\frac{1}{q}\right)} \int_{-\infty}^{\infty} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right) f(x)\mathrm{d}x\right] \left[|\, y\,|^{1-\lambda_2\alpha-\frac{1}{q}}\, g(y)\right] \mathrm{d}y \\
&\leqslant \left\{\int_{-\infty}^{\infty} |\, y\,|^{p\lambda_2\alpha-1} \left[\int_{-\infty}^{\infty} K_\delta\left(x^{\lambda_1}y^{\lambda_2}\right) f(x)\mathrm{d}x\right]^p \mathrm{d}y\right\}^{\frac{1}{p}} \|g\|_{q,v}.
\end{aligned} \tag{3.13}
$$

Substituting (3.10) into (3.13), we get (3.1). Therefore, (3.1) and (3.10) are equivalent, and the constant factor in the right hand side of (3.10) is obviously the best possible from the equivalence of (3.1) and (3.10). Theorem 3.2 is proved. □

4. Some Corollaries

We first present the following two equations [32]:

$$
\sum_{j=0}^{\infty} \frac{1}{(j+1)^{2k}} = \frac{B_k}{2(2k)!}(2\pi)^{2k}, \tag{4.1}
$$

$$
\sum_{j=0}^{\infty} \frac{1}{(2j+1)^{2k}} = \frac{B_k}{2(2k)!}(2^{2k}-1)\pi^{2k}, \tag{4.2}
$$

where B_k is a Bernoulli number, $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $\cdots$.

Setting $\delta = \alpha = n = 0$, $\lambda_1 = \lambda_2 = \tau_2 = 1$, $\beta = -(4m+2)$ $(m \in \mathbb{N})$ and $\gamma = 2k-1$ $(k \in \mathbb{N}^+)$ in Theorem 3.1, and using (4.1) and (4.2), we can obtain the following corollary.

COROLLARY 4.1. *Let* $\tau_1 \in \{-1,1\}$, $m \in \mathbb{N}$, $k \in \mathbb{N}^+$, $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Suppose that* $\mu(x) =| x |^{p-1}$, $\nu(y) =| y |^{q-1}$, *and* $f(x)$, $g(y) \geqslant 0$ *with* $f(x) \in L^p_\mu(\mathbb{R})$, $g(y) \in L^q_\nu(\mathbb{R})$. *Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(| \log | xy ||)^{2k-1}}{| 1+\tau_1 (xy)^{2m+1} |} (\min\{1,| xy |\})^{4m+2} f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{(2\pi)^{2k} B_k}{2k(2m+1)^{2k}} \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{4.3}$$

Let $m = 0$ in (4.3), then we obtain (1.6). Let $k = 1$ in (1.6), then it follows that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{| \log | xy ||}{| 1 \pm xy |} (\min\{1,| xy |\})^2 f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{\pi^2}{3} \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{4.4}$$

Setting $\delta = \alpha = n = 0$, $\lambda_1 = \tau_2 = 1$, $\lambda_2 = -1$, $\beta = -(4m+2)$ $(m \in \mathbb{N})$ and $\gamma = 2k-1$ $(k \in \mathbb{N}^+)$ in Theorem 3.1, and replacing $g(y) | y |^{-(2m+1)}$ with $g(y)$, we have Corollary 4.2.

COROLLARY 4.2. *Let* $\tau_1 \in \{-1,1\}$, $m \in \mathbb{N}$, $k \in \mathbb{N}^+$, $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Suppose that* $\mu(x) =| x |^{p-1}$, $\nu(y) =| y |^{2q(m+1)-1}$, *and* $f(x)$, $g(y) \geqslant 0$ *with* $f(x) \in L^p_\mu(\mathbb{R})$, $g(y) \in L^q_\nu(\mathbb{R})$. *Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{| \log | \frac{x}{y} ||^{2k-1}}{| x^{2m+1} + \tau_1 y^{2m+1} |} (\min\{| x |,| y |\})^{4m+2} f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{(2\pi)^{2k} B_k}{2k(2m+1)^{2k}} \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{4.5}$$

Setting $\delta = n = 0$, $\lambda_1 = \lambda_2 = \tau_2 = 1$, $\beta = -(2m+1)$ $(m \in \mathbb{N})$ and $\gamma = 2k-1$ $(k \in \mathbb{N}^+)$ in Theorem 3.1, by the use of (2.20), we can obtain Corollary 4.3.

COROLLARY 4.3. *Let* $\tau_1 \in \{-1,1\}$, $m \in \mathbb{N}$, $k \in \mathbb{N}^+$, $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* α *be such that* $-(2m+1) < \alpha < 2m+1$. *Suppose that* $\psi(t) = \tan t$, $\mu(x) =| x |^{p(1-\alpha)-1}$, $\nu(y) =| y |^{q(1-\alpha)-1}$, *and* $f(x)$, $g(y) \geqslant 0$ *with* $f(x) \in L^p_\mu(\mathbb{R})$, $g(y) \in L^q_\nu(\mathbb{R})$. *Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{| \log | xy ||^{2k-1}}{| 1+\tau_1 (xy)^{2m+1} |} (\min\{1,| xy |\})^{2m+1} f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{2\pi^{2k}}{(4m+2)^{2k}} \psi^{(2k-1)}\left(\frac{\alpha\pi}{4m+2}\right) \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{4.6}$$

Letting $\alpha = 0$ in (4.6), and using (2.20) and (4.2), we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\log|xy||^{2k-1}}{|1\pm(xy)^{2m+1}|}(\min\{1,|xy|\})^{2m+1}f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{(2^{2k}-1)\pi^{2k}}{k(2m+1)^{2k}}B_k\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{4.7}$$

where $\mu(x) = |x|^{p-1}$, $\nu(y) = |y|^{q-1}$.
Letting $m = 0$, $k = 1$, $\alpha = \frac{1}{2}$, $p = q = 2$ in (4.6), we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\log|xy||}{|1\pm xy|}\min\{1,|xy|\}f(x)g(y)\mathrm{d}x\mathrm{d}y < \pi^2\|f\|_2\|g\|_2. \tag{4.8}$$

Setting $\alpha = \delta = 0$, $\lambda_1 = \lambda_2 = 1$, $\tau_2 = -1$, $\beta = -(2m+1)$ $(m \in \mathbb{N})$ and $\gamma = 2k$ $(k \in \mathbb{N})$ in Theorem 3.1, by the use of (2.19), we can obtain another corollary.

COROLLARY 4.4. *Let* $\tau_1 \in \{-1,1\}$, $m,n,k \in \mathbb{N}$ *and* $m \geqslant n > 0$. *Let* $p > 1$ *and* $\frac{1}{p}+\frac{1}{q} = 1$. *Suppose that* $\psi(t) = \tan t$, $\mu(x) = |x|^{p-1}$, $\nu(y) = |y|^{q-1}$, *and* $f(x)$, $g(y) \geqslant 0$ *with* $f(x) \in L^p_\mu(\mathbb{R})$, $g(y) \in L^q_\nu(\mathbb{R})$. *Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|1-(xy)^{2n}|}{|1+\tau_1(xy)^{2m+1}|}\frac{|\log|xy||^{2k}}{(\min\{1,|xy|\})^{-(2m+1)}}f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{2\pi^{2k+1}}{(4m+2)^{2k+1}}\psi^{(2k)}\left(\frac{n\pi}{2m+1}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.9}$$

Let $k = 0$ in (4.9), then it follows that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|1-(xy)^{2n}|}{|1+\tau_1(xy)^{2m+1}|}(\min\{1,|xy|\})^{2m+1}f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{\pi}{2m+1}\psi\left(\frac{n\pi}{2m+1}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.10}$$

Let $m = n = 1$ in (4.10), then we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|1-\tau_1 xy|}{|1-\tau_1 xy+x^2y^2|}(\min\{1,|xy|\})^3 f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{\sqrt{3}\pi}{3}\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.11}$$

Setting $\delta = \tau_1 = \lambda_1 = \lambda_2 = 1$, $\beta = -(2n+1)$ $(n \in \mathbb{N})$ and $\gamma = 2k$ $(k \in \mathbb{N})$ in Theorem 3.1, in view of (2.23), we can obtain Corollary 4.5.

COROLLARY 4.5. *Let* $\tau_2 \in \{-1,1\}$, *and* $m,n,k \in \mathbb{N}$. *Let* $p > 1$ *and* $\frac{1}{p}+\frac{1}{q} = 1$. *Let* α *be such that* $-(2n+1) < \alpha < 2m-2n+1$. *Suppose that* $\phi(t) = \sec t$, $\mu(x) =$

$|\,x\,|^{p(1-\alpha)-1}$, $\nu(y) = |\,y\,|^{q(1-\alpha)-1}$, *and* $f(x)$, $g(y) \geqslant 0$ *with* $f(x) \in L^p_\mu(\mathbb{R})$, $g(y) \in L^q_\nu(\mathbb{R})$. *Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|\,1+\tau_2\,(xy)^{2n+1}\,|}{1+(xy)^{2m+2}} \frac{|\log|\,xy\,||^{2k}}{(\min\{1,|\,xy\,|\})^{-(2n+1)}} f(x)g(y)\mathrm{d}x\mathrm{d}y$$
$$< \frac{2\pi^{2k+1}}{(2m+2)^{2k+1}}\phi^{(2k)}\left(\frac{m-2n-\alpha}{2m+2}\pi\right)\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.12}$$

Let $k=0$, then (4.12) reduces to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|\,1\pm(xy)^{2n+1}\,|}{1+(xy)^{2m+2}}(\min\{1,|\,xy\,|\})^{2n+1} f(x)g(y)\mathrm{d}x\mathrm{d}y$$
$$< \frac{\pi}{m+1}\phi\left(\frac{m-2n-\alpha}{2m+2}\pi\right)\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.13}$$

Letting $m=n=0$, $\alpha=\frac{1}{2}$, and $p=q=2$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|\,1\pm xy\,|}{1+(xy)^2}\min\{1,|\,xy\,|\} f(x)g(y)\mathrm{d}x\mathrm{d}y < \sqrt{2}\pi\|f\|_2\|g\|_2. \tag{4.14}$$

Observing that [32]

$$\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^{2k+1}} = \frac{\pi^{2k+1}}{2^{2k+2}(2k)!}E_k,$$

where E_k is the Euler number, $E_0=1$, $E_1=1$, $E_2=5$, $\cdots$, it follows from (2.23) that $\phi^{(2k)}(0)=E_k$. Therefore, Letting $m=n=\alpha=0$ in (4.12), we obtain (1.7).

Setting $\delta=\tau_1=\lambda_1=\lambda_2=1$, $\alpha=m-2n$, $\beta=-(2n+1)$ $(m,n\in\mathbb{N})$ and $\gamma=1$ in Theorem 3.1, in view of the representation of the Catalan constant:

$$L_0=\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^2}=0.91596\cdots,$$

we can obtain Corollary 4.6.

COROLLARY 4.6. *Let* $\tau_2\in\{-1,1\}$ *and* $m,n\in\mathbb{N}$. *Let* $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Suppose that* $\mu(x)=|\,x\,|^{p(1+2n-m)-1}$, $\nu(y)=|\,y\,|^{q(1+2n-m)-1}$, *and* $f(x)$, $g(y)\geqslant 0$ *with* $f(x)\in L^p_\mu(\mathbb{R})$, $g(y)\in L^q_\nu(\mathbb{R})$. *Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|\,1+\tau_2\,(xy)^{2n+1}\,|}{1+(xy)^{2m+2}} \frac{|\log|\,xy\,||}{(\min\{1,|\,xy\,|\})^{-(2n+1)}} f(x)g(y)\mathrm{d}x\mathrm{d}y$$
$$< \frac{4L_0}{(m+1)^2}\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.15}$$

Let $m=n=0$ in (4.15), we get (1.8).

Setting $\delta=\lambda_1=\lambda_2=1$, $\tau_1=-1$, $\beta=-(2n+1)$ $(n\in\mathbb{N})$ and $\gamma=2k-1$ $(k\in\mathbb{N}^+)$ in Theorem 3.1, and using (2.20), we obtain the last corollary.

COROLLARY 4.7. *Let* $\tau_2 \in \{-1,1\}$, $m,n \in \mathbb{N}$, $k \in \mathbb{N}^+$, $p > 1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Let* α *be such that* $-(2n+1) < \alpha < 2m-2n+1$. *Suppose that* $\psi(t)=\tan t$, $\mu(x)=|x|^{p(1-\alpha)-1}$, $\nu(y)=|y|^{q(1-\alpha)-1}$, *and* $f(x)$, $g(y) \geqslant 0$ *with* $f(x) \in L^p_\mu(\mathbb{R})$, $g(y) \in L^q_\nu(\mathbb{R})$. *Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|1+\tau_2 (xy)^{2n+1}|}{|1-(xy)^{2m+2}|} \frac{|\log|xy||^{2k-1}}{(\min\{1,|xy|\})^{-(2n+1)}} f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{2\pi^{2k}}{(2m+2)^{2k}} \psi^{(2k-1)}\left(\frac{m-2n-\alpha}{2m+2}\pi\right) \|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.16}$$

Let $m=2n$, (4.16) reduces to (4.6). Let $m=0$, $n=1$, $\tau_2=-1$, then $-3<\alpha<-1$, and (4.15) is transfomed to the following inequality:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|\log|xy||^{2k-1}}{(\min\{1,|xy|\})^{-3}} \left(1+\frac{x^2y^2}{1\pm xy}\right) f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{\pi^{2k}}{2^{2k-1}} \psi^{(2k-1)}\left(\frac{\alpha\pi}{2}\right) \|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{4.17}$$

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(Received July 15, 2021)

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