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GRONWALL–TYPE MOMENT INEQUALITIES FOR A STOCHASTIC PROCESS

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Abstract. The main purpose of this paper is to demonstrate the moment inequality theorems of the stochastic process. More specifically, we want to establish some stochastic moment inequalities in the stochastic process by applying the Itô formula and the Gronwall-type inequalities as well as introduce a new proofs of some parts of the Burkholder-Davis-Gundy inequality and induce inverse inequality.

1. Introduction

The stochastic moment inequalities of stochastic integrals have been widely applied in the theory of stochastic differential equations and partial stochastic differential equations to prove the results on existence, uniqueness, boundedness, comparison, continuous dependence, perturbation and stability etc (see [5]–[12]).

Among these moment inequalities, the well-known inequality is the Doob's inequality introduced in the following theorem.

THEOREM 1. [6] (Doob's martingale inequalities) *Let* $\{M_t\}_{t\geqslant 0}$ *be an* R^d*-valued martingale. Let* $[a,b]$ *be a bounded interval in* R_+ .

(i) *Let* $p \geqslant 1$, *and* $M_t \in L^p(\Omega;R^d)$, *then*

$$P\left\{\omega : \sup_{a\leqslant t\leqslant b} |M_t| \geqslant c\right\} \leqslant \frac{E|M_b|^p}{c^p}$$

holds for all $c > 0$.

(ii) *Let* $p > 1$, *and* $M_t \in L^p(\Omega;R^d)$, *then*

$$E\left(\sup_{a\leqslant t\leqslant b} |M_t|^p\right) \leqslant \left(\frac{p}{p-1}\right)^p E|M_b|^p.$$

The Doob's inequalities with martingale process play an important role in characterizing stochastic process theory. If we apply these results to a stochastic process, we obtain the following important moment inequality for stochastic integrals.

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THEOREM 2. [6] *If* $p \geqslant 2$, $g \in \mathscr{M}^2([0,T];R^{d\times m})$ *such that*

$$E\int_0^T |g(s)|^p \, \mathrm{d}s < \infty,$$

then

$$E\left|\int_0^T g(s)\,\mathrm{d}B(s)\right|^p \leqslant \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |g(s)|^p \, \mathrm{d}s.$$

In particular, for $p = 2$, *there is equality.*

If we also apply Doob's inequalities to a stochastic process, we obtain the following Burkholder-Davis-Gundy moment inequality for stochastic integrals.

THEOREM 3. [6] *Let* $g \in \mathscr{L}^2(R_+;R^{d\times m})$. *Define, for* $t \geqslant 0$,

$$x(t) = \int_0^t g(s)\,dB(s) \quad \text{and} \quad A(t) = \int_0^t |g(s)|^2\,ds.$$

Then for every $p > 0$, *there exist universal positive constants* c_p, C_p *(depending only on* p*) such that*

$$c_p E|A(t)|^{p/2} \leqslant E\left(\sup_{0\leqslant s\leqslant t} |x(s)|^p\right) \leqslant C_p E|A(t)|^{p/2}$$

for all $t \geqslant 0$. *In particular, one may take*

$$\begin{array}{lll} c_p = (p/2)^p, & C_p = (32/p)^{p/2} & if \quad 0 < p < 2; \\ c_p = 1, & C_p = 4 & if \quad p = 2; \\ c_p = (2p)^{-p/2}, & C_p = [p^{p+1}/2(p-1)^{p-1}]^{p/2} & if \quad p > 2. \end{array}$$

The moment inequality for stochastic process play an important role in characterizing stochastic process theory because of apply to various fields of stochastic differential equations and stochastic functional differential equations theory (see [5]–[7], [9]–[12]).

Therefore, the moment inequalities for stochastic process have received much attention and some authors studied generalization, sharpness, applicability, and similar inequality of the moment inequality (see [1], [2], [4], [8]).

In particular, Cho [5], Mao [6], Kim [7]–[11] and Park et al. [12] used these moment inequalities to prove the existence and uniqueness theorem of stochastic differential equations and neutral stochastic functional differential equations solutions.

Motivated by [1, 3, 4, 8], in this paper, we investigated a new type moment inequality for stochastic integral will be introduced in Section 3, which includes the m-dimensional Brownian motion on complete probability space. In particular, we aimed to demonstrate the following key results: first, under the Itôs formula, we estimate the bounds of expectation of the stochastic integral for normal integration. Next, we prove some martingale inequality of the stochastic integrals. Finally, we derive the Burkholder-Davis-Gundy inequality type inequality.

2. Preliminary

This section first introduces some terms and definitions for convenience of deployment. Let X be an R^d-valued random variable. Then X induces a probability measure μ_X on the Borel measurable space (R^d, B^d) and μ_X is called the distribution of X. The expectation of X can be expressed as

$$EX = \int_{R^d} x \, d\mu_X(x).$$

More generally, if $g : R^d \to R^m$ is Borel measurable, we then have the following transformation formula

$$Eg(X) = \int_{R^d} g(x) \, d\mu_X(x).$$

For $p \in (0, \infty)$, let $L^p = L^p(\Omega; R^d)$ be the family of all R^d-valued random variable X such that $E|X|^p < \infty$. Moreover, the following Hölder inequality is very useful:

$$|E(X^T Y) \leqslant (E|X|^p)^{1/p} (E|Y|^q)^{1/q}$$

if $p > 1$, $1/p + 1/q = 1$, $X \in L^p$, $Y \in L^q$.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. A filtration is a family $\{\mathscr{F}_t\}_{t \geqslant t_0}$ of increasing sub-σ-algebra $\mathscr{F}$.

A family $\{X_t\}_{t \in I}$ of R^d-valued random variable is called a stochastic process with parameter set I and state space R^d.

A random variable $\tau : \Omega \to [0, \infty]$ (it may take the value ∞) is called and $\{\mathscr{F}_t\}$-stopping time (or simply, stopping time) if $\{\omega : \tau(\omega) \leqslant t\} \in \mathscr{F}_t$ for any $t \geqslant 0$.

An R^d-valued $\{\mathscr{F}_t\}$-adapted integrable process $\{M_t\}_{t \geqslant t_0}$ is called a martingale with respect to $\{\mathscr{F}_t\}$ (or simply, martingale) if

$$E(M_t | \mathscr{F}_s) = M_s \quad a.s.$$

for all $0 \leqslant s < t < \infty$. An R^d-valued $\{\mathscr{F}_t\}$-adapted integrable process $\{M_t\}_{t \geqslant t_0}$ is called a submartingale with respect to $\{\mathscr{F}_t\}$ if

$$E(M_t | \mathscr{F}_s) \geqslant M_s \quad a.s.$$

for all $0 \leqslant s < t < \infty$.

To describe the Brownian motion mathematically it is natural to use the concept of a stochastic process $B_t(\omega)$, interpreted as the position of the pollen grain ω at time t. The following is a well-known mathematical definition of the Brownian motion.

DEFINITION 1. [6] Let $(\Omega, \mathscr{F}, P)$ be a probability space with a filtration $\{\mathscr{F}_t\}_{t \geqslant 0}$. A one dimensional Brownian motion is a real-valued continuous $\{\mathscr{F}_t\}$-adapted process $\{B_t\}_{t \geqslant 0}$ with the following properties:

(i) $B_0 = 0$ a. s.;

(ii) for $0 \leqslant s < t < \infty$, the increment $B_t - B_s$ is normally distributed with mean zero and variance $t - s$;

(iii) for $0 \leqslant s < t < \infty$, the increment $B_t - B_s$ is independent of $\{\mathscr{F}_s\}$.

A m-dimensional process $\{B_t = (B_t^1, B_t^2, \dots, B_t^m)\}_{t \geqslant 0}$ is called a m-dimensional Brownian motion if every $\{B_t^i\}$ is a one dimensional Brownian motion, and $\{B_t^1\}, \dots, \{B_t^m\}$ are independent.

Let $0 \leqslant a < b < \infty$. Denote by $\mathscr{M}^2([a,b];R)$ the space of all real valued measurable $\{\mathscr{F}_t\}$-adapted process $f = \{f(t)\}_{a \leqslant t \leqslant b}$ such that

$$||f||_{a,b}^2 = E \int_a^b |f(t)|^2 \, \mathrm{d}t < \infty.$$

Denote by $\mathscr{M}_0([a,b];R)$ the family of all real valued stochastic simple(or step) processes $g = \{g(t)\}_{a \leqslant t \leqslant b}$.

For a simple process g in $\mathscr{M}_0([a,b];R)$ and bounded random variable ξ_i , define

$$\int_a^b g(t)\, dB_t = \sum_{i=0}^{k-1} \xi_i (B_{t_i+1} - B_{t_i})$$

and call it the stochastic integral of g with respect to the Brownian motion $\{B_t\}$ or Itô integral. The Itô integral of stochastic process f in $\mathscr{M}^2([a,b];R)$ with respect to $\{B_t\}$ is defined by

$$\int_a^b f(t)\, dB_t = \lim_{n \to \infty} \int_a^b g_n(t)\, dB_t \quad in \quad L^2(\Omega;R),$$

where $\{g_n\}$ is a sequence of simple processes such that

$$\lim_{n \to \infty} E \int_a^b |f(t) - g_n(t)|^2 \, dt = 0.$$

The stochastic integral has many nice properties. The following theorem is part of such stochastic integral properties:

THEOREM 4. [6] *Let $f, g \in \mathscr{M}^2([a,b];R)$ and let α, β be two real numbers. Then*
(i) $E \int_a^b f(t)\, dB_t = 0$;
(ii) $E|\int_a^b f(t)\, dB_t|^2 = E \int_a^b |f(t)|^2 \, dt$;
(iii) $\int_a^b [\alpha f(t) + \beta g(t)]\, dB_t|^2 = \alpha \int_a^b f(t)\, dB_t + \beta \int_a^b g(t)\, dB_t$.

Let $f \in \mathscr{M}^2([a,b];R)$ and let τ be an $\{\mathscr{F}_t\}$-stopping time such that $0 \leqslant \tau \leqslant T$. Then $\{I_{[[0,\tau]]}(t) f(t)\}_{0 \leqslant t \leqslant T} \in \mathscr{M}^2([a,b];R)$ clearly, and we define

$$\int_0^\tau f(s)\, dB_s = \int_0^T I_{[[0,\tau]]}(s) f(s)\, dB_s.$$

We shall now extend the Itô stochastic integral to the multi-dimensional case. Let $\{B_t = (B_t^1, B_t^2, \dots, B_t^m)^T\}_{t \geqslant 0}$ is called a m-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathscr{F}, P)$ adapted to the filtration $\{\mathscr{F}_t\}$. Denote by $\mathscr{M}^2([a,b];R^{d \times m})$ denote the family of all $d \times m$-matrix-valued measurable $\{\mathscr{F}_t\}$-adapted process $f = \{(f_{ij}(t))_{d \times m}\}_{0 \leqslant t \leqslant T}$ such that

$$E \int_0^T |f(t)|^2 \, \mathrm{d}t < \infty.$$

Here, if A is a vector or a matrix, its transpose is denoted by A^T; $|A|$ will denote the trace norm for matrix A, that is $|A| = \sqrt{\text{trace}(A^T A)}$.

And let $\mathscr{L}^2(R_+;R^{d\times m})$ denote the family of all $d\times m$-matrix-valued measurable $\{\mathscr{F}\}$-adapted processes $f \in \mathscr{L}^2(R_+;R^{d\times m})$ such that $\int_0^T |f(t)|^2\,dt < \infty$.

And let $\mathscr{M}^2(R_+;R^{d\times m})$ denote the family of all processes $\{f(t)\}_{t\geqslant 0}$ such that $E\int_0^T |f(t)|^2\,dt < \infty$.

A one-dimensional Itô process is a continuous process $x(t)$ on $t\geqslant 0$ of the form

$$x(t) = x(0) + \int_0^t f(s)\,\mathrm{d}s + \int_0^t g(s)\,\mathrm{d}B_s,$$

where $f \in \mathscr{L}^1(R_+;R)$ and $g \in \mathscr{L}^2(R_+;R)$. We shall say that $x(t)$ has stochastic differential $dx(t)$ on $t\geqslant 0$ given by

$$\mathrm{d}x(t) = f(t)\,dt + g(t)\,dB_t.$$

Let $C^{2,1}(R^d\times R_+;R)$ denote the family of all real-valued functions $V(x,t)$ defined $R^d\times R_+$ such that they are continuously twice differentiable in x and once in t. The following theorem is a stochastic version of the chain rule for the Itôs integrals, which is known as one-dimensional Itôs formula.

THEOREM 5. [6] (The one-dimensional Itôs formula) *Let $x(t)$ be an Itô process on $t\geqslant 0$ with the stochastic differential given by*

$$dx(t) = f(t)\,dt + g(t)\,dB_t,$$

where $f\in\mathscr{L}^1(R_+;R)$ and $g\in\mathscr{L}^2(R_+;R)$. And let $V\in C^{2,1}(R\times R_+;R)$. Then $V(x(t),t)$ is again an Itô process with the stochastic differential given by

$$\begin{aligned} dV(x(t),t) = {} & [V_t(x(t),t) + V_x(x(t),t)f(t) + \frac{1}{2}V_{xx}(x(t),t)g^2(t)]dt \\ & + V_x(x(t),t)g(t)\,dB_t \quad a.s. \end{aligned}$$

We shall now extend the 1-dimensional Itô stochastic formula to the multi-dimensional case. Let $\{B_t = (B_1(t), B_2(t), \ldots, (B_m(t))^T\}_{t\geqslant 0}$ is called a m-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathscr{F}, P)$ adapted to the filtration $\{\mathscr{F}_t\}$.

A d-dimensional Itô process is an R^d-value continuous process $x(t) = (x_1(t),\cdots, x_d(t))^T$ on $t\geqslant 0$ of the form

$$x(t) = x(0) + \int_0^t f(s)\,\mathrm{d}s + \int_0^t g(s)\,\mathrm{d}B(s),$$

where $f = (f_1,\cdots,f_d)^T \in \mathscr{L}^1(R_+;R^d)$ and $g = (g_{ij})_{d\times m} \in \mathscr{L}^2(R_+;R^{d\times m})$. We shall say that $x(t)$ has stochastic differential $dx(t)$ on $t\geqslant 0$ given by

$$\mathrm{d}x(t) = f(t)\,dt + g(t)\,dB(t).$$

The following theorem is a stochastic version of the chain rule for the Itôs integrals, which is known as multi-dimensional Itôs formula.

THEOREM 6. [6] (The multi-dimensional Itôs formula) *Let $x(t)$ be a d-dimensional Itô process on $t \geqslant 0$ with the stochastic differential given by*

$$dx(t) = f(t)\,dt + g(t)\,dB(t),$$

where $f = (f_1, \cdots, f_d)^T \in \mathscr{L}^1(R_+; R^d)$ and $g = (g_{ij})_{d\times m} \in \mathscr{L}^2(R_+; R^{d\times m})$. And let $V \in C^{2,1}(R^d \times R_+; R)$. Then $V(x(t),t)$ ia again an Itô process with the stochastic differential given by

$$\begin{aligned} dV(x(t),t) = {} & [V_t(x(t),t) + V_x(x(t),t)f(t) + \frac{1}{2} trace(g^T(t)V_{xx}(x(t),t)g(t))]dt \\ & + V_x(x(t),t)g(t)\,dB(t) \quad a.s. \end{aligned}$$

3. Gronwall type moment inequalities

In this section, we want to establish some stochastic moment inequalities in the stochastic process by applying the Itô formula and the Gronwall-type inequalities. For the convenience of the proofs of main results, we introduce the following well known inequality of the Gronwall-type.

LEMMA 1. [3] *Let $a(t), b(t)$ and $u(t)$ be continuous functions in $J = [\alpha, \beta]$, and let $a(t)$ be nondecreasing in J and $b(t)$ be nonnegative in J. Suppose that*

$$u(t) \leqslant a(t) + \int_\alpha^t b(s)u(s)\,ds,\ t \in J.$$

Then,

$$u(t) \leqslant a(t) \exp\left(\int_\alpha^t b(s)\,ds\right), \quad t \in J.$$

Now we shall apply Itô's formula to describe some very interesting moment inequalities for stochastic integrals as well as the exponential martingale inequality.

THEOREM 7. *If $p \geqslant 2, g \in \mathscr{M}^2([0,T]; R^{d\times m})$ such that*

$$E\int_0^T |g(s)|^p \,\mathrm{d}s < \infty,$$

then

$$E\left|\int_0^T g(s)\,\mathrm{d}B(s)\right|^p \leqslant (p-1)\left[E\int_0^T |g(s)|^p\,\mathrm{d}s\right] \exp\left(\frac{(p-1)(p-2)}{2}T\right). \tag{1}$$

Proof. For $p = 2$ the required result follows form Theorem 4 so we only need to show the theorem for the case of $p > 2$. For $0 \leqslant t \leqslant T$, set

$$x(t) = \int_0^t g(s)\,\mathrm{d}B(s).$$

By Itô's formula and Theorem 4,

$$E|x(t)|^p = \frac{p(p-1)}{2}E\int_0^t |x(s)|^{p-2}|g(s)|^2\,\mathrm{d}s + pE\int_0^t |x(s)|^{p-1}g(s)\,\mathrm{d}B(s)$$
$$= \frac{p(p-1)}{2}E\int_0^t |x(s)|^{p-2}|g(s)|^2\,\mathrm{d}s. \tag{2}$$

Using the Hölder's inequality, we have

$$E|x(t)|^p = \frac{p(p-1)}{2}E\int_0^t |x(s)|^{p-2}|g(s)|^2\,\mathrm{d}s$$
$$\leqslant \frac{p(p-1)}{2}\left(E\int_0^t |x(s)|^p\,\mathrm{d}s\right)^{\frac{p-2}{p}}\left(E\int_0^t |g(s)|^p\,\mathrm{d}s\right)^{\frac{2}{p}}.$$

By the Young's inequality implies the estimate

$$E|x(t)|^p \leqslant (p-1)\left(E\int_0^t |g(s)|^p\,\mathrm{d}s\right) + \frac{(p-1)(p-2)}{2}\left(E\int_0^t |x(s)|^p\,\mathrm{d}s\right). \tag{3}$$

By Lemma 1, this yields

$$E|x(t)|^p \leqslant (p-1)\left(E\int_0^t |g(s)|^p\,\mathrm{d}s\right)\exp\left(\frac{(p-1)(p-2)}{2}t\right),$$

and the required (1) follows by replacing t with T. □

THEOREM 8. *Under the same assumptions Theorem 7, we have following moment inequality for stochastic integrals*

$$E\left(\sup_{0\leqslant t\leqslant T}\left|\int_0^t g(s)\,\mathrm{d}B(s)\right|^p\right)$$
$$\leqslant \left(\frac{p^p}{(p-1)^{p-1}}\right)\left(E\int_0^t |g(s)|^p\,\mathrm{d}s\right)\exp\left(\frac{(p-1)(p-2)}{2}T\right).$$

Proof. Recall that the stochastic integral $\int_0^t g(s)\,\mathrm{d}B(s)$ is an R^d-valued continuous martingale. Hence, by Doob's martingale inequality (i.e. Theorem 1), we have

$$E\left(\sup_{0\leqslant t\leqslant T}\left|\int_0^t g(s)\,\mathrm{d}B(s)\right|^p\right) \leqslant \left(\frac{p}{(p-1)}\right)^p E\left|\int_0^T g(s)\,\mathrm{d}B(s)\right|^p.$$

In view of Theorem 7, we have obtain the desired result. □

In following theorem, using the Gronwall-type moment inequality, we introduce a new proof of some parts of the Burkholder-Davis-Gundy inequality.

THEOREM 9. *Let $g \in \mathscr{L}^2([0,T];R^{d\times m})$. Define, for $t \geqslant 0$,*

$$x(t) = \int_0^t g(s)\,dB(s) \quad and \quad A(t) = \int_0^t |g(s)|^2\,ds.$$

Then, we have

$$E|A(t)| \leqslant E\left(\sup_{0\leqslant s\leqslant t} |x(s)|^2\right) \leqslant 4E|A(t)| \tag{4}$$

for all $t \geqslant 0$.

Proof. We may assume without loss of generality that both $x(t)$ and $A(t)$ are bounded. Otherwise, for each integer $n \geqslant 1$, define the stopping time

$$\tau_n = \inf\{t \geqslant 0 : |x(t)| \vee A(t) \geqslant n\}.$$

If we can show (4) for the stooped process $x(t\wedge\tau_n)$ and $A(t\wedge\tau_n)$, then the general case follows upon letting $n \to \infty$. It follows from Theorem 4 that

$$\int_0^t |g(s)|^2\,\mathrm{d}s = E\left|\int_0^t g(s)\,\mathrm{d}B(s)\right|^2 \leqslant E\left(\sup_{0\leqslant s\leqslant t} |x(s)|^2\right)$$

which is the left-hand-side inequality of (4). On the other hand, by the Theorem 8 yields

$$E\left(\sup_{0\leqslant s\leqslant t} |x(s)|^2\right) \leqslant 4\,E\left|\int_0^t |g(s)|^2\,\mathrm{d}s\right|$$

which is the right-hand-side inequality of (4). This required the inequality. □

THEOREM 10. *Let $g \in \mathscr{L}^2([0,T];R^{d\times m})$. Define, for $t \geqslant 0$,*

$$x(t) = \int_0^t g(s)\,dB(s) \quad and \quad A(t) = \int_0^t |g(s)|^2\,ds.$$

Then for every $p > 2$, we have, for all $t \geqslant 0$

$$E\left(\sup_{0\leqslant s\leqslant t} |x(s)|^p\right) \leqslant D_p \exp\left(\frac{p^p(p-2)}{2(p-1)^{p-1}}t\right) E\int_0^t |g(s)|^p\,\mathrm{d}s, \tag{5}$$

where $D_p = p^p/(p-1)^{p-1}$.

Proof. We may assume without loss of generality that both $x(t)$ and $A(t)$ are bounded. Otherwise, for each integer $n \geqslant 1$, define the stopping time

$$\tau_n = \inf\{t \geqslant 0 : |x(t)| \vee A(t) \geqslant n\}.$$

If we can show (5) for the stooped process $x(t\wedge\tau_n)$ and $A(t\wedge\tau_n)$, then the general case follows upon letting $n \to \infty$. By Itô's formula and Theorem 4,

$$E|x(t)|^p = \frac{p(p-1)}{2} E\int_0^t |x(s)|^{p-2}|g(s)|^2\,\mathrm{d}s.$$

Using the Hölder's inequality, we have

$$E|x(t)|^p \leqslant \frac{p(p-1)}{2}\left(E\int_0^t |x(s)|^p\,\mathrm{d}s\right)^{\frac{p-2}{p}}\left(E\int_0^t |g(s)|^p\,\mathrm{d}s\right)^{\frac{2}{p}}.$$

It follows from that

$$E|x(t)|^p \leqslant (p-1)\left(E\int_0^t |g(s)|^p\,\mathrm{d}s\right)+\frac{(p-1)(p-2)}{2}\left(E\int_0^t |x(s)|^p\,\mathrm{d}s\right), \tag{6}$$

where the Young's inequality has been used. On the other hand, by the Doob's martingale inequality yields

$$E\left(\sup_{0\leqslant s\leqslant t}|x(s)|^p\right)\leqslant\left(\frac{p}{p-1}\right)^p E|x(t)|^p.$$

Substituting this into inequality (6) yields

$$E\left(\sup_{0\leqslant s\leqslant t}|x(s)|^p\right)\leqslant\left(\frac{p^p}{(p-1)^{p-1}}\right)E\int_0^t |g(s)|^p\,\mathrm{d}s+\left(\frac{p^p(p-2)}{2(p-1)^{p-1}}\right)E\int_0^t |x(s)|^p\,\mathrm{d}s.$$

This implies

$$\begin{aligned}&E\left(\sup_{0\leqslant s\leqslant t}|x(s)|^p\right)\\ &\leqslant\left(\frac{p^p}{(p-1)^{p-1}}\right)E\int_0^t |g(s)|^p\,\mathrm{d}s+\left(\frac{p^p(p-2)}{2(p-1)^{p-1}}\right)\int_0^t E\sup_{0\leqslant r\leqslant s}|x(r)|^p\,\mathrm{d}s.\end{aligned}$$

By Lemma 1, this yields

$$E\left(\sup_{0\leqslant s\leqslant t}|x(s)|^p\right)\leqslant\frac{p^p}{(p-1)^{p-1}}\exp\left(\frac{p^p(p-2)}{2(p-1)^{p-1}}t\right)E\int_0^t |g(s)|^p\,\mathrm{d}s.$$

This required the inequality. The proof is now complete. □

In following theorem, we induce a inverse inequality of some parts of the Burkholder-Davis-Gundy inequality.

THEOREM 11. *Let $g\in\mathscr{L}^2([0,T];R^{d\times m})$. Define, for $t\geqslant 0$,*

$$x(t)=\int_0^t g(s)\,dB(s)\quad\text{and}\quad A(t)=\int_0^t |g(s)|^2\,ds.$$

Then for every $p>2$ and $2(p-1)^{p-1}<p^p(p-2)$, we have

$$E\left(\sup_{0\leqslant s\leqslant t}|x(s)|^p\right)>\widetilde{D}_p E|A(t)|^{\frac{p}{2}}, \tag{7}$$

where $\widetilde{D}_p=2p^p/[2(p-1)^{p-1}-p^p(p-2)]$.

Proof. We may assume without loss of generality that both $x(t)$ and $A(t)$ are bounded. Otherwise, for each integer $n \geqslant 1$, define the stopping time

$$\tau_n = \inf\{t \geqslant 0 : |x(t)| \vee A(t) \geqslant n\}.$$

If we can show (7) for the stooped process $x(t \wedge \tau_n)$ and $A(t \wedge \tau_n)$, then the general case follows upon letting $n \to \infty$. Besides, for convenience, we set $x^*(t) = \sup_{0 \leqslant s \leqslant t} |x(s)|$. It follows form (2),

$$\begin{aligned} E|x(t)|^p &= \frac{p(p-1)}{2} E\left[|x^*(t)|^{p-2} A(t)\right] \\ &\leqslant \frac{p(p-1)}{2} \left[E|x^*(t)|^p\right]^{\frac{p-2}{p}} \left[|A(t)|^{\frac{p}{2}}\right]^{\frac{2}{p}}, \end{aligned}$$

where the Hölder's inequality has been used. It follows from that

$$E|x(t)|^p \leqslant \frac{(p-1)(p-2)}{2} E|x^*(t)|^p + (p-1)E|A(t)|^{\frac{p}{2}}, \tag{8}$$

where the Young's inequality has been used. On the other hand, by the Doob's martingale inequality yields

$$E|x^*(t)|^p \leqslant \left(\frac{p}{p-1}\right)^p E|x(t)|^p.$$

Substituting this into inequality (8) yields

$$E|x^*(t)|^p \leqslant \frac{p^p(p-2)}{2(p-1)^{p-1}} E|x^*(t)|^p + \frac{p^p}{(p-1)^{p-1}} E|A(t)|^{\frac{p}{2}}.$$

This implies for $2(p-1)^{p-1} < p^p(p-2)$,

$$E|x^*(t)|^p > \frac{2p^p}{2(p-1)^{p-1} - p^p(p-2)} E|A(t)|^{\frac{p}{2}}.$$

This required the inequality. The proof is now complete. □

REMARK. In the Theorem 7 and 8, we established a Gronwall type moment inequality of the stochastic process by applying the Itô formula. In the Theorem 9, we gave a new proofs of a parts of the Burkholder-Davis-Gundy inequality. Also in the Theorem 10, we gave a new moment inequality of the stochastic process. Furthermore, in the Theorem 10, we established a inverse moment inequality of a parts of the Burkholder-Davis-Gundy inequality.

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