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P–ADIC WEAK CENTRAL MORREY SPACES ON DIFFERENTIAL FORMS

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Abstract. In this article, the theory of differential forms on $\mathbb{R}^n$ was extended to the filed $\mathbb{Q}_p^n$ of p-adic numbers. The imbedding inequalities for differential forms were derived on $\mathbb{Q}_p^n$. Then, we show the definitions of p-aidc weak central Morrey spaces and p-adic λ-central BMO spaces on differential forms. The boundedness of Hardy operator and its adjoint operator were given in the new space. Finally, we give the characterization of the two operators in p-adic λ-central BMO spaces by using imbedding inequalities on differential forms.

1. Introduction

For a fixed prime p, a nonzero rational number x can be represented in the form $x = p^{\gamma}m/n$, where p,m and n are coprime to each other and $\gamma = \gamma(x) \in \mathbb{Z}$. The norm is defined as $|x|_p = p^{-\gamma}$. For $x = 0$ we have $|0|_p = 0$. The p-adic valuation $|\cdot|_p$ of $\mathbb{Q}$ satisfies all the conditions of real norm together with so called strong triangular inequality,

$$|x+y|_p \leqslant \max\{|x|_p, |y|_p\}. \tag{1}$$

Furthermore, if $|x|_p \neq |y|_p$, then (1) takes the form:

$$|x \pm y|_p = \max\{|x|_p, |y|_p\}.$$

We denote the field of p-adic numbers by $\mathbb{Q}_p$, and $\mathbb{Q}_p$ is the completion of the field of rational number $\mathbb{Q}$ with respect to ultrametric p-adic norm $|\cdot|_p$. From the standard p-adic analysis [6], we know that any p-adic number $x \in \mathbb{Q}_p \setminus \{0\}$ can also be represented in the canonical form as:

$$x = p^{\gamma} \sum_{j=0}^{\infty} a_j p^j, \tag{2}$$

where $a_j, \gamma \in \mathbb{Z}$, $a_0 \neq 0 \leqslant a_j < p$. The series (2) converges in p-adic norm, because $|p^{\gamma} a_j p^j|_p = p^{-\gamma-j}$ for $a_j \neq 0$.

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The space $\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ consists of points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where $x_k \in \mathbb{Q}_p$, $k = 1, 2, \ldots, n$. The p-adic norm defined on higher dimensional space $\mathbb{Q}_p^n$ as

$$|\mathbf{x}|_p = \max_{1 \leqslant k \leqslant n} |x_k|_p. \tag{3}$$

The symbols $B_\gamma(\mathbf{a})$ and $S_\gamma(\mathbf{a})$ represent, respectively, the ball and the sphere with center at $\mathbf{a} \in \mathbb{Q}_p^n$ and radius p^γ, defined by

$$B_\gamma(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p \leqslant p^\gamma\}, \quad S_\gamma(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p = p^\gamma\}. \tag{4}$$

It is clear that $S_\gamma(\mathbf{a}) = B_\gamma(\mathbf{a}) \setminus B_{\gamma-1}(\mathbf{a})$, and

$$B_\gamma(\mathbf{a}) = \bigcup_{k \leqslant \gamma} S_k(\mathbf{a}).$$

We set $B_\gamma(\mathbf{0}) = B_\gamma$ and $S_\gamma(\mathbf{0}) = S_\gamma$. Also, for each $\mathbf{a}_0 \in \mathbb{Q}_p^n$, $\mathbf{a}_0 + B_\gamma = B_\gamma(\mathbf{a}_0)$ and $\mathbf{a}_0 + S_\gamma = S_\gamma(\mathbf{a}_0)$.

The locally compact commutative group under addition of $\mathbb{Q}_p^n$ makes sure the existence of additive positive Haar measure $d\mathbf{x}$ on $\mathbb{Q}_p^n$. It is unique up to a positive constant factor and is translation invariant. We normalize the measure $d\mathbf{x}$ by the equality

$$\int_{B_0(\mathbf{0})} d\mathbf{x} = |B_0(\mathbf{0})|_H = 1,$$

where $|B|_H$ indicates the Haar measure of a subset B of $\mathbb{Q}_p^n$, which is measurable. Also, an easy calculation shows $|B_\gamma(\mathbf{a})|_H = p^{n\gamma}$, $|S_\gamma(\mathbf{a})|_H = p^{n\gamma}(1 - p^{-n})$, for any $\mathbf{a} \in \mathbb{Q}_p^n$.

The p-adic analysis is a key tool to describe Kohlrausch-Williams-Watts law, the power decay law and the logarithmic decay law, see [1]. It has also cemented its role in p-aidc pseudo-differential equations and stochastic process, see [10].

2. Differential forms in $\mathbb{Q}_p^n$

In this section, the extension of differential forms in $\mathbb{Q}_p^n$ was introduced and the imbedding inequalities for differential forms in $\mathbb{Q}_p^n$ were derived.

The spaces of all l-forms in $\mathbb{Q}_p^n$ is denoted by $\wedge^l(\mathbb{Q}_p^n)$, spanned by exterior products $e_I = e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_l}$, for all ordered l-tuples $I = (i_1, i_2, \cdots, i_l)$, $1 \leqslant i_1 < \cdots < i_l \leqslant n$. In particular, $\wedge^1(\mathbb{Q}_p^n)$ is the dual of $\mathbb{Q}_p^n$. The basis of dual space is denoted by $\{e_1, e_2, \cdots, e_n\}$. The direct sum $\wedge(\mathbb{Q}_p^n) = \oplus_{l=0}^n \wedge^l (\mathbb{Q}_p^n)$, where $\wedge^0(\mathbb{Q}_p^n) = \mathbb{Q}_p^n$, is a graded algebra with respect to the exterior product.

If u is a differential l-form in $\mathbb{Q}_p^n$ and it is differentiable, we see the derivative mapping $u'(\mathbf{x}) : \mathbb{Q}_p^n \to \wedge^l(\mathbb{Q}_p^n)$. Then $u'(\mathbf{x})\theta_i$ is an l-antisymmetric metric function on $\mathbb{Q}_p^n \times \cdots \times \in \mathbb{Q}_p^n$, where $\theta_i \in \mathbb{Q}_p^n$ for $i = 1, 2 \cdots, l+1$. The exterior differential $du(\mathbf{x})$ is an $(l+1)$-form which is defined by

$$\begin{aligned} du(\mathbf{x})(\theta_1, \theta_2, \ldots, \theta_{l+1}) :&= du(\mathbf{x}; \theta_1, \theta_2, \ldots, \theta_{l+1}) \\ &= \sum_{i=1}^{l+1} (-1)^{i-1} [u'(\mathbf{x})\theta_i](\theta_1, \ldots, \hat{\theta}_i, \ldots, \theta_{l+1}), \end{aligned} \tag{5}$$

where $\hat{\theta}_i$ denotes that θ_i is absent from $(\theta_1,\ldots,\hat{\theta}_i,\ldots,\theta_{l+1})$.

The coordinate functions $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n$ in $\mathbb{Q}_p^n$ are considered to be differential forms of degree 0. The 1-forms $d\mathbf{x}_1,d\mathbf{x}_2,\ldots,d\mathbf{x}_n$ are constant functions from $\mathbb{Q}_p^n$ into $\wedge^l(\mathbb{Q}_p^n)$. The constant value of $d\mathbf{x}_i$ is simply e_i, $i=1,2,\ldots,n$. Then l-form u from $\mathbb{Q}_p^n$ to $\wedge^l(\mathbb{Q}_p^n)$ and exterior differential du can be written as

$$u(\mathbf{x})=\sum_{1\leqslant i_1<\cdots<i_l\leqslant n} u_{i_1i_2\cdots i_l}(\mathbf{x})d\mathbf{x}_{i_1}\wedge\cdots\wedge d\mathbf{x}_{i_l}$$

and

$$du(\mathbf{x})=\sum_{k=1}^{n}\sum_{1\leqslant i_1\cdots<i_l\leqslant n}\frac{\partial u_{i_1\cdots i_l}(\mathbf{x})}{\partial \mathbf{x}_k}d\mathbf{x}_k\wedge d\mathbf{x}_{i_1}\wedge\cdots\wedge d\mathbf{x}_{i_l},$$

respectively. Moreover, we denote by $C^\infty(\wedge^l,\mathbb{Q}_p^n)$ the space of differential l-forms on $\mathbb{Q}_p^n$ for all l-tuples $I=\{i_1i_2\cdots i_l\}$ whose coefficient functions u_I are infinitely differentiable functions.

We shall denote by $L^q(\wedge^l,\mathbb{Q}_p^n)$ the space of differential l-forms on $\mathbb{Q}_p^n$ for all l-tuples I and with finite norm

$$\|u\|_{L^q(\wedge^l,\mathbb{Q}_p^n)}=\Big(\int_{\mathbb{Q}_p^n}\Big(\sum_I|u_I(\mathbf{y})|^2\Big)^{\frac{q}{2}}d\mathbf{y}\Big)^{\frac{1}{q}}.$$

A differential l-form u is called a closed form if $du=0$ in $\mathbb{Q}_p^n$. Similarly, a differential $(l+1)$-form v is called a coclosed form if $d^\star v=0$. From the Poincaré lemma, $ddu=0$, we know that du is a closed form. The operator $\star:\wedge^l(\mathbb{Q}_p^n)\to\wedge^{n-l}(\mathbb{Q}_p^n)$ is the Hodge-star operator which is an isometric isomorphism. The Hodge codifferential operator $d^\star:\wedge^{l+1}(\mathbb{Q}_p^n)\to\wedge^l(\mathbb{Q}_p^n)$, the formal adjoint of d, is defined by $d^\star=(-1)^{nl+1}\star d\star$, see [12] for more introduction. By $A\lesssim B$ we mean that $A\leqslant CB$ with some positive constant C independent of appropriate quantities. If $A\lesssim B$ and $B\lesssim A$, we write $A\sim B$, then we say that A and B are equivalent. Now, we prove the basic estimates for differential forms in $\mathbb{Q}_p^n$.

THEOREM 1. *Let B_γ be a ball of radius p^γ centered at $\mathbf{0}\in\mathbb{Q}_p^n$. For each $\mathbf{y}\in B_\gamma$ there exists a linear operator $\mathscr{K}_{\mathbf{y}}:C^\infty(B_\gamma,\wedge^l)\to C^\infty(B_\gamma,\wedge^{l-1})$ defined by*

$$(\mathscr{K}_{\mathbf{y}}u)(\mathbf{x};\theta_1,\ldots,\theta_{l-1})=\int_{B_0}|t|_p^{l-1}u((1-|t|_p)\mathbf{x}+|t|_p\mathbf{y};\mathbf{x}-\mathbf{y},\theta_1,\ldots,\theta_{l-1})dt \quad (6)$$

and the following decomposition

$$u=d(\mathscr{K}_{\mathbf{y}}u)+\mathscr{K}_{\mathbf{y}}(du), \quad (7)$$

where $\mathbf{y}$ be any point in $B_\gamma\subset\mathbb{Q}_p^n$.

Proof. From (5), we have

$$du(\mathbf{x};\theta_1,\theta_2,\ldots,\theta_{l+1})=[u'(\mathbf{x})\theta_0](\theta_1,\ldots,\theta_l)+\sum_{i=1}^{l+1}(-1)^{i-1}[u'(\mathbf{x})\theta_i](\theta_1,\ldots,\hat{\theta}_i,\ldots,\theta_l)$$

for all $\mathbf{x} \in B_\gamma$ and $\theta_i \in \mathbb{Q}_p^n$, where $i = 0, 1, 2, \ldots, l$.

Then

$$\begin{aligned} &du((1-|t|_p)\mathbf{x}+|t|_p\mathbf{y};\mathbf{x}-\mathbf{y},\theta_1,\theta_2,\ldots,\theta_l) \\ &= [u'((1-|t|_p)\mathbf{x}+|t|_p\mathbf{y})(\mathbf{x}-\mathbf{y})](\theta_1,\ldots,\theta_l) \\ &\quad + \sum_{i=1}^{l}(-1)^{i-1}[u'((1-|t|_p)\mathbf{x}+|t|_p\mathbf{y})\theta_i](\mathbf{x}-\mathbf{y},\theta_1,\ldots,\hat{\theta}_i,\ldots,\theta_l). \end{aligned}$$

By using (6), we obtain

$$\begin{aligned} &(\mathscr{K}_{\mathbf{y}}du)(\mathbf{x};\theta_1,\ldots,\theta_l) \\ &= \int_{B_0} |t|_p^l [u'((1-|t|_p)\mathbf{x}+|t|_p\mathbf{y})(\mathbf{x}-\mathbf{y})](\theta_1,\ldots,\theta_l)dt \\ &\quad + \sum_{i=1}^{l}(-1)^{i-1}\int_{B_0} |t|_p^l [u'((1-|t|_p)\mathbf{x}+|t|_p\mathbf{y})\theta_i](\mathbf{x}-\mathbf{y},\theta_1,\ldots,\hat{\theta}_i,\ldots,\theta_l)dt. \end{aligned} \tag{8}$$

Similarly, for $\mathscr{K}_{\mathbf{y}}u$ is an $l-1$ form, we have

$$\begin{aligned} &[(\mathscr{K}_{\mathbf{y}}u)'(x)\theta_i](\theta_1,\ldots,\hat{\theta}_i,\ldots,\theta_{l-1}) \\ &\qquad = \int_{B_0} |t|_p^l [u'((1-|t|_p)\mathbf{x}+|t|_p\mathbf{y})\theta_i](\mathbf{x}-\mathbf{y},\theta_1,\ldots,\hat{\theta}_i,\ldots,\theta_l)dt \\ &\qquad\quad + \int_{B_0} |t|_p^{l-1} u((1-|t|_p)\mathbf{x}+|t|_p\mathbf{y};\theta_i,\theta_1,\ldots,\hat{\theta}_i,\ldots,\theta_{l-1})dt. \end{aligned}$$

For $\mathscr{K}_{\mathbf{y}} : C^\infty(B_\gamma, \wedge^l) \to C^\infty(B_\gamma, \wedge^{l-1})$ and $d : \wedge^l(\mathbb{Q}_p^n) \to \wedge^{l+1}(\mathbb{Q}_p^n)$, we get $d(\mathscr{K}_{\mathbf{y}}u)$ is an l-form given by

$$(d\mathscr{K}_{\mathbf{y}}u)(\mathbf{x};\theta_1,\ldots,\theta_l) = \sum_{i=1}^{l}(-1)^{i-1}[(\mathscr{K}_y u)'(\mathbf{x})\theta_i](\theta_1,\ldots,\hat{\theta}_i,\ldots,\theta_l).$$

Combining these two facts, we deduce

$$\begin{aligned} &(d\mathscr{K}_{\mathbf{y}}u)(\mathbf{x};\theta_1,\ldots,\theta_l) \\ &= \sum_{i=1}^{l}(-1)^{i-1}\int_{B_0} |t|_p^l [u'((1-|t|_p)\mathbf{x}+|t|_p\mathbf{y})\theta_i](\mathbf{x}-\mathbf{y},\theta_1,\ldots,\hat{\theta}_i,\ldots,\theta_l)dt \\ &\quad + \sum_{i=1}^{l}(-1)^{i-1}\int_{B_0} |t|_p^{l-1} u((1-|t|_p)\mathbf{x}+|t|_p\mathbf{y};\theta_i,\theta_1\ldots,\hat{\theta}_i,\ldots,\theta_l)dt. \end{aligned} \tag{9}$$

It is easy to see that the second part of (8) is counteracted by the first part of (9).

Therefore,

$$
\begin{aligned}
&(d\mathscr{K}_{\mathbf{y}}u+\mathscr{K}_{\mathbf{y}}du)(\mathbf{x};\theta_1,\theta_2,\ldots,\theta_l)\\
=&\int_{B_0}|t|_p^l[u'((1-|t|_p)\mathbf{x}+|t|_p\mathbf{y})(\mathbf{x}-\mathbf{y})](\theta_1,\ldots,\theta_l)dt\\
&+l\int_{B_0}|t|_p^{l-1}u((1-|t|_p)\mathbf{x}+|t|_p\mathbf{y};\theta_1,\ldots,\theta_l)\\
=&\int_{B_0}\frac{d}{dt}[|t|_p^l u((1-|t|_p)\mathbf{x}+|t|_p\mathbf{y};\theta_1,\ldots,\theta_l)]dt\\
=&\,u(\mathbf{x};\theta_1,\ldots,\theta_l).
\end{aligned}
\tag{10}
$$

The proof of Theorem 1 has been completed. □

The following imbedding inequalities are derived as follow.

THEOREM 2. *Let* $\mathscr{T}:C^\infty(B_\gamma,\wedge^l)\to C^\infty(B_\gamma,\wedge^{l-1})$ *is a homotopy operator for* $l=3,4\ldots,n,\ 1\leqslant q<\infty$. *For* u *be a differential* l*-form, we have*

$$\mathscr{T}u=\int_{B_\gamma}\varphi(\mathbf{y})\mathscr{K}_{\mathbf{y}}ud\mathbf{y},$$

where $\varphi\in C_0^\infty(B_\gamma)$ *satisfies* $\int_{B_\gamma}\varphi(\mathbf{y})d\mathbf{y}=1$. *Then* u *has the following decomposition:*

$$u=d(\mathscr{T}u)+\mathscr{T}(du). \tag{11}$$

and the following inequality

$$\|\mathscr{T}u\|_{L^q(\wedge^l,B_\gamma)}\lesssim|B_\gamma|_H^{1+1/n}\|\varphi\|_{L^\infty}\|u\|_{L^q(\wedge^l,B_\gamma)}. \tag{12}$$

Proof. For $\theta_1,\ldots,\theta_{l-1}\in\mathbb{Q}_p^n$, we see

$$\mathscr{T}u(\mathbf{x};\theta_1,\ldots,\theta_{l-1})=\int_{B_0}|t|_p^{l-1}\int_{B_\gamma}\varphi(\mathbf{y})u((1-|t|_p)\mathbf{x}+|t|_p\mathbf{y};\mathbf{x}-\mathbf{y},\theta_1,\ldots,\theta_{l-1})d\mathbf{y}dt.$$

Let $\mathbf{a}=(1-|t|_p)\mathbf{x}+|t|_p\mathbf{y}$ and $\mathbf{b}=\mathbf{x}-\mathbf{a}$. We consider a mapping $\Gamma:\mathbb{Q}_p^n\times\mathbb{Q}_p^n\to\mathbb{R}^n$, whose valuation is defined by

$$|\Gamma(\mathbf{a},\mathbf{b})|=|\mathbf{b}|_{f_p}\left|\int_{B_0}|t|_p^{l-3}\varphi\Big(\mathbf{a}-\Big(\frac{1-|t|_p}{|t|_p}\Big)\mathbf{b}\Big)dt\right|,$$

where the mapping f_p: $\mathbb{Q}_p^n\to\mathbb{R}^n$ is given by $f_p(\mathbf{z})=|\mathbf{z}|_{f_p}:=(|z_1|_p,|z_2|_p,\ldots,|z_n|_p)$ for any $\mathbf{z}\in\mathbb{Q}_p^n$. By using Hölder inequality, we estimate

$$
\begin{aligned}
|\Gamma(\mathbf{a},\mathbf{b})|&\leqslant|f_p(\mathbf{b})|\|\varphi\|_{L^\infty}\int_{B_0}|t|_p^{l-3}dt\\
&\leqslant\frac{|t|_p|f_p(\mathbf{x}-\mathbf{y})|}{l-2}\|\varphi\|_{L^\infty}\\
&\lesssim|B_\gamma|_H^{1/n}\|\varphi\|_{L^\infty},
\end{aligned}
$$

and

$$|\mathscr{T}u(\mathbf{x})| = \left| \int_{B_\gamma} u(\mathbf{a};\Gamma(\mathbf{a},\mathbf{b}),\theta_1,\ldots,\theta_{l-1})d\mathbf{y} \right| \lesssim |B_\gamma|_H^{1/n} \|\varphi\|_{L^\infty} \int_{B_\gamma} |u(\mathbf{y})|d\mathbf{y}.$$

Thus, we conclude that

$$\|\mathscr{T}u\|_{L^q(\wedge^l,B_\gamma)} \lesssim |B_\gamma|_H^{1+1/n} \|\varphi\|_{L^\infty} \|u\|_{L^q(\wedge^l,B_\gamma)}. \tag{13}$$

The proof of Theorem 2 has been completed. □

COROLLARY 1. *Let $u \in L^q_{loc}(\wedge^l, B_\gamma)$ is a differential l-form and $du \in L^q_{loc}(\wedge^{l+1}, B_\gamma)$, $1 \leqslant q < \infty$, $l = 3,4,\ldots,n$. Then, we have the following result*

$$\|d(\mathscr{T}u)\|_{L^q_{loc}(\wedge^l,B_\gamma)} \leqslant \|u\|_{L^q_{loc}(\wedge^l,B_\gamma)} + |B_\gamma|_H^{1+1/n} \|\varphi\|_{L^\infty} \|du\|_{L^q_{loc}(\wedge^{l+1},B_\gamma)}$$

for each B_γ is a ball in $\mathbb{Q}_p^n$, and $\varphi \in C_0^\infty(B_\gamma)$ satisfies $\int_{B_\gamma} \varphi(\mathbf{y})d\mathbf{y} = 1$.

Similar to the integral average in function spaces, we denote the integral average of u over $B_\gamma \subset \mathbb{Q}_p^n$ by u_{B_γ} which satisfies

$$u_{B_\gamma} = \begin{cases} \frac{1}{|B_\gamma|_H} \int_{B_\gamma} u(\mathbf{y})d\mathbf{y}, & l = 0 \\ d(\mathscr{T}u), & l = 1,2\cdots,n \end{cases}$$

and

$$u = \mathscr{T}d(u) + u_{B_\gamma}. \tag{14}$$

From (14), we can obtain the following result in $\mathbb{Q}_p^n$.

THEOREM 3. *Let $u \in L^q(\wedge^l, \mathbb{Q}_p^n)$ be a differential l-form and $du \in L^q(\wedge^{l+1}, \mathbb{Q}_p^n)$. Then $u - u_\gamma$ satisfies the following inequality*

$$\|u - u_\gamma\|_{L^q(\wedge^l,B_\gamma)} \leqslant |B_\gamma|_H^{1+1/n} \|\varphi\|_{L^\infty} \|du\|_{L^q(\wedge^{l+1},B_\gamma)}$$

for $\varphi \in C_0^\infty(B_\gamma)$ satisfies $\int_{B_\gamma} \varphi(\mathbf{y})d\mathbf{y} = 1$ and B_γ is a ball in $\mathbb{Q}_p^n$.

Next, we show the definitions of p-adic Morrey spaces and p-adic central Morrey spaces on differential forms. The p-adic weak Lebesgue space $WL^q(\wedge^l, \mathbb{Q}_p^n)$ for which the set of all differential l-forms u is defined as:

$$\|u\|_{WL^q(\wedge^l,\mathbb{Q}_p^n)} := \sup_{\lambda>0} \lambda \left| \{\mathbf{x} \in \mathbb{Q}_p^n : |u(\mathbf{x})| > \lambda\} \right|^{1/q} < \infty. \tag{15}$$

More details of p-adic weak Lebesgue space were introduced in [14].

DEFINITION 1. Let u be a differential l-form on $\mathbb{Q}_p^n$. For $1 \leqslant q < \infty$ and $-1/q \leqslant \lambda < \infty$, we denote the p-adic Morrey space on differential forms by $L^{q,\lambda}(\wedge^l, \mathbb{Q}_p^n)$ with norm

$$\|u\|_{L^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)} = \sup_{\mathbf{a}\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{a})|_p^{1/q+\lambda}} \|u\|_{L^q(\wedge^l,B_\gamma(\mathbf{a}))} < \infty.$$

It is easy to see that $L^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)$ coincides with $L^q(\wedge^l,\mathbb{Q}_p^n)$ when $\lambda=-1/q$.

DEFINITION 2. Let $1\leqslant q<\infty$ and $-1/q\leqslant\lambda<\infty$. The p-adic central Morrey space on differential forms $\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)$ is defined as

$$\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n):=\sup_{\gamma\in\mathbb{Z}}\frac{1}{|B_\gamma|_H^{1/q+\lambda}}\|u\|_{L^q(\wedge^l,B_\gamma)}, \tag{16}$$

where $B_\gamma=B_\gamma(\mathbf{0})$ is a ball in $\mathbb{Q}_p^n$ centered at $\mathbf{0}$ and radius of p^γ.

It is clear that $L^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)\subset\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)$. When $\lambda<-1/q$, the space $\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)$ reduces to $\{0\}$.

DEFINITION 3. Let $1\leqslant q<\infty$ and $-1/q\leqslant\lambda<\infty$. The p-adic weak central Morrey space on differential forms $\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)$ is defined as

$$W\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n):=\{u:\|u\|_{W\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}<\infty\},$$

where

$$\|u\|_{W\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}=\sup_{\gamma\in\mathbb{Z}}\frac{1}{|B_\gamma|_H^{1/q+\lambda}}\|u\|_{WL^q(\wedge^l,B_\gamma)},$$

and $\|u\|_{WL^q(\wedge^l,B_\gamma)}$ is the local p-adic L^q-norm of $u(\mathbf{x})$ respect to (15).

Also, $\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)\subset W\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)$ for $1\leqslant q<\infty$ and $-1/q<\lambda<0$. The definition of $\mathrm{CMO}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)$ is as follows.

DEFINITION 4. For $1\leqslant q<\infty$, let u be a differential l-form on $\mathbb{Q}_p^n$, $l=0,1,\cdots,n$ and $\lambda\in(-\infty,n)$. The space $\mathrm{CMO}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)$ is defined by

$$\|u\|_{\mathrm{CMO}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}:=\sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}|u(\mathbf{x})-u_{B_\gamma}|^q d\mathbf{x}\right)^{1/q}<\infty. \tag{17}$$

REMARK 1. If $l=0$, the space of $\mathrm{CMO}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)$ coincides with $\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$ which intoduced in [20]. The formulas (16) and (17) yield that $\dot{B}^{q,\lambda}$ is a Banach space continuously included in $\mathrm{CMO}^{q,\lambda}$.

Next, we show a important property for differential forms in $\mathrm{CMO}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)$.

THEOREM 4. *Assume that $u\in L^q(\wedge^l,\mathbb{Q}_p^n)$ be a differential l-form and $du\in L^q(\wedge^{l+1},\mathbb{Q}_p^n)$, $1\leqslant q<\infty$. Then we have the following characterization:*

(i) *If $\|u\|_{\mathrm{CMO}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}=0$, then u is a closed form.*

(ii) *Let $\lambda'=\lambda-1-1/n$, then we have*

$$\|u\|_{\mathrm{CMO}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}\lesssim\|\varphi\|_{L^\infty}\|du\|_{\dot{B}^{q,\lambda'}(\wedge^{l+1},\mathbb{Q}_p^n)},$$

where $\varphi\in C_0^\infty(B_\gamma)$ satisfies $\int_{B_\gamma}\varphi(\mathbf{y})d\mathbf{y}=1$.

Proof. From (i), it is clear to see that $u = u_{B_\gamma}$. In fact, the definition of u_{B_γ} coincides with (14) and according to the Poincaré lemma

$$du_{B_\gamma} = du - d(\mathscr{T}du) = \mathscr{T}(ddu) + d\mathscr{T}(du) - d\mathscr{T}(du) = 0.$$

Then, both u and u_{B_γ} are closed form. Next, we show that

$$\begin{aligned}
\|u\|_{\mathrm{CMO}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)} &= \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |u(\mathbf{x}) - u_{B_\gamma}|^q d\mathbf{x}\right)^{1/q} \\
&= \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |\mathscr{T}du(\mathbf{x})|^q d\mathbf{x}\right)^{1/q} \\
&\lesssim \|\varphi\|_{L^\infty} \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma|_H^{1/q+\lambda}} |B_\gamma|_H^{1+1/n} \|du\|_{L^q(\wedge^{l+1},B_\gamma)} \\
&\lesssim \|\varphi\|_{L^\infty} \|du\|_{\dot{B}^{q,\lambda'}(\wedge^{l+1},\mathbb{Q}_p^n)}.
\end{aligned}$$

The proof of Theorem 4 has been completed. □

3. Estimates of fractional p-adic Hardy operator

For a locally integrable function f on $\mathbb{R}^+$, the one-dimensional Hardy operator is defined as:

$$Hf(x) = \frac{1}{x}\int_0^x f(y)dy, \qquad x > 0,$$

which satisfies the following integral inequality:

$$\|Hf\|_{L^q} \leqslant \frac{q}{q-1}\|f\|_{L^q}, \qquad 1 < q < \infty.$$

More details for Hardy operator were introduced in [21, 22]. For $f \in L_{loc}(\mathbb{Q}_p^n)$ and $0 < \alpha < n$, the p-adic fractional Hardy operator is defined as:

$$H_\alpha f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{n-\alpha}} \int_{|\mathbf{y}|_p \leqslant |\mathbf{x}|_p} f(\mathbf{y})d\mathbf{y}, \qquad \mathbf{x} \in \mathbb{Q}_p^n \setminus \{\mathbf{0}\}.$$

If $\alpha = 0$, the fractional p-adic Hardy operator is reduced to p-adic Hardy operator, see [13] for more detials. Now we give the fractional p-adic Hardy operator on differential forms:

$$\mathscr{H}_\alpha u(\mathbf{x}) := \frac{1}{|\mathbf{x}|_p^{n-\alpha}} \sum_I \int_{|\mathbf{y}|_p \leqslant |\mathbf{x}|_p} u_I(\mathbf{y})d\mathbf{y}d\mathbf{x}_I, \qquad \mathbf{x} \in \mathbb{Q}_p^n \setminus \{\mathbf{0}\}, \tag{18}$$

and its adjoint operator on differential forms:

$$\mathscr{H}_\alpha^* u(\mathbf{x}) = \sum_I \int_{|\mathbf{y}|_p > |\mathbf{x}|_p} \frac{u_I(\mathbf{y})}{|\mathbf{y}|_p^{n-\alpha}} d\mathbf{y}d\mathbf{x}_I, \qquad \mathbf{x} \in \mathbb{Q}_p^n \setminus \{\mathbf{0}\}, \tag{19}$$

where $u(\mathbf{x}) \in L_{loc}(\wedge^l, \mathbb{Q}_p^n)$ is a differential l-form and each u_I is a locally integrable function on $\mathbb{R}^+$. In this section, we give the boundedness of fractional p-adic Hardy operator in p-adic weak central Morrey spaces on differential forms.

THEOREM 5. *Let* $0<\alpha<n,\ 1\leqslant q<\infty$ *and* $-1/q\leqslant\lambda<n$. *If* $u\in L^q_{loc}(\wedge^l,\mathbb{Q}_p^n)$, *then*

$$\|\mathscr{H}_\alpha u(\mathbf{x})\|_{\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}\lesssim\|u\|_{\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}\sup_{\gamma\in\mathbb{Z}}\left(\frac{p^{\alpha q\gamma}(1-p^{-n})}{1-p^{-(n+\alpha q+n\lambda q)}}\right)^{1/q}$$

and

$$\|\mathscr{H}_\alpha u(\mathbf{x})\|_{W\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}\leqslant\|u\|_{\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}.$$

Proof. By using Fubini's theorem and the elementary inequality $|\sum_{i=1}^n s_i|^q\leqslant n^{q-1}\sum_{i=1}^n|s_i|^q$ for $q,n>0$, we deduce that

$$\begin{aligned}|\mathscr{H}_\alpha u(\mathbf{x})|&=\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\left|\sum_I\int_{|\mathbf{y}|_p\leqslant|\mathbf{x}|_p}u_I(\mathbf{y})d\mathbf{y}d\mathbf{x}_I\right|\\&\leqslant\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\left|\int_{|\mathbf{y}|_p\leqslant|\mathbf{x}|_p}\sum_I u_I(\mathbf{y})d\mathbf{x}_Id\mathbf{y}\right|\\&\leqslant\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leqslant|\mathbf{x}|_p}\left(\Big|\sum_I u_I(\mathbf{y})\Big|^2\right)^{1/2}d\mathbf{y}\\&\leqslant\binom{n}{l}\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leqslant|\mathbf{x}|_p}(\sum_I|u_I(\mathbf{y})|^2)^{1/2}d\mathbf{y},\end{aligned}$$

where n and l are fixed integers. In fact, we know

$$\dim(\wedge^l)=\binom{n}{l}$$

and for $l=0,1,2,\cdots,n$, we get

$$\dim(\wedge)=\sum_{l=0}^n\dim(\wedge^l)=\sum_{l=0}^n\binom{n}{l}=2^n.$$

Hence, by using Hölder's inequality, we get

$$\begin{aligned}|\mathscr{H}_\alpha u(\mathbf{x})|&=\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\left|\sum_I\int_{|\mathbf{y}|_p\leqslant|\mathbf{x}|_p}u_I(\mathbf{y})d\mathbf{y}d\mathbf{x}_I\right|\\&\lesssim\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{B(\mathbf{0},|\mathbf{x}|_p)}(\sum_I|u_I(\mathbf{y})|^2)^{1/2}d\mathbf{y}\\&\lesssim\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\left(\int_{B(\mathbf{0},|\mathbf{x}|_p)}(\sum_I|u_I(\mathbf{y})|^2)^{q/2}d\mathbf{y}\right)^{1/q}\left(\int_{B(\mathbf{0},|\mathbf{x}|_p)}d\mathbf{y}\right)^{1/q'}\\&\lesssim|\mathbf{x}|_p^{\alpha+n\lambda}\|u\|_{\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)},\end{aligned}$$

and

$$\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}|\mathbf{x}|_p^{\alpha q+n\lambda q}d\mathbf{x}=p^{-\gamma n(1+\lambda q)}\sum_{k=-\infty}^{\gamma}\int_{S_k}p^{k\alpha q+kn\lambda q}d\mathbf{x}$$
$$=\frac{p^{\alpha q\gamma}(1-p^{-n})}{1-p^{-(n+\alpha q+n\lambda q)}}.$$

Hence

$$\|\mathscr{H}_\alpha u(\mathbf{x})\|_{\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}\lesssim\|u\|_{\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}\sup_{\gamma\in\mathbb{Z}}\left(\frac{p^{\alpha q\gamma}(1-p^{-n})}{1-p^{-(n+\alpha q+n\lambda q)}}\right)^{1/q}. \tag{20}$$

Let $M=\|u\|_{\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}$ is a constant. We have

$$\|\mathscr{H}_\alpha u(\mathbf{x})\|_{W\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}=\sup_{\gamma\in\mathbb{Z}}\sup_{t>0}t|B|_H^{-\lambda-1/q}|\{|\mathbf{x}|_p\in B_\gamma:M|\mathbf{x}|_p^{\alpha+n\lambda}<t\}|^{1/q}$$
$$\leqslant\sup_{\gamma\in\mathbb{Z}}\sup_{t>0}t|B|_H^{-\lambda-1/q}|\{|\mathbf{x}|_p\leqslant p^\gamma:|\mathbf{x}|_p<(t/M)^{\frac{1}{\alpha+n\lambda}}\}|^{1/q}.$$

If $\gamma\leqslant\log_p(t/M)^{\frac{1}{\alpha+n\lambda}}$, for $\lambda<0$,

$$\sup_{t>0}\sup_{\gamma\leqslant\log_p(t/M)^{\frac{1}{\alpha+n\lambda}}}t|B|_H^{-\lambda-1/q}\{|\mathbf{x}|_p\leqslant p^\gamma:|\mathbf{x}|_p<(t/M)^{\frac{1}{\alpha+n\lambda}}\}|^{1/q}$$
$$\leqslant\sup_{t>0}\sup_{\gamma\leqslant\log_p(t/M)^{\frac{1}{\alpha+n\lambda}}}tp^{-\gamma(\alpha+n\lambda)}$$
$$=\|u\|_{\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}.$$

If $\gamma>\log_p(t/M)^{\frac{1}{\alpha+n\lambda}}$,

$$\sup_{t>0}\sup_{\gamma>\log_p(t/M)^{\frac{1}{\alpha+n\lambda}}}t|B|_H^{-\lambda-1/q}\{|\mathbf{x}|_p\leqslant p^\gamma:|\mathbf{x}|_p<(t/M)^{\frac{1}{\alpha+n\lambda}}\}|^{1/q}$$
$$\leqslant\sup_{t>0}\sup_{\gamma>\log_p(t/M)^{\frac{1}{\alpha+n\lambda}}}tp^{-\gamma n(-\lambda-1/q)}(t/M)^{\frac{1}{\alpha+n\lambda}}$$
$$=\|u\|_{\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}.$$

Then, we obtain

$$\|\mathscr{H}_\alpha u(\mathbf{x})\|_{W\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}\leqslant\|u\|_{\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}. \tag{21}$$

The proof of Theorem 5 has been completed. □

Then, we show a property of fractional p-adic Hardy operator in $\mathrm{CMO}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)$.

THEOREM 6. *Let* $1 \leqslant q < \infty$, $\lambda \in (-\infty, n)$ *and* $\varphi \in C_0^\infty(B_\gamma)$ *satisfies* $\int_{B_\gamma} \varphi(\mathbf{y}) d\mathbf{y} = 1$. *Let also* $u \in L^q_{loc}(\wedge^l, \mathbb{Q}_p^n)$ *and* $du \in L^q_{loc}(\wedge^{l+1}, \mathbb{Q}_p^n)$, $l = 3, 4, \cdots, n-1$. *Then, we have*

$$\|\mathscr{H}_\alpha u\|_{\mathrm{CMO}^{q,\lambda}(\wedge^l, \mathbb{Q}_p^n)} \lesssim \|\varphi\|_{L^\infty} \|d\mathscr{H}_\alpha u\|_{\dot{B}^{q,\lambda'}(\wedge^{l+1}, \mathbb{Q}_p^n)}, \tag{22}$$

where $\lambda' = \lambda - 1 - 1/n$.

Proof. Clearly we know that $\mathscr{H}_\alpha u$ be a differential l-form in $\mathbb{Q}_p^n$. Then, we have

$$\mathscr{H}_\alpha u = d\mathscr{T}(\mathscr{H}_\alpha u) + \mathscr{T} d(\mathscr{H}_\alpha u).$$

From (14) and Definition 4, we have

$$\begin{aligned}
\|\mathscr{H}_\alpha u\|_{\mathrm{CMO}^{q,\lambda}(\wedge^l, \mathbb{Q}_p^n)} &= \sup_{\gamma \in \mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |\mathscr{H}_\alpha u(\mathbf{x}) - (\mathscr{H}_\alpha u)_{B_\gamma}|^q d\mathbf{x} \right)^{1/q} \\
&= \sup_{\gamma \in \mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |\mathscr{T} d(\mathscr{H}_\alpha u(\mathbf{x}))|^q d\mathbf{x} \right)^{1/q} \\
&= \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma|_H^{1/q+\lambda}} \|\mathscr{T} d(\mathscr{H}_\alpha u(\mathbf{x}))\|_{L^q(\wedge^l, B_\gamma)} \\
&\lesssim \|\varphi\|_{L^\infty} \|d\mathscr{H}_\alpha u\|_{\dot{B}^{q,\lambda'}(\wedge^{l+1}, \mathbb{Q}_p^n)}.
\end{aligned}$$

According to Theorem 2, we obtain

$$\begin{aligned}
&\|\mathscr{H}_\alpha u\|_{\mathrm{CMO}^{q,\lambda}(\wedge^l, \mathbb{Q}_p^n)} \\
&\lesssim \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma|_H^{1/q+\lambda}} |B_\gamma|_H^{1+1/n} \|\varphi\|_{L^\infty} \|u\|_{L^q(\wedge^l, B_\gamma)} \\
&= \|\varphi\|_{L^\infty} \|d\mathscr{H}_\alpha u\|_{\dot{B}^{q,\lambda'}(\wedge^{l+1}, \mathbb{Q}_p^n)},
\end{aligned}$$

where $\lambda' = \lambda - 1 - 1/n$. The proof of Theorem 6 has been completed. □

Next, we show the boundedness of fractional p-adic adjoint Hardy operator in p-adic weak central Morrey space.

THEOREM 7. *Let* $0 < \alpha < n$, $1 \leqslant q < \infty$ *and* $-1/q \leqslant \lambda < n$. *Let also* $u \in L^q_{loc}(\wedge^l, \mathbb{Q}_p^n)$. *Then* $\mathscr{H}_\alpha^* u(\mathbf{x})$ *is bounded from* $\dot{B}^{q,\lambda}(\wedge^l, \mathbb{Q}_p^n)$ *to* $W\dot{B}^{q,\lambda}(\wedge^l, \mathbb{Q}_p^n)$.

Proof. By using Hölder's inequality, we have

$$\begin{aligned}
|\mathscr{H}_\alpha^* u(\mathbf{x})| &= \left| \sum_I \int_{|\mathbf{y}|_p > |\mathbf{x}|_p} \frac{u_I(\mathbf{y})}{|\mathbf{y}|_p^{n-\alpha}} d\mathbf{y} d\mathbf{x} \right| \\
&\lesssim \int_{|\mathbf{y}|_p > |\mathbf{x}|_p} \frac{(\sum_I |u_I(\mathbf{y})|^2)^{1/2}}{|\mathbf{y}|_p^{n-\alpha}} d\mathbf{y} \\
&\lesssim \left(\int_{|\mathbf{y}|_p > |\mathbf{x}|_p} (\sum_I |u_I(\mathbf{y})|^2)^{q/2} d\mathbf{y} \right)^{1/q} \left(\int_{|\mathbf{y}|_p > |\mathbf{x}|_p} |\mathbf{y}|^{(\alpha-n)q'} d\mathbf{y} \right)^{1/q'}.
\end{aligned} \tag{23}$$

Then, we estimate

$$\left(\int_{|\mathbf{y}|_p>|\mathbf{x}|_p}|\mathbf{y}|^{(\alpha-n)q'}d\mathbf{y}\right)^{1/q'}=\left(\sum_{k=\log_p|\mathbf{x}|_p+1}^{\infty}\int_{S_k}p^{kq'(\alpha-n)}d\mathbf{y}\right)^{1/q'}$$
$$=(1-p^{-n})^{1/q'}\left(\sum_{k=\log_p|\mathbf{x}|_p}^{\infty}p^{kq'(\alpha-n)}\right)^{1/q'}$$
$$=\left(\frac{1-p^{-n}}{1-p^{q'(\alpha-n)}}\right)^{1/q'}|\mathbf{x}|_p^{\alpha-n}. \tag{24}$$

Combining with (23) and (24), we obtain

$$|\mathscr{H}_\alpha^* u(\mathbf{x})|\lesssim\left(\frac{1-p^{-n}}{1-p^{q'(\alpha-n)}}\right)^{1/q'}|\mathbf{x}|_p^{\alpha+n\lambda}\|u\|_{\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}.$$

Let $M_1=\left(\frac{p^{\alpha q\gamma}(1-p^{-n})}{1-p^{-(n+\alpha q+n\lambda q)}}\right)^{1/q}\|u\|_{\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}$ is a constant. We get

$$\|\mathscr{H}_\alpha^* u(\mathbf{x})\|_{W\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}=\sup_{\gamma\in\mathbb{Z}}\sup_{t>0}t|B|_H^{-\lambda-1/q}|\{|\mathbf{x}|_p\in B_\gamma:M_1|\mathbf{x}|_p^{\alpha+n\lambda}<t\}|^{1/q}$$
$$\leqslant\sup_{\gamma\in\mathbb{Z}}\sup_{t>0}t|B|_H^{-\lambda-1/q}|\{|\mathbf{x}|_p\leqslant p^\gamma:|\mathbf{x}|_p<(t/M_1)^{\frac{1}{\alpha+n\lambda}}\}|^{1/q}.$$

Hence

$$\|\mathscr{H}_\alpha^* u(\mathbf{x})\|_{W\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}\leqslant\begin{cases}\sup\limits_{t>0}\ \sup\limits_{\gamma\leqslant\log_p(t/M_1)^{\frac{1}{\alpha+n\lambda}}} tp^{-\gamma(\alpha+n\lambda)}\\ \sup\limits_{t>0}\ \sup\limits_{\gamma>\log_p(t/M_1)^{\frac{1}{\alpha+n\lambda}}} tp^{-\gamma n(-\lambda-1/q)}(t/M_1)^{\frac{1}{\alpha+n\lambda}}\end{cases}.$$

Finally, we obtain

$$\|\mathscr{H}_\alpha^* u(\mathbf{x})\|_{W\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}\leqslant\|u\|_{\dot{B}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}.$$

The proof of Theorem 7 has been completed. □

COROLLARY 2. *Let* $1<q<\infty$ *and* $\lambda\in(-\infty,n)$. *Let also* $u\in L_{loc}^q(\wedge^l,\mathbb{Q}_p^n)$ *and* $du\in L_{loc}^q(\wedge^{l+1},\mathbb{Q}_p^n)$, $l=3,4,\cdots,n-1$. *Then, we have*

$$\|\mathscr{H}_\alpha^* u\|_{\mathrm{CMO}^{q,\lambda}(\wedge^l,\mathbb{Q}_p^n)}\lesssim\|\varphi\|_{L^\infty}\|d\mathscr{H}_\alpha^* u\|_{\dot{B}^{q,\lambda'}(\wedge^{l+1},\mathbb{Q}_p^n)},$$

where $\lambda'=\lambda-1-1/n$ *and* $\varphi\in C_0^\infty(B_\gamma)$ *satisfies* $\int_{B_\gamma}\varphi(\mathbf{y})d\mathbf{y}=1$.

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