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EQUIVALENT CONDITIONS OF OPTIMAL HALF DISCRETE HILBERT TYPE MULTIPLE INTEGRAL INEQUALITIES WITH QUASI HOMOGENEOUS KERNEL AND APPLICATIONS

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Abstract. Let $G(u,v)$ be a λ-order homogeneous function. In this paper, by discussing optimal matching parameters of the half discrete Hilbert type multiple integral inequality with quasi-homogeneous kernel $K(n,||x||_{\rho,m}) = G(n^{\lambda_1},||x||^{\lambda_2}_{\rho,m})$, several equivalent conditions of the optimal matching parameters are obtained, and a basic theoretical problem of the half discrete Hilbert type inequality is solved. Finally, their applications to operator boundedness and operator norm are discussed.

1. Introduction

Let $m \in \mathbb{N}$, $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $\rho>0$, α, $\beta \in \mathbb{R}$, $x=(x_1, \cdots, x_m) \in \mathbb{R}^m_+$, $||x||_{\rho,m}=(x_1^{\rho}+\cdots+x_m^{\rho})^{1/\rho}$. Defined spaces L and l respectively:

$$L_q^{\beta}(\mathbb{R}^m_+)=\left\{f(x): ||f||_{q,\beta}=\left(\int_{\mathbb{R}^m_+}||x||^{\beta}_{\rho,m}|f(x)|^q\mathrm{d}x\right)^{1/q}<+\infty\right\},$$

$$l_p^{\alpha}=\left\{\tilde{a}=\{a_n\}: ||\tilde{a}||_{p,\alpha}=\left(\sum_{n=1}^{\infty}n^{\alpha}|a_n|^p\right)^{1/p}<+\infty\right\}.$$

For $K(n,||x||_{\rho,m}) \geqslant 0$, $\tilde{a}=\{a_n\}\in l_p^{\alpha}$, $f(x)\in L_q^{\beta}(\mathbb{R}^m_+)$, we call

$$\int_{\mathbb{R}^m_+}\sum_{n=1}^{\infty}K(n,||x||_{\rho,m})a_nf(x)\mathrm{d}x \leqslant M||\tilde{a}||_{p,\alpha}||f||_{q,\beta} \tag{1}$$

the half discrete Hilbert type multiple integral inequality. Define series operator T_1 and integral operator T_2 respectively:

$$T_1(\tilde{a})(x)=\sum_{n=1}^{\infty}K(n,||x||_{\rho,m})a_n, T_2(f)_n=\int_{\mathbb{R}^m_+}K(n,||x||_{\rho,m})f(x)\mathrm{d}x. \tag{2}$$

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By the basic theories of Hilbert type inequality, it can be easily proved that (1) is equivalent to

$$||T_1(\tilde{a})||_{p,\beta(1-p)} \leqslant M||\tilde{a}||_{p,\alpha}, ||T_2(f)||_{q,\alpha(1-q)} \leqslant M||f||_{q,\beta}.$$

Therefore, the discussion of (1) is of great significance to study the boundedness and norms of operators T_1 and T_2.

One of the most important topics of Hilbert type inequality is to select appropriate matching parameters to construct various exquisite inequalities with the best constant factors. The idea of weight functions proposed by Xu in [1] is the main method to solve this problem. Its core is: by introducing two matching parameters a and b, and using Hölder's inequality, we can obtain the following forms of inequality

$$\int_{\mathbb{R}_+^m} \sum_{n=1}^{\infty} K(n, ||x||_{\rho,m}) a_n f(x) \mathrm{d}x \leqslant M(a,b) ||\tilde{a}||_{p,\alpha(a,b)} ||f||_{q,\beta(a,b)}, \tag{3}$$

where the constant factor $M(a,b)$ is related to the matching parameters. Generally speaking, $M(a,b)$ is not the best constant factor of (3). Only by selecting some specific a and b, can $M(a,b)$ be optimal. By using this method, numerous papers [2–15] have been published at home and abroad, and many Hilbert type inequalities with optimal constant factors are obtained. However, there are few literatures on the law of optimal matching parameters. In this paper, we discuss the law of optimal matching parameters for half discrete Hilbert type multiple integral inequality with quasi-homogeneous kernel, and obtain some equivalent conditions.

Let $\lambda_1\lambda_2 > 0$, $G(u,v)$ be a λ-order homogeneous function, we say $K(u,v) = G(u^{\lambda_1}, v^{\lambda_2})$ is a quasi-homogeneous kernel with parameters $\{\lambda, \lambda_1, \lambda_2\}$. For $t > 0$, there are some properties of $K(u,v)$ as follows:

$$K(tu,v) = t^{\lambda\lambda_1} K(u, t^{-\lambda_1/\lambda_2} v), K(u,tv) = t^{\lambda\lambda_2} K(t^{-\lambda_2/\lambda_1} u, v).$$

2. Preliminary Lemmas

The following lemmas are used in this paper.

LEMMA 1. ([16]) *Assume that* $t_i \geqslant 0$, $\alpha_i > 0$, $i = 1,2,\cdots,m$, $\varphi(u)$ *is continuous. Then*

$$\int_{t_1+\cdots+t_m \leqslant 1} \varphi(t_1+\cdots+t_m) t_1^{\alpha_1-1} \cdots t_m^{\alpha_m-1} \mathrm{d}t_1 \cdots \mathrm{d}t_m = \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_m)}{\Gamma(\alpha_1+\cdots\alpha_m)} \int_0^1 \varphi(u) u^{\alpha_1+\cdots\alpha_m-1} \mathrm{d}u.$$

By using lemma 1, it is not difficult to prove that: if $\rho > 0$, $r > 0$, $x = (x_1,\cdots,x_m) \in \mathbb{R}_+^m$, then

$$\int_{||x||_{\rho,m} \leqslant r} \varphi\left(||x||_{\rho,m}\right) \mathrm{d}x = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_0^r \varphi(u) u^{m-1} \mathrm{d}u,$$

$$\int_{||x||_{\rho,m}\geqslant r} \varphi\left(||x||_{\rho,m}\right) \mathrm{d}x = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_r^{+\infty} \varphi(u) u^{m-1} \mathrm{d}u.$$

In particular,

$$\int_{\mathbb{R}_+^m} \varphi\left(||x||_{\rho,m}\right) \mathrm{d}x = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_0^{+\infty} \varphi(u) u^{m-1} \mathrm{d}u.$$

LEMMA 2. *Let* $\psi(t)$ *be measurable,* $\psi(t) > 0$ *a.e. in* $(0, +\infty)$, $\int_0^{+\infty} t^{c_1} \psi(t) \mathrm{d}t < +\infty$, *and*

$$\int_0^{+\infty} t^{c_1} \psi(t) \mathrm{d}t = \int_0^{+\infty} t^{c_2} \psi(t) \mathrm{d}t.$$

Then $c_1 = c_2$.

Proof. Suppose that $c_2 = c_1 + c$, $\frac{1}{r} + \frac{1}{s} = 1$ $(0 < r < 1,\ s < 0)$. According to the inverse Hölder's inequality, one has

$$\begin{aligned} &\int_0^{+\infty} t^{c_1} \psi(t) \mathrm{d}t = \int_0^{+\infty} t^{c_2} \psi(t) \mathrm{d}t = \int_0^{+\infty} t^c t^{c_1} \psi(t) \mathrm{d}t \\ &= \int_0^{+\infty} 1 \cdot t^c \psi(t) t^{c_1} \mathrm{d}t \geqslant \left(\int_0^{+\infty} 1^r \psi(t) t^{c_1} \mathrm{d}t\right)^{1/r} \left(\int_0^{+\infty} t^{cs} \psi(t) t^{c_1} \mathrm{d}t\right)^{1/s}. \end{aligned}$$

Due to $\psi(t) > 0$ a.e. in $(0, +\infty)$, we get $0 < \int_0^{+\infty} t^{c_1} \psi(t) \mathrm{d}t < +\infty$, therefore

$$\int_0^{+\infty} t^{c_1} \psi(t) \mathrm{d}t \geqslant \int_0^{+\infty} t^{cs} \psi(t) t^{c_1} \mathrm{d}t.$$

If $c > 0$, then $cs < 0$ and

$$\int_0^{+\infty} t^{c_1} \psi(t) \mathrm{d}t \geqslant \int_0^{\frac{1}{2}} t^{cs} \psi(t) t^{c_1} \mathrm{d}t \geqslant \left(\frac{1}{2}\right)^{cs} \int_0^{\frac{1}{2}} \psi(t) t^{c_1} \mathrm{d}t > 0.$$

Let $s \to -\infty$, then $\int_0^{+\infty} t^{c_1} \psi(t) \mathrm{d}t = +\infty$, which contradicts the conditions. Hence $c > 0$ is invalid.

If $c < 0$, then $cs > 0$ and

$$\int_0^{+\infty} t^{c_1} \psi(t) \mathrm{d}t \geqslant \int_2^{+\infty} t^{cs} \psi(t) t^{c_1} \mathrm{d}t \geqslant 2^{cs} \int_2^{+\infty} t^{c_1} \psi(t) \mathrm{d}t > 0.$$

Note that $\int_0^{+\infty} t^{c_1} \psi(t) \mathrm{d}t = +\infty$ as $s \to -\infty$, which also contradicts the conditions. Thus $c < 0$ is invalid too.

To sum up, it is proved that $c = 0$, that is $c_1 = c_2$. □

LEMMA 3. *Assume that* $m \in \mathbb{N}$, $\rho > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ $(p > 1)$, a, $b \in \mathbb{R}$, $\lambda_1 \lambda_2 > 0$, $G(u,v)$ *is a homogeneous non-negative measurable function of* λ*-order,* $x = (x_1, \cdots, x_m)$

$\in \mathbb{R}_+^m$, $K(n,||x||_{\rho,m}) = G(n^{\lambda_1},\ ||x||_{\rho,m}^{\lambda_2})$, $K(t,1)t^{-aq}$ *is monotonically decreasing in* $(0,+\infty)$. *Then*

$$\omega_1(n) = \int_{\mathbb{R}_+^m} K(n,||x||_{\rho,m})||x||_{\rho,m}^{-bp}\mathrm{d}x$$
$$= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} n^{\lambda\lambda_1-\frac{\lambda_1}{\lambda_2}(bp-m)} \int_0^{+\infty} K(1,t)t^{-bp+m-1}\mathrm{d}t,$$

$$\omega_2(x) = \sum_{n=1}^{\infty} K(n,||x||_{\rho,m})n^{-aq} \leqslant ||x||_{\rho,m}^{\lambda\lambda_2-\frac{\lambda_2}{\lambda_1}(aq-1)} \int_0^{+\infty} K(t,1)t^{-aq}\mathrm{d}t.$$

Proof. It follows from Lemma 1 that

$$\omega_1(n) = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_0^{+\infty} K(n,u)u^{-bp+m-1}\mathrm{d}u$$
$$= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} n^{\lambda\lambda_1} \int_0^{+\infty} K(1,n^{-\lambda_1/\lambda_2}u)u^{-bp+m-1}\mathrm{d}u$$
$$= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} n^{\lambda\lambda_1-\frac{\lambda_1}{\lambda_2}(bp-m)} \int_0^{+\infty} K(1,t)t^{-bp+m-1}\mathrm{d}t.$$

According to the monotone decreasing property of $K(t,1)t^{-aq}$ in $(0,+\infty)$, we have

$$\omega_2(x) = ||x||_{\rho,m}^{\lambda\lambda_2} \sum_{n=1}^{\infty} K(||x||_{\rho,m}^{-\frac{\lambda_2}{\lambda_1}} n,1)n^{-aq}$$
$$= ||x||_{\rho,m}^{\lambda\lambda_2-\frac{\lambda_2}{\lambda_1}aq} \sum_{n=1}^{\infty} K(||x||_{\rho,m}^{-\frac{\lambda_2}{\lambda_1}} n,1)\left(||x||_{\rho,m}^{-\frac{\lambda_2}{\lambda_1}} n\right)^{-aq}$$
$$\leqslant ||x||_{\rho,m}^{\lambda\lambda_2-\frac{\lambda_2}{\lambda_1}aq} \int_0^{+\infty} K(||x||_{\rho,m}^{-\frac{\lambda_2}{\lambda_1}} u,1)\left(||x||_{\rho,m}^{-\frac{\lambda_2}{\lambda_1}} u\right)^{-aq}\mathrm{d}u$$
$$= ||x||_{\rho,m}^{\lambda\lambda_2-\frac{\lambda_2}{\lambda_1}(aq-1)} \int_0^{+\infty} K(t,1)t^{-aq}\mathrm{d}t. \quad \square$$

3. Equivalent conditions of optimal matching parameters

The main theorems and their proofs are given below.

THEOREM 1. *Suppose that if* $m \in \mathbb{N}$, $\rho > 0$, $\frac{1}{p}+\frac{1}{q} = 1\ (p > 1)$, $a,\ b \in \mathbb{R}$, $\lambda_1\lambda_2 > 0$, $\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp-\left(\lambda+\frac{1}{\lambda_1}+\frac{m}{\lambda_2}\right) = c$, $K(u,v)$ *is a quasi-homogeneous non-negative measurable function with parameters* $\{\lambda,\lambda_1,\lambda_2\}$, $K(t,1) > 0$ *a.e. in* $(0,+\infty)$,

$K(t,1)t^{-aq}$ and $K(t,1)t^{-aq+\frac{\lambda_1 c}{p}}$ are monotonically decreasing in $(0,+\infty)$, $x=(x_1,\cdots,x_m)\in\mathbb{R}_+^m$, and

$$W_1(b,p,m)=\int_0^{+\infty}K(1,t)t^{-bp+m-1}\mathrm{d}t, W_2(a,q)=\int_0^{+\infty}K(t,1)t^{-aq}\mathrm{d}t$$

are both convergent. Then

(i) denote that

$$\alpha=\lambda_1\left[\lambda+\frac{m}{\lambda_2}+p\left(\frac{a}{\lambda_1}-\frac{b}{\lambda_2}\right)\right],\qquad \beta=\lambda_2\left[\lambda+\frac{1}{\lambda_1}+q\left(\frac{b}{\lambda_2}-\frac{a}{\lambda_1}\right)\right],$$

one has

$$\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}K(n,||x||_{\rho,m})a_nf(x)\mathrm{d}x \leqslant\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}W_1^{1/p}(b,p,m)W_2^{1/q}(a,q)||\tilde{a}||_{p,\alpha}||f||_{q,\beta}, \tag{4}$$

where $\tilde{a}=\{a_n\}\in l_p^{\alpha}$, $f(x)\in L_q^{\beta}(\mathbb{R}_+^m)$.

(ii) The following three conditions are equivalent:

(a) The constant factor of (4) is the best;

(b) $\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda+\frac{1}{\lambda_1}+\frac{m}{\lambda_2}$;

(c) $\frac{1}{\lambda_1}W_1(b,p,m)=\frac{1}{\lambda_2}W_2(a,\ q)$.

(iii) For $\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda+\frac{1}{\lambda_1}+\frac{m}{\lambda_2}$, (4) becomes

$$\begin{aligned}&\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}K(n,||x||_{\rho,m})a_nf(x)\mathrm{d}x\\ &\leqslant\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1(b,p,m)||\tilde{a}||_{p,apq-1}||f||_{q,bpq-m}\\ &=\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\lambda_1}{\lambda_2}\right)^{1/p}W_2(a,q)||\tilde{a}||_{p,apq-1}||f||_{q,bpq-m}.\end{aligned} \tag{5}$$

Proof. (i) It follows from mixed type Hölder's inequality, Lemma 3 and the introduction of matching parameters a, b that

$$\begin{aligned}&\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}K(n,||x||_{\rho,m})a_nf(x)\mathrm{d}x\\ &\leqslant\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}\left(\frac{n^a}{||x||_{\rho,m}^b}|a_n|\right)\left(\frac{||x||_{\rho,m}^b}{n^a}|f(x)|\right)K(n,||x||_{\rho,m})\mathrm{d}x\end{aligned}$$

$$\leqslant \left(\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}\frac{n^{ap}}{||x||_{\rho,m}^{bp}}|a_n|^p K(n,||x||_{\rho,m})\mathrm{d}x\right)^{1/p}$$
$$\times\left(\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}\frac{||x||_{\rho,m}^{bq}}{n^{aq}}|f(x)|^q K(n,||x||_{\rho,m})\mathrm{d}x\right)^{1/q}$$
$$=\left(\sum_{n=1}^{\infty}n^{ap}|a_n|^p\omega_1(n)\right)^{1/p}\left(\int_{\mathbb{R}_+^m}||x||_{\rho,m}^{bq}|f(x)|^q\omega_2(x)\mathrm{d}x\right)^{1/q}$$
$$\leqslant\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}W_1^{1/p}(b,p,m)W_2^{1/q}(a,q)$$
$$\times\left(\sum_{n=1}^{\infty}n^{ap+\lambda\lambda_1-\frac{\lambda_1}{\lambda_2}(bp-m)}|a_n|^p\right)^{1/p}\left(\int_{\mathbb{R}_+^m}||x||_{\rho,m}^{bq+\lambda\lambda_2-\frac{\lambda_2}{\lambda_1}(aq-1)}|f(x)|^q\mathrm{d}x\right)^{1/q}$$
$$=\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}W_1^{1/p}(b,p,m)W_2^{1/q}(a,q)||\tilde{a}||_{p,\alpha}||f||_{q,\beta}.$$

Hence (4) holds.

(ii) First, we will prove (b)$\Leftrightarrow$(c):

Suppose that $\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda+\frac{1}{\lambda_1}+\frac{m}{\lambda_2}$, then

$$\begin{aligned}W_1(b,p,m)&=\int_0^{+\infty}K(t^{-\lambda_2/\lambda_1},1)t^{\lambda\lambda_2-bp+m-1}\mathrm{d}t\\&=\frac{\lambda_1}{\lambda_2}\int_0^{+\infty}K(u,1)u^{-\frac{\lambda_1}{\lambda_2}(\lambda\lambda_2-bp+m-1)-\frac{\lambda_1}{\lambda_2}-1}\mathrm{d}u\\&=\frac{\lambda_1}{\lambda_2}\int_0^{+\infty}K(u,1)u^{-aq}\mathrm{d}u=\frac{\lambda_1}{\lambda_2}W_2(a,q).\end{aligned}$$

Therefore, $\frac{1}{\lambda_1}W_1(b,p,m)=\frac{1}{\lambda_2}W_2(a,q)$.

Conversely, assume that $\frac{1}{\lambda_1}W_1(b,p,m)=\frac{1}{\lambda_2}W_2(a,q)$, then

$$\int_0^{+\infty}K(t,1)t^{-aq}\mathrm{d}t=W_2(a,q)=\frac{\lambda_2}{\lambda_1}W_1(b,p,m)$$
$$=\frac{\lambda_2}{\lambda_1}\int_0^{+\infty}K(1,t)t^{-bp+m-1}\mathrm{d}t=\frac{\lambda_2}{\lambda_1}\int_0^{+\infty}K(t^{-\lambda_2/\lambda_1},1)t^{\lambda\lambda_2-bp+m-1}\mathrm{d}t$$
$$=\int_0^{+\infty}K(u,1)u^{-\frac{\lambda_1}{\lambda_2}(\lambda\lambda_2-bp+m-1)-\frac{\lambda_1}{\lambda_2}-1}\mathrm{d}u=\int_0^{+\infty}K(t,1)t^{-\lambda_1\left(\lambda-\frac{1}{\lambda_2}bp+\frac{m}{\lambda_2}\right)-1}\mathrm{d}t.$$

Since $K(t,1)>0$ a.e. in $(0,+\infty)$ and $\int_0^{+\infty}K(t,1)t^{-aq}\mathrm{d}t<+\infty$, it follows from Lemma 2 that $-aq=-\lambda_1\left(\lambda-\frac{1}{\lambda_2}bp+\frac{m}{\lambda_2}\right)-1$. Thus $\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda+\frac{1}{\lambda_1}+\frac{m}{\lambda_2}$.

Second, we prove (a)$\Leftrightarrow$(b):

Suppose that $\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda+\frac{1}{\lambda_1}+\frac{m}{\lambda_2}$, then it follows from $(b)\Rightarrow(c)$ that $\frac{1}{\lambda_1}W_1(b,p,m)=\frac{1}{\lambda_2}W_2(a,q)$. By simple calculation, we obtain $\alpha=apq-1$, $\beta=bpq-m$, thus, (4) becomes (5).

If the constant factor in (5) is not optimal, then there exists a constant $M_0 > 0$, such that

$$M_0 < \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q} W_1(b,p,m), \tag{6}$$

$$\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty} K(n,||x||_{\rho,m})a_n f(x)\mathrm{d}x \leqslant M_0||\tilde{a}||_{p,apq-1}||f||_{q,bpq-m}. \tag{7}$$

For sufficiently small $\varepsilon > 0$ and $\delta > 0$, take

$$a_n = n^{(-apq-|\lambda_1|\varepsilon)/p} \quad (n = 1,2,\cdots),$$

$$f(x) = \begin{cases} ||x||_{\rho,m}^{(-bpq-|\lambda_2|\varepsilon)/q}, & ||x||_{\rho,m} \geqslant \delta, \\ 0, & 0 < ||x||_{\rho,m} < \delta. \end{cases}$$

By Lemma 1, one has

$$\begin{aligned} ||\tilde{a}||_{p,apq-1}||f||_{q,bpq-m} &= \left(\sum_{n=1}^{\infty} n^{-1-|\lambda_1|\varepsilon}\right)^{1/p}\left(\int_{||x||_{\rho,m}\geqslant\delta} ||x||_{\rho,m}^{-m-|\lambda_2|\varepsilon}\mathrm{d}x\right)^{1/q} \\ &\leqslant \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\left(\int_0^{+\infty} t^{-1-|\lambda_1|\varepsilon}\mathrm{d}t\right)^{1/p}\left(\int_\delta^{+\infty} u^{-1-|\lambda_2|\varepsilon}\mathrm{d}u\right)^{1/q} \\ &= \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\frac{\delta^{-|\lambda_2|\varepsilon/q}}{\varepsilon|\lambda_1|^{1/p}|\lambda_2|^{1/q}}, \end{aligned} \tag{8}$$

$$\begin{aligned} &\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty} K(n,||x||_{\rho,m})a_n f(x)\mathrm{d}x \\ &= \sum_{n=1}^{\infty} n^{(-apq-|\lambda_1|\varepsilon)/p}\int_{||x||_{\rho,m}\geqslant\delta} K(n,||x||_{\rho,m})||x||_{\rho,m}^{(-bpq-|\lambda_2|\varepsilon)/q}\mathrm{d}x \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{(-apq-|\lambda_1|\varepsilon)/p}\int_\delta^{+\infty} K(n,u)u^{\frac{-bpq-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}u \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{\lambda\lambda_1+\frac{-apq-|\lambda_1|\varepsilon}{p}}\int_\delta^{+\infty} K(1,n^{-\lambda_1/\lambda_2}u)u^{\frac{-bpq-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}u \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{-1-|\lambda_1|\varepsilon}\int_{\delta n^{-\lambda_1/\lambda_2}}^{+\infty} K(1,t)t^{\frac{-bpq-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}t \\ &\geqslant \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{-1-|\lambda_1|\varepsilon}\int_\delta^{+\infty} K(1,t)t^{\frac{-bpq-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}t \\ &\geqslant \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_1^{+\infty} t^{-1-|\lambda_1|\varepsilon}\mathrm{d}t\int_\delta^{+\infty} K(1,t)t^{\frac{-bpq-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}t \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\frac{1}{|\lambda_1|\varepsilon}\int_\delta^{+\infty} K(1,t)t^{\frac{-bpq-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}t. \end{aligned} \tag{9}$$

It follows from (7), (8) and (9) that

$$\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\frac{1}{|\lambda_1|}\int_\delta^{+\infty}K(1,t)t^{\frac{-bpq-|\lambda_2|\varepsilon}{q}+m-1}\mathrm{d}t\leqslant\frac{M_0\delta^{-|\lambda_2|\varepsilon/q}}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}.$$

Let $\varepsilon\to0^+$ and then $\delta\to0^+$, we get

$$\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1(b,p,m)\leqslant M_0,$$

this is in contradiction with (6). Hence the constant factor of (5) is the best, and the same as (4).

On the contrary, assume that the constant factor of (4) is the best. In view of $\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp-\left(\lambda+\frac{1}{\lambda_1}+\frac{m}{\lambda_2}\right)=c$, note that $a-\frac{\lambda_1c}{pq}=a'$, $b-\frac{\lambda_2c}{pq}=b'$, then $\frac{1}{\lambda_1}a'q+\frac{1}{\lambda_2}b'p=\lambda+\frac{1}{\lambda_1}+\frac{m}{\lambda_2}$, $\alpha=a'pq-1$, $\beta=b'pq-m$ and

$$\begin{aligned}W_2(a,q)&=\int_0^{+\infty}K(t,1)t^{-aq}\mathrm{d}t=\int_0^{+\infty}K(1,t^{-\lambda_1/\lambda_2})t^{\lambda\lambda_1-aq}\mathrm{d}t\\&=\frac{\lambda_2}{\lambda_1}\int_0^{+\infty}K(1,u)u^{-\frac{\lambda_2}{\lambda_1}(\lambda\lambda_1-aq)-\frac{\lambda_2}{\lambda_1}-1}\mathrm{d}u\\&=\frac{\lambda_2}{\lambda_1}\int_0^{+\infty}K(1,t)t^{-bp+m-1+\lambda_2c}\mathrm{d}t.\end{aligned}$$

So (4) is equivalent to

$$\begin{aligned}&\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}K(n,||x||_{\rho,m})a_nf(x)\mathrm{d}x\\&\leqslant\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1^{1/p}(b,p,m)\\&\quad\times\left(\int_0^{+\infty}K(1,t)t^{-bp+m-1+\lambda_2c}\mathrm{d}t\right)^{1/q}||\tilde{a}||_{p,a'pq-1}||f||_{q,b'pq-m}.\end{aligned}\tag{10}$$

It is assumed that the constant factor of (4) is the best, thus the best constant factor of (10) is

$$\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1^{1/p}(b,p,m)\left(\int_0^{+\infty}K(1,t)t^{-bp+m-1+\lambda_2c}\mathrm{d}t\right)^{1/q}.$$

Notice that $\frac{1}{\lambda_1}a'q+\frac{1}{\lambda_2}b'p=\lambda+\frac{1}{\lambda_1}+\frac{m}{\lambda_2}$ and $K(t,1)t^{-a'q}=K(1,t)t^{-aq+\frac{\lambda_1c}{p}}$ is monotonically decreasing in $(0,+\infty)$, it follows from the proof of (b)$\Rightarrow$(a) that the best constant of (10) is

$$\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1(b',p,m).$$

Therefore,

$$W_1^{1/p}(b,p,m)\left(\int_0^{+\infty} K(1,t)t^{-bp+m-1+\lambda_2 c}\mathrm{d}t\right)^{1/q} = W_1(b',p,m).$$

Consequently,

$$\int_0^{+\infty} K(1,t)t^{-b'p+m-1}\mathrm{d}t = \int_0^{+\infty} K(1,t)t^{-bp+m-1+\frac{\lambda_2 c}{q}}\mathrm{d}t$$
$$= \left(\int_0^{+\infty} K(1,t)t^{-bp+m-1}\mathrm{d}t\right)^{1/p}\left(\int_0^{+\infty} K(1,t)t^{-bp+m-1+\lambda_2 c}\mathrm{d}t\right)^{1/q}. \qquad (11)$$

It follows from the Hölder's inequality that

$$\int_0^{+\infty} K(1,t)t^{-bp+m-1+\frac{\lambda_2 c}{q}}\mathrm{d}t = \int_0^{+\infty} 1\cdot t^{\frac{\lambda_2 c}{q}}K(1,t)t^{-bp+m-1}\mathrm{d}t$$
$$\leqslant \left(\int_0^{+\infty} 1^p K(1,t)t^{-bp+m-1}\mathrm{d}t\right)^{1/p}\left(\int_0^{+\infty} t^{\lambda_2 c}K(1,t)t^{-bp+m-1}\mathrm{d}t\right)^{1/q}$$
$$= \left(\int_0^{+\infty} K(1,t)t^{-bp+m-1}\mathrm{d}t\right)^{1/p}\left(\int_0^{+\infty} K(1,t)t^{-bp+m-1+\lambda_2 c}\mathrm{d}t\right)^{1/q}. \qquad (12)$$

From (11), (12) takes equal sign. According to the condition of equal sign in Hölder's inequality, we can get $t^{\lambda_2 c}$ =constant, so $c=0$, that is $\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\lambda+\frac{1}{\lambda_1}+\frac{m}{\lambda_2}$.

It has been proved that (b)⇔(c) and (a)⇔(b), hence (a), (b) and (c) are equivalent to each other.

(iii) It can be obtained by the proof of (b)⇒(a) in (ii). □

4. Applications in operator theory

According to the relation between (1) and corresponding operators, the following Theorem 2 equivalent to Theorem 1 can be obtained.

THEOREM 2. *Suppose that* $m\in\mathbb{N}$, $\rho>0$, $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, a, $b\in\mathbb{R}$, $\lambda_1\lambda_2>0$, $\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp-\left(\lambda+\frac{1}{\lambda_1}+\frac{m}{\lambda_2}\right)=c$, $K(u,v)$ *is a quasi-homogeneous non-negative measurable function with parameters* $\{\lambda,\lambda_1,\lambda_2\}$, $K(t,1)>0$ *a.e. in* $(0,+\infty)$, $K(t,1)t^{-aq}$ *and* $K(t,1)t^{-aq+\frac{\lambda_1 c}{p}}$ *are monotonically decreasing in* $(0,+\infty)$, $x=(x_1,\cdots,x_m)\in\mathbb{R}_+^m$, $W_1(b,p,m)$ *and* $W_2(a,q)$ *are defined as in Theorem 1 and they are both convergent. Series operator* T_1 *and integral operator* T_2 *are defined by (2). Then*

(i) denote that

$$\alpha=\lambda_1\left[\lambda+\frac{m}{\lambda_2}+p\left(\frac{a}{\lambda_1}-\frac{b}{\lambda_2}\right)\right], \qquad \beta=\lambda_2\left[\lambda+\frac{1}{\lambda_1}+q\left(\frac{b}{\lambda_2}-\frac{a}{\lambda_1}\right)\right],$$

then operators $T_1 : l_p^{\alpha} \to L_p^{\beta(1-p)}(\mathbb{R}_+^m)$ *and* $T_2 : L_q^{\beta}(\mathbb{R}_+^m) \to l_q^{\alpha(1-q)}$ *are bounded, and*

$$||T_1|| \leqslant \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} W_1^{1/p}(b,p,m)W_2^{1/q}(a,q),$$

$$||T_2|| \leqslant \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} W_1^{1/p}(b,p,m)W_2^{1/q}(a,q).$$

(ii) The following three conditions are equivalent:

(a) $T_1 : l_p^{\alpha} \to L_p^{\beta(1-p)}(\mathbb{R}_+^m)$ *and* $T_2 : L_q^{\beta}(\mathbb{R}_+^m) \to l_q^{\alpha(1-q)}$ *are bounded, and*

$$||T_1|| = ||T_2|| = \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} W_1^{1/p}(b,p,m)W_2^{1/q}(a,q);$$

(b) $\frac{1}{\lambda_1}aq + \frac{1}{\lambda_2}bp = \lambda + \frac{1}{\lambda_1} + \frac{m}{\lambda_2}$;

(c) $\frac{1}{\lambda_1}W_1(b,p,m) = \frac{1}{\lambda_2}W_2(a,\ q)$.

(iii) If $\frac{1}{\lambda_1}aq + \frac{1}{\lambda_2}bp = \lambda + \frac{1}{\lambda_1} + \frac{m}{\lambda_2}$, *then* $\alpha = apq - 1$, $\beta = bpq - m$, *the operator norms of* $T_1 : l_p^{apq-1} \to L_p^{(bpq-m)(1-p)}(\mathbb{R}_+^m)$ *and* $T_2 : L_q^{bpq-m}(\mathbb{R}_+^m) \to l_q^{(apq-1)(1-q)}$ *are*

$$||T_1|| = ||T_2|| = \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \left(\frac{\lambda_2}{\lambda_1}\right)^{1/q} W_1(b,p,m).$$

COROLLARY 1. *Assume that* $m \in \mathbb{N}$, $\rho > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ $(p > 1)$, $a \geqslant 0$, $b \geqslant 0$, $a \neq b$, $0 < \lambda_1 < 1$, $\lambda_2 > 0$, $\alpha = p(1-\lambda_1) - 1$, $\beta = q(m-\lambda_2) - m$, $x = (x_1, \cdots, x_m) \in \mathbb{R}_+^m$, T_1 *and* T_2 *are defined by respectively*

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty} \frac{a_n}{\left(n^{\lambda_1} + a||x||_{\rho,m}^{\lambda_2}\right)^2 + \left(n^{\lambda_1} + b||x||_{\rho,m}^{\lambda_2}\right)^2},$$

$$T_2(f)_n = \int_{\mathbb{R}_+^m} \frac{f(x)}{\left(n^{\lambda_1} + a||x||_{\rho,m}^{\lambda_2}\right)^2 + \left(n^{\lambda_1} + b||x||_{\rho,m}^{\lambda_2}\right)^2} \mathrm{d}x.$$

Then operators $T_1 : l_p^{\alpha} \to L_p^{\beta(1-p)}(\mathbb{R}_+^m)$ *and* $T_2 : L_q^{\beta}(\mathbb{R}_+^m) \to l_q^{\alpha(1-q)}$ *are bounded, and*

$$||T_1|| = ||T_2|| = \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \frac{1}{\lambda_1^{1/p}\lambda_2^{1/q}|a-b|} \left(\frac{\pi}{2} - \arctan\frac{a+b}{|a-b|}\right).$$

Proof. (i) Let $a_0 pq - 1 = \alpha = p(1-\lambda_1) - 1$, $b_0 pq - m = \beta = q(m-\lambda_2) - m$, then $a_0 = \frac{1}{q}(1-\lambda_1)$, $b_0 = \frac{1}{p}(m-\lambda_2)$ and

$$\frac{1}{\lambda_1}a_0 q + \frac{1}{\lambda_2}b_0 p = \frac{1}{\lambda_1}(1-\lambda_1) + \frac{1}{\lambda_2}(m-\lambda_2) = -2 + \frac{1}{\lambda_1} + \frac{m}{\lambda_2}. \tag{13}$$

Take again

$$K(n,||x||_{\rho,m}) = \frac{1}{\left(n^{\lambda_1}+a||x||_{\rho,m}^{\lambda_2}\right)^2+\left(n^{\lambda_1}+b||x||_{\rho,m}^{\lambda_2}\right)^2},$$

then $K(u,v)$ is a quasi-homogeneous non-negative function with parameters $\{-2,\lambda_1,\lambda_2\}$. According to (13), a_0 and b_0 are the best matching parameters.

It follows from $a \geqslant 0$, $b \geqslant 0$, $a \neq b$ and $\lambda_2 > 0$ that

$$\begin{aligned} &W_1(b_0,p,m) = \int_0^{+\infty} K(1,t)t^{-b_0p+m-1}\mathrm{d}t \\ &= \int_0^{+\infty} \frac{t^{\lambda_2-1}}{\left(1+at^{\lambda_1}\right)^2+\left(1+bt^{\lambda_2}\right)^2}\mathrm{d}t = \frac{1}{\lambda_2}\int_0^{+\infty}\frac{\mathrm{d}u}{(1+au)^2+(1+bu)^2} \\ &= \frac{1}{\lambda_2}\int_0^{+\infty}\frac{\mathrm{d}u}{(a^2+b^2)u^2+2(a+b)u+2} = \frac{1}{\lambda_2}\frac{1}{|a-b|}\left(\frac{\pi}{2}-\arctan\frac{a+b}{|a-b|}\right) < +\infty. \end{aligned}$$

Since $0<\lambda_1<1$, we know that

$$K(t,1)t^{-a_0q} = \frac{1}{\left(1+at^{\lambda_1}\right)^2+\left(1+bt^{\lambda_2}\right)^2}t^{\lambda_1-1}$$

is monotonically decreasing in $(0,+\infty)$. Therefore, the corollary holds according to theorem 2. □

COROLLARY 2. *Assume that* $m \in \mathbb{N}$, $\rho > 0$, $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $\lambda_1>0$, $\lambda_2>0$, $\lambda_1(1-\frac{m}{q})-\frac{\lambda_2}{q}<\sigma<\lambda_1(1-\frac{m}{q})-\frac{\lambda_2}{q}+\lambda_1\lambda_2$, $\sigma<\frac{m\lambda_1+\lambda_2}{p}$, $s=\frac{1}{\lambda_1\lambda_2}\left[\frac{\lambda_2}{q}+\lambda_1(\frac{m}{q}-1)+\sigma\right]$, $\alpha=\frac{1}{\lambda_2}(m\lambda_1-p\sigma)$, $\beta=\frac{1}{\lambda_1}(\lambda_2+q\sigma)$, $x=(x_1,\cdots,x_m)\in\mathbb{R}_+^m$, *the operators* T_1 *and* T_2 *are respectively*

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty}\frac{\ln\left(n^{\lambda_1}\Big/||x||_{\rho,m}^{\lambda_2}\right)}{n^{\lambda_1}\Big/||x||_{\rho,m}^{\lambda_2}-1}a_n,$$

$$T_2(f)_n = \int_{\mathbb{R}_+^m}\frac{\ln(n^{\lambda_1}/||x||_{\rho,m}^{\lambda_2})}{n^{\lambda_1}\Big/||x||_{\rho,m}^{\lambda_2}-1}f(x)\mathrm{d}x.$$

Then operators $T_1: l_p^{\alpha}\to L_p^{\beta(1-p)}(\mathbb{R}_+^m)$ *and* $T_2: L_q^{\beta}(\mathbb{R}_+^m)\to l_q^{\alpha(1-q)}$ *are bounded, and the operator norms are*

$$||T_1|| = ||T_2|| = \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\frac{1}{\lambda_1^{1/p}\lambda_2^{1/q}}\left(\frac{\pi}{\sin\pi s}\right)^2.$$

Proof. Let $K(n,||x||_{\rho,m}) = \ln\left(n^{\lambda_1}\Big/||x||_{\rho,m}^{\lambda_2}\right)\Big/\left(n^{\lambda_1}\Big/||x||_{\rho,m}^{\lambda_2}-1\right)$, then $K(u,v)$ is a quasi-homogeneous non-negative function with parameters $\{0,\lambda_1,\lambda_2\}$. Take again $apq-1=\alpha=\frac{1}{\lambda_2}(m\lambda_1-p\sigma)$, $bpq-m=\beta=\frac{1}{\lambda_1}(\lambda_2+q\sigma)$, it follows that $a=\frac{1}{\lambda_2 pq}(m\lambda_1-p\sigma)+\frac{1}{pq}$, $b=\frac{1}{\lambda_1 pq}(\lambda_2+q\sigma)+\frac{m}{pq}$. Notice that

$$\frac{1}{\lambda_1}aq+\frac{1}{\lambda_2}bp=\frac{1}{\lambda_1\lambda_2 p}(m\lambda_1-p\sigma)+\frac{1}{\lambda_1 p}+\frac{1}{\lambda_1\lambda_2 q}(\lambda_2+q\sigma)+\frac{m}{\lambda_2 q}=\frac{1}{\lambda_1}+\frac{m}{\lambda_2},$$

then a and b are the best matching parameters.

Since $\lambda_1(1-\frac{m}{q})-\frac{\lambda_2}{q}<\sigma<\lambda_1(1-\frac{m}{q})-\frac{\lambda_2}{q}+\lambda_1\lambda_2$, we have $0<s<1$. It follows from $\sigma<\frac{m\lambda_1+\lambda_2}{p}$ that $-\frac{1}{\lambda_2}(\frac{m\lambda_1}{p}-\sigma)-\frac{1}{p}<0$. Therefore,

$$\begin{aligned}W_1(b,p,m)&=\int_0^{+\infty}K(1,t)t^{-bp+m-1}\mathrm{d}t=\int_0^{+\infty}\frac{\ln t^{-\lambda_2}}{t^{-\lambda_2}-1}t^{-\frac{1}{\lambda_1}\left(\frac{\lambda_2}{q}+\sigma\right)-\frac{m}{q}}\mathrm{d}t\\&=\frac{1}{\lambda_2}\int_0^{+\infty}\frac{\ln u}{u-1}u^{\frac{1}{\lambda_1\lambda_2}\left[\frac{\lambda_2}{q}+\lambda_1\left(\frac{m}{q}-1\right)+\sigma\right]-1}\mathrm{d}u\\&=\frac{1}{\lambda_2}\int_0^{+\infty}\frac{\ln u}{u-1}u^{s-1}\mathrm{d}u=\frac{1}{\lambda_2}\left(\frac{\pi}{\sin\pi s}\right)^2<+\infty.\end{aligned}$$

Note that $h(u)=\ln u/(u-1)$ $(u>0$, let $h(1)=\lim_{u\to 1}\frac{\ln u}{u-1}=1)$, $g(u)=u-1-u\ln u$, we obtain

$$\begin{aligned}h'(u)&=\frac{u-1-u\ln u}{u(u-1)^2}=\frac{g(u)}{u(u-1)^2},\\g'(u)&=-\ln u, g''(u)=-\frac{1}{u}.\end{aligned}$$

Let $g'(u)=0$, then $u=1$. It follows from $g''(1)=-1<0$ that $u=1$ is the maximum point of $g(u)$. Hence $g(u)\leqslant g(1)=0$ and $h'(u)\leqslant 0$. Therefore $h(u)$ is monotonically decreasing in $(0,+\infty)$. What's more, notice that $\lambda_1>0$, $-\frac{1}{\lambda_2}(\frac{m\lambda_1}{p}-\sigma)-\frac{1}{p}<0$, one knows that

$$K(t,1)t^{-aq}=\frac{\ln t^{\lambda_1}}{t^{\lambda_1}-1}t^{-\frac{1}{\lambda_2}\left(\frac{m\lambda_1}{p}-\sigma\right)-\frac{1}{p}}$$

is monotonically decreasing in $(0,+\infty)$.

To sum up and in view of Theorem 2, $T_1: l_p^{\alpha}\to L_p^{\beta(1-p)}(\mathbb{R}_+^m)$ and $T_2: L_q^{\beta}(\mathbb{R}_+^m)\to l_q^{\alpha(1-q)}$ are bounded operators, and their norms are

$$\begin{aligned}||T_1||=||T_2||&=\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{\lambda_2}{\lambda_1}\right)^{1/q}W_1(b,p,m)\\&=\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\frac{1}{\lambda_1^{1/p}\lambda_2^{1/q}}\left(\frac{\pi}{\sin\pi s}\right)^2.\quad\square\end{aligned}$$

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