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THE NONEXISTENCE OF EXTREMALS FOR THE HARDY–TRUDINGER–MOSER INEQUALITY IN THE HYPERBOLIC SPACE

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Abstract. Let $\mathbb{B}$ be the unit disc in $\mathbb{R}^2$, $\mathscr{H}$ be the completion of $C_0^\infty(\mathbb{B})$ under the norm

$$||u||_{\mathscr{H}} = \Big(\int_{\mathbb{B}} |\nabla u|^2 dx - \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} dx\Big)^{\frac{1}{2}}, \quad \forall u \in C_0^\infty(\mathbb{B}).$$

We prove that the supremum in the following inequality

$$\sup_{u\in\mathscr{H}, ||u||_{\mathscr{H}}\leqslant 1} \int_{\mathbb{B}} \exp\{4\pi(1+\alpha||u||_2^2)u^2\} dx < +\infty$$

can not be achieved by any functions in the function space $\mathscr{H}$ when α is sufficiently close to λ_1^-, i.e., $0 < \lambda_1 - \alpha \ll 1$, where

$$\lambda_1(\mathbb{B}) = \inf_{u\in\mathscr{H}, u\neq 0} \frac{||u||_{\mathscr{H}}^2}{||u||_2^2}.$$

Evidently, this conclusion is complementary to that of [12, Theorem 1.1 (ii)].

1. Document preamble

In our paper, we let $\mathbb{B} \subset \mathbb{R}^2$ be the unit disc, let $\mathbb{B}(\sigma)$ be the disc with radius σ centered at origin, let $||\cdot||_p$ denote the usual L^p-norm of the classic L^p-Space, and let $W_0^{1,2}(\Omega)$ be the classic Sobolev space. It is well known that the famous Trudinger-Moser inequality [16, 17, 18, 20, 30] is

$$\sup_{u\in W_0^{1,2}(\mathbb{B}), ||\nabla u||_2\leqslant 1} \int_{\mathbb{B}} \exp\{\gamma u^2\} dx < \infty, \quad \forall \gamma \leqslant 4\pi. \tag{1.1}$$

When $\gamma > 4\pi$ the supremum is infinite, although the above integrals are still finite. Owing to the importance of the Trudinger-Moser inequality and the well-known Hardy inequality

$$\int_{\mathbb{B}} |\nabla u|^2 dx \geqslant \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} dx, \quad \forall u \in W_0^{1,2}(\mathbb{B}), \tag{1.2}$$

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in analysis, Wang-Ye [21] improved the Trudinger-Moser inequality (1.1) into the Hardy-Trudinger-Moser inequality

$$\sup_{u\in\mathscr{H},||u||_{\mathscr{H}}\leqslant 1}\int_{\mathbb{B}}\exp\{4\pi u^2\}\mathrm{d}x<+\infty.$$

Moreover, the above supremum is also achieved by some extremals and the function space $\mathscr{H}$ is the completion of $C_0^\infty(\mathbb{B})$ under the norm

$$||u||_{\mathscr{H}}=\Big(\int_{\mathbb{B}}|\nabla u|^2\mathrm{d}x-\int_{\mathbb{B}}\frac{u^2}{(1-|x|^2)^2}\mathrm{d}x\Big)^{\frac{1}{2}}. \tag{1.3}$$

In fact, (1.3) can be defined obviously by the inequality

$$\int_{\mathbb{B}}|\nabla u|^2\mathrm{d}x-\int_{\mathbb{B}}\frac{u^2}{(1-|x|^2)^2}\mathrm{d}x\geqslant C\int_{\mathbb{B}}u^2\mathrm{d}x,\quad \forall u\in W_0^{1,2}(\mathbb{B}),$$

which is an improvement of (1.2) given by Brezis-Marcus [4]. Moreover, $\mathscr{H}$ is also a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ induced by the norm $\|\cdot\|_{\mathscr{H}}$ on it. Another improvement of (1.1) was also proved by P. L. Lions [10]. That is, for any smooth bounded domain $\Omega\subset\mathbb{R}^2$, if $u_\varepsilon\in W_0^{1,2}(\Omega)$ satisfies $||\nabla u_\varepsilon||_2=1$ and $u_\varepsilon\rightharpoonup u_0$ weakly in $W_0^{1,2}(\Omega)$, $q<1/(1-||\nabla u_0||_2^2)$, there holds

$$\limsup_{\varepsilon\to 0}\int_{\Omega}\exp\{4\pi q u_\varepsilon^2\}\mathrm{d}x<+\infty$$

when $u_0\not\equiv 0$. For the case $u_0\equiv 0$, another modified Trudinger-Moser inequality of (1.1) involving L^2-norm was also verified by Adimurthi-Druet [1]. Namely, letting $\Omega\subset\mathbb{R}^2$ be the smooth bounded domain and

$$\lambda_1(\Omega)=\inf_{u\in W_0^{1,2}(\Omega),u\not\equiv 0}\frac{||\nabla u||_2^2}{||u||_2^2}>0$$

be the first eigenvalue of the Laplacian with Dirichlet boundary condition in Ω, then, for any $0\leqslant\alpha<\lambda_1(\Omega)$,

$$\sup_{u\in W_0^{1,2}(\Omega),||\nabla u||_2=1}\int_{\Omega}\exp\{4\pi u^2(1+\alpha||u||_2^2)\}\mathrm{d}x<+\infty. \tag{1.4}$$

Moreover, for any $\alpha\geqslant\lambda_1(\Omega)$, the supremum in (1.4) is equal to positive infinity. Furthermore, Lu-Yang [11] extended the results of Adimurthi-Druet [1] into L^p-norm, $p>1$. Going a step further, Lu-Yang [11] also considered the existence of extremal function for the modified Moser-Trudinger inequality involving L^p-norm. That is to say, with the facts that the first eigenvalue

$$\lambda_p(\Omega)=\inf_{u\in W_0^{1,2}(\Omega),u\not\equiv 0}\frac{||\nabla u||_2^2}{||u||_p^2}>0$$

and the smooth bounded domain $\Omega \subset \mathbb{R}^2$, for any fixed $p > 1$ and sufficiently small $\alpha > 0$, there holds

$$\begin{aligned}&\int_\Omega \exp\{4\pi(1+\alpha||u_\alpha||_p^2)u_\alpha^2\}\mathrm{d}x \\ &= \sup_{u\in W_0^{1,2}(\Omega),||\nabla u||_2=1} \int_\Omega \exp\{4\pi(1+\alpha||u||_p^2)u^2\}\mathrm{d}x,\end{aligned} \tag{1.5}$$

where $u_\alpha \in W_0^{1,2}(\Omega)\cap C^2(\Omega)$ satisfying $||\nabla u_\alpha||_2 = 1$. Correspondingly, Mancini-Thizy [15] researched the counterpart result of Lu-Yang [11] in the case $p = 2$, and verified that (1.5) did not hold water when α is sufficiently close to the eigenvalue $\lambda_1^-(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian operator $\Delta = -\partial_{xx} - \partial_{yy}$ with zero Dirichlet boundary condition. Analogy to that of Lu-Yang [11], Luo [12] studied the Adimurthi-Druet type inequality on the space $\mathscr{H}$, and found that, for sufficiently small $\alpha > 0$, the supremum

$$\sup_{u\in\mathscr{H},||u||_{\mathscr{H}}\leqslant 1} \int_{\mathbb{B}} \exp\{4\pi u^2(1+\alpha||u||_p^2)\}\mathrm{d}x$$

can be obtained by some extremals satisfying $u_\alpha \in \mathscr{H}\cap C^2(\mathbb{B})$, $||u_\alpha||_{\mathscr{H}} = 1$. Apart from the papers on the Trudinger-Moser inequality mentioned above, Yang-Zhu [29] extended the claim of (1.1) to the space $\mathscr{H}$ as well, however, with an equivalent norm to that of (1.3), which improved the conclusion of Wang-Ye [21] to the version involving the first eigenvalue of the Hardy-Laplacian operator $-\Delta - 1/(1-|x|^2)^2$. Using the method of energy estimate, Wang [23] extended the result of Mancini-Thizy [15] into the case of L^p norms, $p > 1$; In addition, Wang [22] reproved the results of Carleson-Chang [5], Flucher [8], Li [9] and Su [19], which evidently includes the result of (1.1) when the parameter α in the paper is equal to zero; Yang [26] reproved the conclusion of Carleson-Chang [5] in the case of the smooth bounded domain $\Omega \subset \mathbb{R}^2$; The results of Wang-Ye [21] and Yang-Zhu [29] were reproved by [25]. Besides, other literatures based on the method of energy estimate, I refer the readers to [28], [24]. Another important technique we use in our paper is blow-up analysis which will help us to get one of the expression of the energy identity. Pioneer works about blow-up analysis can be found in [2], [6] and [5]. In this aspect, a lot of work have been well done. In addition to Lu-Yang [11], Wang-Ye [21] and Yang-Zhu [29], etc, combining the result of Carleson-Chang [5] with the blow-up analysis, Yang [27] improved the classic result of (1.1) with the new norm $||u||_\alpha = \int_\Omega |\nabla u|^2\mathrm{d}x - \alpha\int_\Omega u^2\mathrm{d}x$, where $0 \leqslant \alpha < \lambda_1(\Omega)$ and $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian operator with the Dirichlet boundary condition. Later, Yang also considered the similar problem on the Riemannian surface. In view of Trudinger-Moser inequalities can be considered on Riemannian manifolds (Trudinger-Moser inequalities on Riemannian manifolds were due to T. Aubin [3]), it is interesting to ask whether or not our problem discussed in this paper can be hold on Riemannian manifolds. Inspired by Mancini-Thizy [15] and Luo [12], in our paper, we aim to get an complementary conclusion to that of Luo [12, Theorem 1.1 (ii)] as below.

THEOREM 1.1. *Let $\mathbb{B}\subset\mathbb{R}^2$ be the unit disc. Then, the supremum of the Trudinger-Moser inequality of Adimurthi-Druet type*

$$\sup_{u\in\mathscr{H},||u||_{\mathscr{H}}\leqslant 1}\int_{\mathbb{B}}\exp\{4\pi(1+\alpha||u||_2^2)u^2\}\mathrm{d}x<+\infty$$

can not be achieved by any functions in the function space $\mathscr{H}$ when α is sufficiently close to λ_1^-, i.e., $0<\lambda_1-\alpha\ll 1$, where $\lambda_1(\mathbb{B})=\inf_{u\in\mathscr{H},u\not\equiv 0}\frac{||u||_{\mathscr{H}}^2}{||u||_2^2}$.

To prove Theorem 1.1, we mainly follow the train of thought of [15] and use the method of energy estimate, which was developed by [13] and [14] et al. Besides, some critical results derived from the technique of non-increasing symmetrization are just simply applied to our paper, for more information about this aspect, I refer the readers to [21, 29].

The remaining part of this paper is organized as below: we mainly decompose our proof into three parts. In part 2.1 and part 2.2, we get two different expressions of the extremal function respectively. In part 2.3, we point out, by simple computation, that the two different expressions are actually contradictory, which finally gives an end to our demonstration.

2. The proof of Theorem 1.1

We know from Wang-Ye [21] or [29] that our problem can be equivalently discussed in the space of non-increasing, radially symmetric functions due to the technique of non-increasing symmetrization. Precisely, we let

$$\mathscr{S}_0=\{u\in C_0^\infty(\mathbb{B}):u(x)=u(r),u'(r)\leqslant 0,r=|x|\}$$

and let $\mathscr{S}$ be the completion of $\mathscr{S}_0$ under the norm $||\cdot||_{\mathscr{H}}$ defined by (1.3).

Besides, we let $C_\alpha(\mathbb{B})$ be defined by

$$C_\alpha(\mathbb{B})\triangleq\sup_{u\in\mathscr{H},||u||_{\mathscr{H}}\leqslant 1}\int_{\mathbb{B}}\exp\{4\pi(1+\alpha||u||_2^2)u^2\}\mathrm{d}x.$$

From Luo [12, Theorem 1.1], we have known that $C_\alpha(\mathbb{B})$ can be obtained by some extremal functions when $\alpha>0$ is sufficiently small. Therefore, by contradiction, we can assume that there exists a positive real number α_0 satisfying $0<\alpha_0\leqslant\lambda_1$ such that when $0<\alpha_0\leqslant\alpha_i<\lambda_1$, $C_{\alpha_i}(\mathbb{B})$ can be achieved by some extremal function u_{α_i}, where $u_{\alpha_i}\geqslant 0$ is an radially symmetric function. Besides, owing to $C_{\lambda_1}(\mathbb{B})=+\infty$, we know $C_{\alpha_i}(\mathbb{B})\to+\infty$ when $\alpha_i\to\lambda_1^-$. For simplicity, we use the indexes α in place of α_i. Then, by a direct computation, we have the following equation

$$\begin{cases}-\Delta u_\alpha=A_\alpha u_\alpha+2\beta_\alpha\lambda_\alpha u_\alpha\exp\{\beta_\alpha u_\alpha^2\}+\dfrac{u_\alpha}{(1-|x|^2)^2}\ \text{in}\ \mathbb{B},\\ ||u_\alpha||_{\mathscr{H}}=1,\quad u_\alpha>0\ \text{in}\ \mathbb{B},\\ \beta_\alpha=4\pi(1+\alpha||u_\alpha||_2^2),\\ A_\alpha=\dfrac{\alpha}{1+2\alpha||u_\alpha||_2^2},\quad \lambda_\alpha=\dfrac{(1+\alpha||u_\alpha||_2^2)}{\left(1+2\alpha||u_\alpha||_2^2\right)\int_{\mathbb{B}}u_\alpha^2\exp\{\beta_\alpha u_\alpha^2\}\mathrm{d}x}>0.\end{cases}\tag{2.1}$$

Furthermore, by Wang-Ye [21, Lemma 1 and Lemma 5] or Luo [12, Lemma 2.3], we know that, up to a subsequence (in the sequel, for simplicity, we do not distinguish between the sequence and its subsequence), $u_\alpha \rightharpoonup 0$ weakly in $\mathscr{S} \subset \mathscr{H}$; $u_\alpha \to 0$ strongly in $L^p(\mathbb{B}), \forall p \geqslant 1$; $u_\alpha \to 0$ a.e. in $\mathbb{B}$ as $\alpha \to \lambda_1^-$. Hence, it is easy to get

$$\beta_\alpha \to 4\pi \tag{2.2}$$

and

$$A_\alpha \to \lambda_1^-, \tag{2.3}$$

as $\alpha \to \lambda_1^-$. Let $v_\alpha = \sqrt{\beta_\alpha} u_\alpha$, then we rewrite (2.1) as

$$\begin{cases} -\Delta v_\alpha = A_\alpha v_\alpha + \Lambda_\alpha v_\alpha \exp\{v_\alpha^2\} + \dfrac{v_\alpha}{(1-|x|^2)^2} \text{ in } \mathbb{B}, \\ \|v_\alpha\|_{\mathscr{H}} = \sqrt{\beta_\alpha}, \quad v_\alpha > 0 \text{ in } \mathbb{B}, \\ \beta_\alpha = 4\pi(1 + \dfrac{\alpha}{\beta_\alpha}\|v_\alpha\|_2^2), \\ A_\alpha = \dfrac{\alpha}{1+2\frac{\alpha}{\beta_\alpha}\|v_\alpha\|_2^2}, \\ \Lambda_\alpha = 2\beta_\alpha\lambda_\alpha > 0. \end{cases} \tag{2.4}$$

In the upcoming two subsections, we will be dedicated to reaching two different expressions of $\|v_\alpha\|_{\mathscr{H}}^2$ respectively, which are contradictory to each other. Hence, we know that our assumption established before can not be true and the proof of Theorem 1.1 is finished.

2.1. The first expression of $\|v_\alpha\|_{\mathscr{H}}^2$

Basing on the standard quadratic-root formula and the Taylor expansion, we get the first expression of $\|v_\alpha\|_{\mathscr{H}}^2$ from the third line of (2.4)

$$\begin{aligned} \|v_\alpha\|_{\mathscr{H}}^2 = \beta_\alpha &= 2\pi\Big(1 + \sqrt{1 + \frac{\alpha\int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x}{\pi}}\Big) \\ &= 4\pi\Big(1 + \frac{\alpha\int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x}{4\pi} - \frac{\alpha^2\Big(\int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x\Big)^2}{16\pi^2} + o\Big(\Big(\int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x\Big)^2\Big)\Big). \end{aligned} \tag{2.5}$$

Besides, noticing that

$$\int_{\mathbb{B}} \exp\{v_\alpha^2\}\mathrm{d}x = \int_{\mathbb{B}} \exp\{\beta_\alpha u_\alpha^2\}\mathrm{d}x = C_\alpha(\Omega) \to +\infty$$

and

$$\begin{aligned} \Lambda_\alpha \int_{\mathbb{B}} v_\alpha^2 \exp\{v_\alpha^2\}\mathrm{d}x &= \int_{\mathbb{B}} |\nabla v_\alpha|^2 \mathrm{d}x - \int_{\mathbb{B}} \frac{v_\alpha^2}{(1-|x|^2)^2}\mathrm{d}x - A_\alpha \int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x \\ &= 4\pi + o(1), \end{aligned}$$

as $\alpha \to \lambda_1^-$, we get, by the elementary inequality $\exp\{t\} \leqslant 1 + t\exp\{t\}$, $t \geqslant 0$, that

$$\Lambda_\alpha \to 0, \quad \text{as } \alpha \to \lambda_1^-. \tag{2.6}$$

2.2. The second expression of $\|v_\alpha\|^2_{\mathscr{H}}$

We will make the blow-up analysis on (2.4) to get the second expression of $\|v_\alpha\|^2_{\mathscr{H}}$ which is included in the following Proposition 2.1.

PROPOSITION 2.1. *Let $\{v_\alpha\}_\alpha \subset \mathscr{H} \bigcap C^1(\mathbb{B})$ be a sequence of solutions of*

$$-\Delta v_\alpha = A_\alpha v_\alpha + \Lambda_\alpha v_\alpha \exp\{v_\alpha^2\} + \frac{v_\alpha}{(1-|x|^2)^2}, \quad v_\alpha > 0 \text{ in } \mathbb{B}, \tag{2.7}$$

where $A_\alpha \in [0,\lambda_1)$, $\Lambda_\alpha > 0$, $\beta_\alpha > 0$ and $\alpha\,(0 < \lambda_1 - \alpha \ll 1)$ are given by (2.4). Besides, we assume that the blow-up of v_α occurs, i.e.,

$$\gamma_\alpha \triangleq \max_{\mathbb{B}} v_\alpha = v_\alpha(0) \to +\infty, \tag{2.8}$$

and that (2.2), (2.3) and (2.6) hold true. Then, we have

$$\Lambda_\alpha = o\Big(\frac{1}{\gamma_\alpha^2}\Big), \tag{2.9}$$

and

$$\|v_\alpha\|^2_{\mathscr{H}} = \beta_\alpha = 4\pi\Big(1 + \frac{A_\alpha \int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x}{4\pi} + o\Big(\Big(\int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x\Big)^2\Big)\Big), \tag{2.10}$$

as $\alpha \to \lambda_1^-$.

As in [7], we define μ_α as below and have

$$\frac{1}{\mu_\alpha^2} \triangleq \frac{\Lambda_\alpha \gamma_\alpha^2}{4} \exp\{\gamma_\alpha^2\} \to +\infty, \text{ as } \alpha \to \lambda_1^-. \tag{2.11}$$

Moreover, there exists a positive number sequence $\{R_\alpha\}_\alpha$ satisfying $R_\alpha \to +\infty$, $R_\alpha\mu_\alpha \ll 1$, such that

$$\|\gamma_\alpha(\gamma_\alpha - v_\alpha(\mu_\alpha x)) - T_0\|_{C^2(\mathbb{B}(R_\alpha))} \to 0, \text{ as } \alpha \to \lambda_1^-, \tag{2.12}$$

where $T_0(x) = \log(1+|x|^2)$ is the solution of the Liouville equation

$$\Delta T_0 = 4\exp\{-2T_0\} \quad \text{in } \mathbb{R}^2. \tag{2.13}$$

Observing (2.7), (2.8) and $\|v_\alpha\|_{\mathscr{H}} = O(1)$, we get

$$\Lambda_\alpha \exp\{\gamma_\alpha^2\} \to +\infty, \text{ as } \alpha \to \lambda_1^-.$$

Besides, we let

$$t_\alpha(x) \triangleq \log\Big(1 + \frac{|x|^2}{\mu_\alpha^2}\Big) = T_0\Big(\frac{x}{\mu_\alpha}\Big).$$

Because of $|z| = r$ around the origin and the radial symmetry of functions, in the following, we will rewrite sometimes $v_\alpha(z), t_\alpha(z)$ as $v_\alpha(r), t_\alpha(r)$ respectively. For any $\delta \in (0,1)$, we let $r_{\alpha,\delta} > 0$ be determined by

$$t_\alpha(r_{\alpha,\delta}) = \delta\gamma_\alpha^2. \tag{2.14}$$

Then, we have

$$r_{\alpha,\delta}^2 = \mu_\alpha^2 \exp\{\delta\gamma_\alpha^2 + o(1)\} \gg \mu_\alpha^2, \tag{2.15}$$

as $\alpha \to \lambda_1^-$. In order to verify Proposition 2.1, we need to examine the asymptotic behavior of v_α in $\mathbb{B}(r_{\alpha,\delta})$ and in $\mathbb{B}\setminus\mathbb{B}(r_{\alpha,\delta})$ respectively, as $\alpha \to \lambda_1^-$. For this objective, we will separate our proof into four steps as below. We first consider the case in $\mathbb{B}(r_{\alpha,\delta})$, namely, the asymptotic behavior near the blow-up point 0.

STEP 1. We have

$$|\log \Lambda_\alpha| = o(\gamma_\alpha^2), \text{ as } \alpha \to \lambda_1^-. \tag{2.16}$$

Proof. We make $\mathbb{B}_\alpha = \mathbb{B}\setminus\mathbb{B}(\mu_\alpha)$ and let V_α be the unique harmonic function defined in $\mathbb{B}_\alpha$ which satisfies $V_\alpha(x) = v_\alpha(x)$ for $x \in \partial\mathbb{B}_\alpha$. Then, for all α, there holds

$$\int_{\mathbb{B}_\alpha} |\nabla V_\alpha|^2 \mathrm{d}x \leqslant \int_{\mathbb{B}_\alpha} |\nabla v_\alpha|^2 \mathrm{d}x. \tag{2.17}$$

For $\check{A}_\alpha > 0$, we define $\Psi_\alpha \triangleq \check{A}_\alpha \log|x|^{-1}$ such that Ψ_α coincides with $\gamma_\alpha - t_\alpha/\gamma_\alpha$ on $\partial\mathbb{B}(\mu_\alpha)$. Then, by (2.11), we have

$$\check{A}_\alpha = \frac{\gamma_\alpha^2 - \log 2}{\gamma_\alpha \log\frac{1}{\mu_\alpha}} = \frac{\gamma_\alpha(1+o(1))}{\log\frac{1}{\mu_\alpha}}. \tag{2.18}$$

Combining (2.12) with the elliptic estimates, for all $\alpha\,(0<\lambda_1-\alpha \ll 1)$, we get

$$|\nabla V_\alpha - \nabla\Psi_\alpha| \leqslant o\Big(\frac{1}{\gamma_\alpha|x|}\Big) \quad \text{in } \mathbb{B}_\alpha\ . \tag{2.19}$$

Then, by (2.18) and (2.19), we have

$$\int_{\mathbb{B}_\alpha} |\nabla V_\alpha|^2 \mathrm{d}x = \pi\check{A}_\alpha^2 \log\frac{1}{\mu_\alpha^2}(1+o(1)) = \frac{4\pi\gamma_\alpha^2(1+o(1))}{\log\frac{1}{\mu_\alpha^2}}. \tag{2.20}$$

Since $\|v_\alpha\|_{\mathscr{H}}^2 = \beta_\alpha \to 4\pi$ and $\int_{\mathbb{B}} v_\alpha^2/(1-|x|^2)^2 \mathrm{d}x = o(1)$ as $\alpha \to \lambda_1^-$, we have $\|v_\alpha\|_{H_0^1}^2 \leqslant 4\pi + o(1)$. Then, combining (2.17) and (2.20) together, we find

$$\log\frac{1}{\mu_\alpha^2} \geqslant (1+o(1))\gamma_\alpha^2,$$

which finishes the proof of (2.16) together with (2.6) and (2.11). □

Now, we fix $\delta \in (0,1)$ and let S_0 be the radial symmetric function around $0 \in \mathbb{R}^2$, which solves

$$-\Delta S_0 - 8\exp\{-2T_0\}S_0 = 4\exp\{-2T_0\}(T_0^2 - T_0) \tag{2.21}$$

and satisfies $S_0(0) = 0$. Then, by [14], we have

$$S_0(r) = -T_0(r) + \frac{2r^2}{1+r^2} - \frac{T_0(r)^2}{2} + \frac{1-r^2}{1+r^2}\int_1^{1+r^2} \frac{\log t}{1-t}\mathrm{d}t$$

and, in particular,

$$S_0(r)=\frac{A_0}{4\pi}\log\frac{1}{r^2}+B_0+O\Big(\frac{(\log r)^2}{r^2}\Big),\qquad \text{as } r\to+\infty, \tag{2.22}$$

where $4\pi=A_0=-\int_{\mathbb{R}^2}\Delta S_0\mathrm{d}x$ and $B_0=\pi^2/6+2$. For further arguments, we define

$$S_\alpha(z)\triangleq S_0\Big(\frac{z}{\mu_\alpha}\Big),$$

where $0<\lambda_1-\alpha\ll 1$.

STEP 2. For all the point sequence $\{z_\alpha\}_\alpha\subset\mathbb{B}(r_{\alpha,\delta})$, we have

$$v_\alpha(z_\alpha)=\gamma_\alpha-\frac{t_\alpha(z_\alpha)}{\gamma_\alpha}+\frac{S_\alpha(z_\alpha)}{\gamma_\alpha^3}+O\Big(\frac{1+t_\alpha(z_\alpha)}{\gamma_\alpha^5}\Big), \tag{2.23}$$

where $0<\lambda_1-\alpha\ll 1$.

Proof. We make $w_{1,\alpha}$ satisfy

$$v_\alpha=\gamma_\alpha-\frac{t_\alpha}{\gamma_\alpha}+\frac{w_{1,\alpha}}{\gamma_\alpha^3} \tag{2.24}$$

and define $\rho_{1,\alpha}$ as

$$\rho_{1,\alpha}\triangleq\sup\{r\in(0,r_{\alpha,\delta}]:|S_\alpha-w_{1,\alpha}|\leqslant 1+t_\alpha \text{ in } [0,r]\}. \tag{2.25}$$

First, we display the asymptotic expansions of $-\Delta w_{1,\alpha}$ in a precise way in $\mathbb{B}(\rho_{1,\alpha})$, as $\alpha\to\lambda_1^-$. We have, by (2.11), (2.14) and (2.16), that

$$\begin{aligned}\frac{\exp\Big\{-2t_\alpha+\frac{t_\alpha^2}{\gamma_\alpha^2}\Big\}}{\mu_\alpha^2}&=\exp\{\log\Lambda_\alpha+o(\gamma_\alpha^2)\}\exp\Big\{\Big(\gamma_\alpha-\frac{t_\alpha}{\gamma_\alpha}\Big)^2\Big\}\\&\geqslant\exp\Big\{\Big(1-\delta\Big)^2\gamma_\alpha^2+o(\gamma_\alpha^2)\Big\}\end{aligned} \tag{2.26}$$

in $\mathbb{B}(r_{\alpha,\delta})$. Evidently, $v_\alpha\leqslant\gamma_\alpha$ in $\mathbb{B}(\rho_{1,\alpha})$. Then, (2.26) gives

$$A_\alpha v_\alpha\leqslant\lambda_1\gamma_\alpha=o\left(\frac{\exp\Big\{-2t_\alpha+\frac{t_\alpha^2}{\gamma_\alpha^2}\Big\}}{\gamma_\alpha^5\mu_\alpha^2}\right) \tag{2.27}$$

uniformly in $\mathbb{B}(\rho_{1,\alpha})$. That is to say $A_\alpha v_\alpha$ is well controlled in $\mathbb{B}(\rho_{1,\alpha})$ because of Step 1. Moreover, observing that $w_{1,\alpha}=O(1+t_\alpha)$ is given by (2.22) and (2.25) in $\mathbb{B}(\rho_{1,\alpha})$, we can get from (2.24) that

$$v_\alpha=\gamma_\alpha-\frac{t_\alpha}{\gamma_\alpha}+O\Big(\frac{1+t_\alpha}{\gamma_\alpha^3}\Big) \tag{2.28}$$

and

$$v_\alpha^2 = \gamma_\alpha^2 - 2t_\alpha + \frac{t_\alpha^2 + 2w_{1,\alpha}}{\gamma_\alpha^2} + O\Big(\frac{1+t_\alpha^2}{\gamma_\alpha^4}\Big) \tag{2.29}$$

in $\mathbb{B}(\rho_{1,\alpha})$. Combining $t_\alpha = O(\gamma_\alpha^2)$ in $\mathbb{B}(r_{\alpha,\delta})$ with the inequality

$$\Big|\exp\{x\} - \sum_{j=0}^{k-1} \frac{x^j}{j!}\Big| \leqslant \frac{|x|^k}{k!}\exp\{|x|\}$$

for all $x \in \mathbb{R}$ and integer $k \geqslant 1$, we have

$$\exp\Big\{\frac{t_\alpha^2 + 2w_{1,\alpha}}{\gamma_\alpha^2} + O\Big(\frac{1+t_\alpha^2}{\gamma_\alpha^4}\Big)\Big\} = 1 + \frac{t_\alpha^2 + 2w_{1,\alpha}}{\gamma_\alpha^2} + O\Big(\frac{(1+t_\alpha^4)\exp\Big\{\frac{t_\alpha^2}{\gamma_\alpha^2}\Big\}}{\gamma_\alpha^4}\Big) \tag{2.30}$$

in $\mathbb{B}(\rho_{1,\alpha})$. Then, by (2.11), (2.28), (2.29) and (2.30), we have

$$\Lambda_\alpha v_\alpha \exp\{v_\alpha^2\} = \frac{4\exp\{-2t_\alpha\}}{\mu_\alpha^2 \gamma_\alpha}\Big[1 + \frac{t_\alpha^2 + 2w_{1,\alpha} - t_\alpha}{\gamma_\alpha^2} + O\Big(\frac{(1+t_\alpha^4)\exp\Big\{\frac{t_\alpha^2}{\gamma_\alpha^2}\Big\}}{\gamma_\alpha^4}\Big)\Big] \tag{2.31}$$

in $\mathbb{B}(\rho_{1,\alpha})$. Besides, by (2.11) and $t_\alpha = O(\gamma_\alpha^2)$, we easily get

$$\frac{v_\alpha \gamma^3}{(1-|x|^2)^2} \cdot \frac{\mu_\alpha^2}{4\exp\{-2t_\alpha\}} = O\Big(\frac{(1+t_\alpha^4)\exp\Big\{\frac{t_\alpha^2}{\gamma_\alpha^2}\Big\}}{\gamma_\alpha^2}\Big) \tag{2.32}$$

in $\mathbb{B}(\rho_{1,\alpha})$. Therefore, by (2.7), (2.13), (2.27),(2.31) and (2.32), we have

$$-\Delta w_{1,\alpha} = \frac{4\exp\{-2t_\alpha\}}{\mu_\alpha^2}\Big[2w_{1,\alpha} + t_\alpha^2 - t_\alpha + O\Big(\frac{(1+t_\alpha^4)\exp\{\frac{t_\alpha^2}{\gamma_\alpha^2}\}}{\gamma_\alpha^2}\Big)\Big] \tag{2.33}$$

in $\mathbb{B}(\rho_{1,\alpha})$.

Second, we assess the growth of $w_{1,\alpha} - S_\alpha$. In the following, we will confine our discussions to $\mathbb{B}(r_{\alpha,\delta})$, which gives that $2 - \frac{t_\alpha}{\gamma_\alpha^2} \geqslant 2 - \delta > 1$. Then,

$$(1+t_\alpha^4)\exp\Big\{-2t_\alpha + \frac{t_\alpha^2}{\gamma_\alpha^2}\Big\} \leqslant C\exp\{-\kappa t_\alpha\}, \quad \text{in } \mathbb{B}(r_{\alpha,\delta}), \tag{2.34}$$

where $1 < \kappa < 2$ and $C > 0$ are certain constants. Noticing that

$$\int_{\mathbb{B}(r)} \Delta(w_{1,\alpha} - S_\alpha)\mathrm{d}x = 2\pi r (w_{1,\alpha} - S_\alpha)'(r) \tag{2.35}$$

and, by (2.21), (2.33), that

$$-\Delta(w_{1,\alpha} - S_\alpha) = \frac{8\exp\{-2t_\alpha\}}{\mu_\alpha^2}\Big[(w_{1,\alpha} - S_\alpha) + O\Big(\frac{(1+t_\alpha^4)\exp\Big\{\frac{t_\alpha^2}{\gamma_\alpha^2}\Big\}}{\gamma_\alpha^2}\Big)\Big] \tag{2.36}$$

for all $0 \leqslant r \leqslant \rho_{1,\alpha}$, we have, by (2.34), that

$$\int_{\mathbb{B}(r)} \frac{8(1+t_\alpha^4)\exp\Big\{-2t_\alpha+\frac{t_\alpha^2}{\gamma_\alpha^2}\Big\}}{\mu_\alpha^2}\mathrm{d}x = \frac{8\pi}{\kappa-1}\Big(1-(1+(r/\mu_\alpha)^2)^{1-\kappa}\Big) \tag{2.37}$$

and, by $|(w_{1,\alpha}-S_\alpha)(r)| \leqslant \|(w_{1,\alpha}-S_\alpha)'\|_{L^\infty([0,\rho_{1,\alpha}])} r$, that

$$\int_{\mathbb{B}(r)} \frac{8\exp\{-2t_\alpha\}}{\mu_\alpha^2}|w_{1,\alpha}-S_\alpha|\mathrm{d}x \leqslant \mu_\alpha h(r/\mu_\alpha)\|(w_{1,\alpha}-S_\alpha)'\|_{L^\infty([0,\rho_{1,\alpha}])}, \tag{2.38}$$

where $h(s)=8\pi\Big(\arctan s - s/(1+s^2)\Big)$, $s \geqslant 0$. Then, by (2.35), (2.37) and (2.38), we have

$$\frac{r|(w_{1,\alpha}-S_\alpha)'(r)|}{C'} \leqslant \frac{(r/\mu_\alpha)^2}{\gamma_\alpha^2(1+(r/\mu_\alpha)^2)} + \frac{\mu_\alpha\|(w_{1,\alpha}-S_\alpha)'\|_{L^\infty([0,\rho_{1,\alpha}])}(r/\mu_\alpha)^3}{1+(r/\mu_\alpha)^3} \tag{2.39}$$

for some constant $C'>1$, $\alpha\,(0<\lambda_1-\alpha\ll 1)$ and all $0 \leqslant r \leqslant \rho_{1,\alpha}$.

Third, we verify that

$$\mu_\alpha\|(w_{1,\alpha}-S_\alpha)'\|_{L^\infty([0,\rho_{1,\alpha}])} = O\Big(\frac{1}{\gamma_\alpha^2}\Big). \tag{2.40}$$

Otherwise, we assume, by contradiction, that there exists a positive real number sequence $\{s_\alpha\}_\alpha \subset (0,\rho_{1,\alpha}]$ such that

$$\gamma_\alpha^2\mu_\alpha\|(w_{1,\alpha}-S_\alpha)'\|_{L^\infty([0,\rho_{1,\alpha}])} = \gamma_\alpha^2\mu_\alpha|(w_{1,\alpha}-S_\alpha)'|(s_\alpha) \to +\infty, \tag{2.41}$$

as $\alpha \to \lambda_1^-$. Up to a subsequence, we assume that

$$\frac{\rho_{1,\alpha}}{\mu_\alpha} \to \eta_0,$$

as $\alpha \to \lambda_1^-$, for some $\eta_0 \in (0,+\infty]$. We know from (2.39) and (2.41) that $s_\alpha = O(\mu_\alpha)$, $\mu_\alpha = O(s_\alpha)$, which implies $\eta_0 > 0$. We define $\tilde{w}_\alpha(s)$ as

$$\tilde{w}_\alpha(s) \triangleq \frac{(w_{1,\alpha}-S_\alpha)(\mu_\alpha s)}{\mu_\alpha\|(w_{1,\alpha}-S_\alpha)'\|_{L^\infty([0,\rho_{1,\alpha}])}}.$$

Then, by (2.39) and (2.41), we have

$$|\tilde{w}_\alpha'(s)| \leqslant \frac{C''}{1+s} \quad \text{in } [0,\rho_{1,\alpha}/\mu_\alpha] \tag{2.42}$$

for all $\alpha\,(0<\lambda_1-\alpha\ll 1)$, where C'' is some positive constant. Then, (2.36), (2.42) and elliptic theory imply that

$$\tilde{w}_\alpha \to \tilde{w} \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{B}(\eta_0)), \text{ as } \alpha \to \lambda_1^-, \tag{2.43}$$

and

$$\begin{cases} -\Delta\tilde{w} = 8\exp\{-2T_0\}\tilde{w} \quad \text{in } \mathbb{B}(\eta_0), \\ \tilde{w}(0) = 0, \\ \tilde{w} \text{ is radially symmetric around } 0 \in \mathbb{R}^2, \end{cases}$$

which indicates

$$\tilde{w} \equiv 0 \text{ in } \mathbb{B}(\eta_0). \tag{2.44}$$

Besides, combining (2.42), (2.43), (2.44) and the dominated convergence theorem, we have

$$\int_{\mathbb{B}(\rho_{1,\alpha})} \frac{\exp\{-2t_\alpha\}}{\mu_\alpha^2} |w_{1,\alpha} - S_\alpha| \mathrm{d}x = o(\mu_\alpha \|(w_{1,\alpha} - S_\alpha)'\|_{L^\infty([0,\rho_{1,\alpha}])}). \tag{2.45}$$

Now, we repeat the way to get (2.39), by replacing (2.38) with (2.45), and using (2.41), we get

$$r|(w_{1,\alpha} - S_\alpha)'(r)| = o(\mu_\alpha \|(w_{1,\alpha} - S_\alpha)'\|_{L^\infty([0,\rho_{1,\alpha}])}) \tag{2.46}$$

for all $0 \leqslant r \leqslant \rho_{1,\alpha}$, as $\alpha \to \lambda_1^-$. Evidently, it is impossible for (2.46) at s_α. Thus, we have finished the proof of (2.40).

Finally, inserting (2.40) into (2.39), using the fundamental theorem of calculus and the fact that $w_{1,\alpha}(0) = S_\alpha(0) = 0$, we have

$$\|w_{1,\alpha} - S_\alpha\|_{L^\infty([0,\rho_{1,\alpha}])} = O\Big(\frac{1+t_\alpha}{\gamma_\alpha^2}\Big), \qquad \text{as } \alpha \to \lambda_1^-,$$

which, by(2.25), implies $\rho_{1,\alpha} = r_{\alpha,\delta}$ and finishes the proof of Step 2. □

We let the functional sequence $\{f_\alpha\}$ be defined by

$$f_\alpha(v_\alpha) \triangleq A_\alpha v_\alpha + \Lambda_\alpha v_\alpha \exp\{v_\alpha^2\} = -\Delta v_\alpha - \frac{v_\alpha}{(1-|x|^2)^2}, \quad v_\alpha > 0 \text{ in } \mathbb{B}.$$

Then, we get the asymptotic expansion of f_α from Step 2 directly that

$$f_\alpha(v_\alpha) = O(\gamma_\alpha) + \Lambda_\alpha v_\alpha \exp\{v_\alpha^2\},$$

which, together with (2.25) and (2.26), leads to the expansion, as in (2.26), (2.27), (2.31), that

$$f_\alpha(v_\alpha) = \frac{4\exp\{-2t_\alpha\}}{\mu_\alpha^2\gamma_\alpha}\Big[1 + \frac{2S_\alpha + t_\alpha^2 - t_\alpha}{\gamma_\alpha^2} + O\Big(\frac{(1+t_\alpha^4)\exp\Big\{\frac{t_\alpha^2}{\gamma_\alpha^2}\Big\}}{\gamma_\alpha^4}\Big)\Big]$$

in $\mathbb{B}(r_{\alpha,\delta})$. Because of $\delta < 1$, as the arguments in (2.34), we can find $\kappa > 1$ such that

$$f_\alpha(v_\alpha) = \frac{4\exp\{-2t_\alpha\}}{\mu_\alpha^2\gamma_\alpha}\Big(1 + \frac{2S_\alpha + t_\alpha^2 - t_\alpha}{\gamma_\alpha^2}\Big) + O\Big(\frac{\exp\{-\kappa t_\alpha\}}{\mu_\alpha^2\gamma_\alpha^4}\Big) \tag{2.47}$$

and

$$v_\alpha f_\alpha(v_\alpha) = \frac{4\exp\{-2t_\alpha\}}{\mu_\alpha^2}\Big(1 + \frac{2S_\alpha + t_\alpha^2 - 2t_\alpha}{\gamma_\alpha^2}\Big) + O\Big(\frac{\exp\{-\kappa t_\alpha\}}{\mu_\alpha^2\gamma_\alpha^4}\Big) \tag{2.48}$$

in $\mathbb{B}(r_{\alpha,\delta})$. We second consider the case in $\mathbb{B}\setminus\mathbb{B}(r_{\alpha,\delta})$, i.e., the asymptotic behavior away from the blow-up point 0. For v_α is a non-increasing, radially symmetric function, we can choose δ' satisfying $0<\delta'<\delta<1$ and can define

$$\tilde{v}_\alpha \triangleq \begin{cases} v_\alpha & \text{in } \mathbb{B}\setminus\mathbb{B}(r_{\alpha,\delta}), \\ \min\{v_\alpha, (1-\delta')\gamma_\alpha\} & \text{in } \mathbb{B}(r_{\alpha,\delta}). \end{cases}$$

Then, by (2.23), we get

$$v_\alpha < (1-\delta')\gamma_\alpha \quad \text{in } \partial\mathbb{B}(r_{\alpha,\delta}),$$

as $0<\lambda_1-\alpha\ll 1$. By the arguments similar to that of [11] or to that of [12], we have that

$$\|\tilde{v}_\alpha\|_{\mathscr{H}} \leqslant 4\pi(1-\delta'+o(1)),$$

which implies, by [12, Proposition 2.2], that there exists $p'>0$ such that $(\exp\{\tilde{v}_\alpha^2\})_\alpha$ is bounded in $L^{p'}(\mathbb{B})$. Using (2.23) again, we get $\tilde{v}_\alpha = v_\alpha$ in $\mathbb{B}\setminus\mathbb{B}(r_{\alpha,\delta}/2)$. Hence, we know that

$$(\exp\{\tilde{v}_\alpha^2\})_\alpha \text{ is bounded in } L^{p'}(\mathbb{B}\setminus\mathbb{B}(r_{\alpha,\delta}/2)). \tag{2.49}$$

In addition, owing to [12, Lemma 2.3] and [21, Remark 1], we know that

$$v_\alpha \longrightarrow 0 \text{ strongly in } L^p(\mathbb{B}). \tag{2.50}$$

From now on, we let $p\geqslant 2$ and $r>1$ be fixed such that

$$\frac{1}{p'} + \frac{1}{p} + \frac{1}{r} = 1, \tag{2.51}$$

and let v be the unique function characterized by

$$\begin{cases} -\Delta v = \lambda_1 v + \frac{v}{(1-|x|^2)^2}, & v>0 \text{ in } \mathbb{B} \\ \|v\|_{\mathscr{H}} = 1. \end{cases} \tag{2.52}$$

STEP 3. For all the point sequence $\{z_\alpha\}_\alpha \subset \mathbb{B}\setminus\mathbb{B}(r_{\alpha,\delta})$, we have

$$v_\alpha(z_\alpha) = \|v_\alpha\|_{\mathscr{H}} v(z_\alpha) + o(\|v_\alpha\|_{\mathscr{H}}) + \frac{1}{\gamma_\alpha}\log\frac{1}{|z_\alpha|^2} + O\Big(\frac{1}{\gamma_\alpha}\Big) \tag{2.53}$$

for all α $(0<\lambda_1-\alpha\ll 1)$, where p and v satisfy (2.51) and (2.52) respectively. Besides, (2.9) holds true.

Proof. Since the item $1/(1-|x|^2)^2$ is singular on the boundary, which makes our problem not trivial, we need to consider the Hardy operator

$$L_{\mathscr{H}} = -\Delta - \frac{\mathscr{I}}{(1-|x|^2)^2},$$

where $\mathscr{I}$ is the identity operator. We let G be the Green's function of $L_{\mathscr{H}}$ in the disc $\mathbb{B} \subset \mathbb{R}^2$. Then, we know from [21, Proposition 2] that $G \in \mathscr{H} + W_0^{1,p''}(\mathbb{B}(1/2))$, $p'' \in [1,2)$, is the unique function such that

$$L_{\mathscr{H}}(G) = \delta_0,$$

and that G is a radial function satisfying

$$G(r) = -\frac{\log r}{2\pi} + C_G + O(r^{1+\tau})$$

for $\tau \in (0,1)$, as $r \to 0$, where δ_0 is the Dirac distribution in the common sense and $C_G \in \mathbb{R}$ is a constant. Furthermore, we have

$$G(r) = -\frac{\log r}{2\pi} + O(1) \tag{2.54}$$

for $0 < r \leqslant 1$. Thus we know that there exists a constant $C > 0$ such that

$$G_x(y) = \frac{1}{2\pi} \log \frac{C}{|x-y|} + O(1), \tag{2.55}$$

for all $x \neq y$ in $\mathbb{B}$. Letting the point sequence $(z_\alpha)_\alpha \subset \mathbb{B} \setminus \mathbb{B}(r_{\alpha,\delta})$ for all α, by the Green's representation formula and (2.7), we have that

$$v_\alpha(z_\alpha) = \int_{\mathbb{B}} G_{z_\alpha}(x) f_\alpha(v_\alpha(x)) \mathrm{d}x. \tag{2.56}$$

Here, we proceed our arguments by splitting the integral in (2.56) into two parts according to $\mathbb{B} = \mathbb{B}(r_{\alpha,\delta}/2) \cup \mathbb{B}(r_{\alpha,\delta}/2)^c$, where $\mathbb{B}(r_{\alpha,\delta}/2)^c = \mathbb{B} \setminus \mathbb{B}(r_{\alpha,\delta}/2)$. By (2.15) and the dominated convergence theorem, we integrate (2.47) and have that

$$\int_{\mathbb{B}(\frac{r_{\alpha,\delta}}{2})} f_\alpha(v_\alpha(x)) \mathrm{d}x = \int_{\mathbb{B}(\frac{r_{\alpha,\delta}}{2})} \frac{4\exp\{-2t_\alpha\}}{\mu_\alpha^2 \gamma_\alpha} \mathrm{d}x + O\Big(\frac{1}{\gamma_\alpha^3}\Big) = \frac{4\pi}{\gamma_\alpha} + O\Big(\frac{1}{\gamma_\alpha^3}\Big) \tag{2.57}$$

for all $\alpha\,(0 < \lambda_1 - \alpha \ll 1)$. Independently, we know from (2.55) that

$$|G_{z_\alpha}(x) - G_{z_\alpha}(0)| \leqslant \frac{C|x|}{r_{\alpha,\delta}} \tag{2.58}$$

for all $x \in \mathbb{B}(r_{\alpha,\delta}/2)$ and some constant $C > 0$. Then, together with (2.47), (2.57) and (2.58), we have

$$\begin{aligned} \int_{\mathbb{B}(\frac{r_{\alpha,\delta}}{2})} G_{z_\alpha}(x) f_\alpha(v_\alpha(x)) \mathrm{d}x = \frac{C}{r_{\alpha,\delta}} & \int_{\mathbb{B}(\frac{r_{\alpha,\delta}}{2})} f_\alpha(v_\alpha(x)) |x| \mathrm{d}x \\ & + \Big(\frac{4\pi}{\gamma_\alpha} + O\Big(\frac{1}{\gamma_\alpha^3}\Big)\Big) G_{z_\alpha}(0). \end{aligned} \tag{2.59}$$

Moreover, by (2.15) and (2.47), we have

$$\frac{C}{r_{\alpha,\delta}}\int_{\mathbb{B}(\frac{r_{\alpha,\delta}}{2})} f_\alpha(v_\alpha(x))|x|\mathrm{d}x = O\Big(\int_{\mathbb{B}(\frac{r_{\alpha,\delta}}{2})}\frac{\exp\{-\kappa t_\alpha\}|x|}{\gamma_\alpha\mu_\alpha^2 r_{\alpha,\delta}}\mathrm{d}x\Big) = o\Big(\frac{1}{\gamma_\alpha}\Big). \tag{2.60}$$

Hence, combining (2.59) and (2.60), we have

$$\int_{\mathbb{B}(\frac{r_{\alpha,\delta}}{2})} G_{z_\alpha}(x) f_\alpha(v_\alpha(x))\mathrm{d}x = \Big(\frac{4\pi}{\gamma_\alpha} + O\Big(\frac{1}{\gamma_\alpha^3}\Big)\Big)G_{z_\alpha}(0) + o\Big(\frac{1}{\gamma_\alpha}\Big). \tag{2.61}$$

As for the domain $\mathbb{B}(r_{\alpha,\delta}/2)^c$, using Hölder's inequality, Minkovsky's inequality, the fact that $\mathscr{H}$ is embedded continuously in $L^t(\mathbb{B})$ for any $t\in[1,+\infty)$, (2.6), (2.49), (2.51) and (2.55), we know that

$$\begin{aligned}
&\int_{\mathbb{B}(\frac{r_{\alpha,\delta}}{2})^c} G_{z_\alpha}(x) f_\alpha(v_\alpha(x))\mathrm{d}x\\
&\leqslant C\|G_{z_\alpha}\|_r\|v_\alpha\|_{\mathscr{H}}\Big(\lambda_1 + \Lambda_\alpha\|\exp\{v_\alpha^2\}\|_{L^{p'}(\mathbb{B}(\frac{r_{\alpha,\delta}}{2})^c)}\Big)\\
&= O(\|v_\alpha\|_{\mathscr{H}})
\end{aligned} \tag{2.62}$$

for all α. Combining (2.55), (2.56), (2.61) and (2.62), we get

$$v_\alpha(z_\alpha) \leqslant (1+o(1))\frac{\log\frac{C}{|z_\alpha|^2}}{\gamma_\alpha} + \overline{C}\|v_\alpha\|_{\mathscr{H}}, \quad \text{as } \alpha\to\lambda_1^-, \tag{2.63}$$

where $C>0$ and $\overline{C}>0$ are certain constants. Evidently, we have

$$\gamma_\alpha\|v_\alpha\|_{\mathscr{H}} \to +\infty, \quad \text{as } \alpha\to\lambda_1^-. \tag{2.64}$$

Now, we verify that

$$\frac{v_\alpha}{\|v_\alpha\|_{\mathscr{H}}} \to v \text{ in } C^1_{\mathrm{loc}}(\mathbb{B}), \quad \text{as } \alpha\to\lambda_1^-. \tag{2.65}$$

Combining (2.6), (2.7), (2.50), (2.63), (2.64) and elliptic theory, we have that $v_\alpha/\|v_\alpha\|_{\mathscr{H}} \to \tilde{v}$ in $C^1_{\mathrm{loc}}(\mathbb{B})$, as $\alpha\to\lambda_1^-$, and $\tilde{v}$ solves

$$-\Delta\tilde{v} - \frac{\tilde{v}}{(1-|x|^2)^2} = \lambda_1\tilde{v} \tag{2.66}$$

in $\mathbb{B}$. However, using (2.63), (2.64) again, we obtain that $0\leqslant\tilde{v}\leqslant\overline{C}$ in $\mathbb{B}$, where $\overline{C}$ is as in (2.63). Besides, $\tilde{v}$ solves (2.66) and satisfies $\|\tilde{v}\|_{\mathscr{H}}=1$. Hence, $\tilde{v}=v$ and (2.65) is proved. $r_{\alpha,\delta}\to 0$ and (2.65) indicate that there exists a positive real number sequence $(\delta_\alpha)_\alpha$ satisfying $\delta_\alpha\geqslant r_{\alpha,\delta}/2$ and $\delta_\alpha\to 0$, as $\alpha\to\lambda_1^-$ and that

$$\Big\|\frac{v_\alpha}{\|v_\alpha\|_{\mathscr{H}}} - v\Big\|_{C^0(\mathbb{B}\setminus\mathbb{B}(\delta_\alpha))} = o(1), \quad \text{as } \alpha\to\lambda_1^-. \tag{2.67}$$

Now, we turn to prove (2.53). Owing to (2.64) and (2.67), we have $O(1/\gamma_\alpha)=o(\|v_\alpha\|_{\mathscr{H}})$ and $v_\alpha = v\|v_\alpha\|_{\mathscr{H}} + o(\|v_\alpha\|_{\mathscr{H}})$ respectively.

Besides, $\liminf_{\alpha\to\lambda_1^-}|z_\alpha|>0$ indicates that $1/\gamma_\alpha\log|z_\alpha|^{-2}=O(1/\gamma_\alpha)$. Hence, if $\liminf_{\alpha\to\lambda_1^-}|z_\alpha|>0$, we have

$$v_\alpha(z_\alpha)=\|v_\alpha\|_{\mathscr{H}}v(z_\alpha)+o(\|v_\alpha\|_{\mathscr{H}})+\frac{1}{\gamma_\alpha}\log\frac{1}{|z_\alpha|^2}+O\Big(\frac{1}{\gamma_\alpha}\Big).$$

Namely, (2.53) is proved. Thus, we only need to prove that if $|z_\alpha|\to 0$ as $\alpha\to\lambda_1^-$, then (2.53) holds. In this case, we know from (2.11) and (2.15) that

$$\log\frac{1}{r_{\alpha,\delta}^2}=(1-\delta+o(1))\gamma_\alpha^2. \tag{2.68}$$

Because of $z_\alpha\in\mathbb{B}(r_{\alpha,\delta})^c$, we obtain, by (2.55), (2.61) and (2.68), that

$$\int_{\mathbb{B}(\frac{r_{\alpha,\delta}}{2})}G_{z_\alpha}(x)f_\alpha(v_\alpha(x))\mathrm{d}x=\frac{\log\frac{1}{|z_\alpha|^2}}{\gamma_\alpha}+O\Big(\frac{1}{\gamma_\alpha}\Big). \tag{2.69}$$

Besides, since $-\Delta v-v/(1-|x|^2)^2=\lambda_1 v$ and (2.3), we have, by (2.6), (2.50), (2.67) and the Green's representation formula, that

$$\begin{aligned}\int_{\mathbb{B}(\delta_\alpha)^c}G_{z_\alpha}(x)f_\alpha(v_\alpha(x))\mathrm{d}x&=\|v_\alpha\|_{\mathscr{H}}\lambda_1\int_{\mathbb{B}}G_{z_\alpha}(x)v(x)\mathrm{d}x+o(\|v_\alpha\|_{\mathscr{H}})\\&=\|v_\alpha\|_{\mathscr{H}}v(z_\alpha)+o(\|v_\alpha\|_{\mathscr{H}}),\end{aligned} \tag{2.70}$$

as $\alpha\to\lambda_1^-$. Now, we deal with the integral on the domain $\Omega_\alpha\triangleq\mathbb{B}(\delta_\alpha)\setminus\mathbb{B}(r_{\alpha,\delta}/2)$. Combining (2.54), (2.63) and (2.64), we have

$$\int_{\Omega_\alpha}G_{z_\alpha}(x)A_\alpha v_\alpha(x)\mathrm{d}x=O\Big(\delta_\alpha^2\log\frac{1}{\delta_\alpha}\|v_\alpha\|_{\mathscr{H}}\Big)+O\Big(\frac{1}{\gamma_\alpha}\Big)=o(\|v_\alpha\|_{\mathscr{H}}), \tag{2.71}$$

as $\alpha\to\lambda_1^-$. Besides, by (2.6), (2.56) and (2.63), we have

$$\begin{aligned}&\int_{\Omega_\alpha}G_{z_\alpha}(x)\Lambda_\alpha v_\alpha(x)\exp\{v_\alpha^2\}\mathrm{d}x\\&=o\Big(\int_{\Omega_\alpha}\log\frac{C}{|z_\alpha-x|}\Big(\frac{1}{\gamma_\alpha}\log\frac{C}{|x|}+\|v_\alpha\|_{\mathscr{H}}\Big)\\&\qquad\times\exp\Big\{\Big(\frac{1+o(1)}{\gamma_\alpha}\log\frac{1}{|x|^2}+o(1)\Big)^2\Big\}\mathrm{d}x\Big)\\&=o(\|v_\alpha\|_{\mathscr{H}}),\end{aligned} \tag{2.72}$$

as $\alpha\to\lambda_1^-$. Clearly, adding up (2.69), (2.70), (2.71) and (2.72) together leads to (2.53). In addition, from (2.68), we obtain

$$\frac{1}{\gamma_\alpha^2}\log\frac{1}{|x|^2}\leqslant 1-\delta+o(1)<1\ \text{in}\ \mathbb{B}\Big(\frac{r_{\alpha,\delta}}{2}\Big)^c \tag{2.73}$$

for all α, $0<\lambda_1-\alpha\ll 1$. Further more, together (2.11) with (2.23), we get

$$v_\alpha(\tilde{z}_\alpha)=\frac{1}{\gamma_\alpha}\Big(\log\frac{1}{|\tilde{z}_\alpha|^2}+\log\frac{1}{\gamma_\alpha^2\Lambda_\alpha}+O(1)\Big) \tag{2.74}$$

for all α, $0<\lambda_1-\alpha\ll 1$, and all $\tilde{z}_\alpha\in\partial\mathbb{B}(r_{\alpha,\delta})$. Choosing $z_\alpha=\tilde{z}_\alpha$ in (2.53) and by (2.64), (2.74), we obtain

$$\log\frac{1}{\gamma_\alpha^2\Lambda_\alpha}=\gamma_\alpha\|v_\alpha\|_{\mathscr{H}}v(0)(1+o(1))\to+\infty,\qquad \text{as } \alpha\to\lambda_1^-,$$

which finishes the proof of (2.9) and, finally, that of Step 3. $\square$

STEP 4. The proof of (2.10).

Multiplying (2.7) by v_α and integrating on $\mathbb{B}$, we have

$$\begin{aligned}\|v_\alpha\|_{\mathscr{H}}^2&=-\int_{\mathbb{B}}v_\alpha\Delta v_\alpha\mathrm{d}x-\int_{\mathbb{B}}\frac{v_\alpha^2}{(1+|x|^2)^2}\mathrm{d}x\\&=A_\alpha\int_{\mathbb{B}}v_\alpha^2\mathrm{d}x+\Lambda_\alpha\int_{\mathbb{B}}v_\alpha^2\exp\{v_\alpha^2\}\mathrm{d}x,\end{aligned}$$

as $\alpha\to\lambda_1^-$. Comparing with the formula (2.10), we know that, for the purpose of proving (2.10), it only needs to prove

$$\Lambda_\alpha\int_{\mathbb{B}}v_\alpha^2\exp\{v_\alpha^2\}\mathrm{d}x=4\pi+o\Big(\Big(\int_{\mathbb{B}}v_\alpha^2\mathrm{d}x\Big)^2\Big),\qquad \text{as } \alpha\to\lambda_1^-. \tag{2.75}$$

Thus, we first use (2.48) and get that

$$\begin{aligned}\Lambda_\alpha\int_{\mathbb{B}(r_{\alpha,\delta})}v_\alpha^2\exp\{v_\alpha^2\}\mathrm{d}x&=\int_{\mathbb{B}(r_{\alpha,\delta})}\frac{4\exp\{-2t_\alpha\}}{\mu_\alpha^2}\Big(1+\frac{2S_\alpha+t_\alpha^2-2t_\alpha}{\gamma_\alpha^2}\Big)\mathrm{d}x\\&\quad+O\Big(\int_{\mathbb{B}(r_{\alpha,\delta})}\frac{\exp\{-\kappa t_\alpha\}}{\mu_\alpha^2\gamma_\alpha^4}\mathrm{d}x\Big)\\&=4\int_{\mathbb{B}(\frac{r_{\alpha,\delta}}{\mu_\alpha})}\exp\{-2T_0\}\Big(1+\frac{2S_0+T_0^2-2T_0}{\gamma_\alpha^2}\Big)\mathrm{d}x\\&\quad+O\Big(\frac{1}{\gamma_\alpha^4}\Big).\end{aligned} \tag{2.76}$$

Noticing $\int_{\mathbb{B}(r_{\alpha,\delta}/\mu_\alpha)}\exp\{-2T_0\}(2S_0+T_0^2-2T_0)\gamma_\alpha^{-2}\mathrm{d}x$ vanishes, by (2.21) and (2.22), we have

$$4\pi=A_0=-\int_{\mathbb{R}^2}\Delta S_0\mathrm{d}x=\int_{\mathbb{R}^2}4\exp\{-2T_0\}T_0\mathrm{d}x.$$

Then, (2.15) and (2.76) give that

$$\Lambda_\alpha\int_{\mathbb{B}(r_{\alpha,\delta})}v_\alpha^2\exp\{v_\alpha^2\}\mathrm{d}x=4\pi+O\Big(\frac{1}{\gamma_\alpha^4}\Big). \tag{2.77}$$

Independently as in (2.72), we have, combining (2.50), (2.53) with (2.73), that

$$\begin{aligned}\int_{\mathbb{B}(r_{\alpha,\delta})^c} v_\alpha^2 \exp\{v_\alpha^2\}\mathrm{d}x = O\Big(\int_{\mathbb{B}(r_{\alpha,\delta})^c} \Big(\frac{1}{\gamma_\alpha^2}\Big(\log\frac{C}{|x|}\Big)^2 + \|v_\alpha\|_{\mathscr{H}}^2\Big) \\ \times \exp\Big\{\Big(\frac{1+o(1)}{\gamma_\alpha}\log\frac{1}{|x|^2} + o(1)\Big)^2\Big\}\mathrm{d}x\Big) \\ = O\Big(\|v_\alpha\|_{\mathscr{H}}^2 + \frac{1}{\gamma_\alpha^2}\Big).\end{aligned} \tag{2.78}$$

In addition, we get from (2.53) and (2.64) that

$$\int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x = \|v_\alpha\|_{\mathscr{H}}^2 \int_{\mathbb{B}} v^2 \mathrm{d}x + o\Big(\|v_\alpha\|_{\mathscr{H}}^2\Big). \tag{2.79}$$

Finally, (2.77), (2.78) and (2.79) together with (2.9) and (2.64) lead to the proof of (2.75).

2.3. The final proof of Theorem 1.1

In the above two sections, we have given two different expressions of $\|v_\alpha\|_{\mathscr{H}}^2$ respectively. Now, we aim to show that the two different expressions are actually contradictory, which finally indicates that our assumption that there exists a sequence of real numbers α_i, $0 < \alpha_0 \leqslant \alpha_i < \lambda_1$, such that $C_{\alpha_i}(\mathbb{B})$ can be achieved by some extremal function $u_{\alpha_i} \geqslant 0$, where u_{α_i} is an radially symmetric function, is not true. Thus, the proof of Theorem 1.1 is finished at last. Noticing that expanding the forth line of (2.4) gives

$$A_\alpha = \alpha - \frac{\alpha^2 \int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x}{2\pi} + O\Big(\Big(\int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x\Big)^2\Big), \tag{2.80}$$

we get, by combining (2.10) and (2.80), that

$$\|v_\alpha\|_{\mathscr{H}}^2 = \beta_\alpha = 4\pi\Big(1 + \frac{\alpha \int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x}{4\pi} - \frac{\alpha^2 (\int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x)^2}{8\pi^2} + o\Big(\Big(\int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x\Big)^2\Big)\Big). \tag{2.81}$$

Then, matching (2.5) and (2.81), we have

$$-\frac{\alpha^2 \Big(\int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x\Big)^2}{16\pi^2} = o\Big(\Big(\int_{\mathbb{B}} v_\alpha^2 \mathrm{d}x\Big)^2\Big), \quad \text{as } \alpha \to \lambda_1^-. \tag{2.82}$$

Evidently, (2.82) is a contradictory conclusion, which indicates that the proof of Theorem 1.1 is finished at last.

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