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SOME DEGENERATE MEAN CONVERGENCE THEOREMS FOR BANACH SPACE VALUED RANDOM ELEMENTS

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Abstract. For an array $\{V_{n,j}, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ of random elements taking values in a real separable Rademacher type p $(1 < p \leqslant 2)$ Banach space and a sequence of positive constants $\{d_n, n \geqslant 1\}$, a theorem is established providing conditions under which the degenerate mean convergence result $\mathbb{E}\|(S_n - \mathbb{E}S_n)/d_n\|^p \to 0$ holds where $S_n = \sum_{j=1}^{n} V_{n,j}$, $n \geqslant 1$. An example is provided showing that the above degenerate mean convergence can fail if the Banach space is not of Rademacher type p where $1 < p \leqslant 2$. Moreover for a general sequence of random elements $\{W_n, n \geqslant 1\}$ which is not structurally of any specific form taking values in a real separable Banach space which is not assumed to be of Rademacher type p for any $p \in (1,2]$, conditions are provided under which the degenerate mean convergence result $\mathbb{E}(g(\|W_n\|)) \to 0$ holds where g is a continuous strictly increasing function with $g(0) = 0$ and $\lim_{x\to\infty} g(x) = \infty$.

1. Introduction

In this article, we obtain two degenerate mean convergence theorems for sequences of random elements. For a sequence of random elements $\{W_n, n \geqslant 1\}$ taking values in a real separable Banach space $\mathfrak{X}$ with norm $\|\cdot\|$ and a measurable function $g : [0,\infty) \to [0,\infty)$, the sequence of random variables $\{g(\|W_n\|), n \geqslant 1\}$ is said to *converge in mean* to 0 if $\mathbb{E}(g(\|W_n\|)) \to 0$. The main results to be presented, Theorems 1 and 2, provide conditions under which convergence in mean to 0 holds.

In Theorem 1, we consider an array $\{V_{n,j}, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ of random elements taking values in a real separable Rademacher type p $(1 < p \leqslant 2)$ Banach space. (Technical definitions will be reviewed in Section 2.) The sequence of random elements under consideration is of the specific form $W_n = (S_n - \mathbb{E}S_n)/d_n$, $n \geqslant 1$ where $S_n = \sum_{j=1}^{k_n} V_{n,j}$, $n \geqslant 1$ and $\{d_n, n \geqslant 1\}$ is a sequence of positive constants. Let g be the function $g(x) = x^p$, $x \geqslant 0$. Conditions are given for $\{\|W_n\|, n \geqslant 1\}$ to converge in mean of order p to 0; that is for $\mathbb{E}\|W_n\|^p \to 0$. In Theorem 1 and Corollary 1, $\{k_n, n \geqslant 1\}$ is a sequence of positive integers with $k_n \to \infty$ as $n \to \infty$ and the array is assumed to be comprised of rowwise independent random elements; that is, the random elements from

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the same row are independent but there are no independence or dependence conditions imposed on the random elements from different rows.

The argument for proving Theorem 1 is a modification of the argument used to prove Theorem 3.3 of Chandra, Li, and Rosalsky (2018) which established a degenerate mean convergence result of order 1 for normed and centered row sums from an array of random variables whose n^{th} row is comprised of pairwise negative quadrant dependent random variables. However, the truncation schemes used in the proofs of Theorem 1 and Theorem 3.3 of Chandra, Li, and Rosalsky (2018) are substantially different.

In contradistinction to Theorem 1, the sequence of random elements $\{W_n, n \geqslant 1\}$ in Theorem 2 is not structurally of any specific form and the underlying Banach space is not assumed to be of Rademacher type p for any $p \in (1,2]$. Conditions are placed on the sequence of random elements $\{W_n, n \geqslant 1\}$ and on a continuous strictly increasing function g with $g(0) = 0$ and $\lim_{x\to\infty} g(x) = \infty$ for $\mathbb{E}(g(\|W_n\|)) \to 0$ to hold. Corollary 2 is the special case $g(x) = x^p$, $x \geqslant 0$ $(0 < p < \infty)$ of Theorem 2 and it extends in several directions a result appearing in Fristedt and Gray (1997, p. 110) which states that if $\{X_n, n \geqslant 1\}$ is a sequence of (real-valued) random variables such that $X_n \to 0$ almost surely (a.s.) and $\sup_{n\geqslant 1} \mathrm{Var} X_n < \infty$, then $\mathbb{E}|X_n|^p \to 0$ for $p = 1$. In Corollary 2, the Fristedt and Gray assumption $\sup_{n\geqslant 1} \mathrm{Var} X_n < \infty$ is replaced by $\sup_{n\geqslant 1} \mathbb{E}\|W_n - w_n\|^q < \infty$ for some $q > p$ and some sequence $\{w_n, n \geqslant 1\}$ in the Banach space. Morover, convergence a.s. is weakened to convergence in probability.

2. Preliminaries

Throughout, let $(\mathfrak{X}, \|\cdot\|)$ be a real separable Banach space equipped with its Borel σ-algebra $\mathscr{B}$ (= the σ-algebra generated by the class of open subsets of $\mathfrak{X}$ determined by $\|\cdot\|$) and all random elements under consideration are defined on a fixed but otherwise arbitrary probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and take values in $\mathfrak{X}$. A *random element* in $\mathfrak{X}$ is an $\mathscr{F}$-measurable transformation from Ω to the measurable space $(\mathfrak{X}, \mathscr{B})$. Let $\mathfrak{X}^*$ be the *(dual)* space of all continuous linear functionals on $\mathfrak{X}$. The symbol C is used to denote a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance.

We now review various technical definitions pertaining to an $\mathfrak{X}$-valued random element V or to the Banach space $\mathfrak{X}$ itself.

The *expected value* or *mean* of an $\mathfrak{X}$-valued random element V, denoted by $\mathbb{E}X$, is defined to be the *Pettis integral* provided it exists; that is, V has expected value $\mathbb{E}V \in \mathfrak{X}$ if $f(\mathbb{E}V) = \mathbb{E}(f(V))$ for every $f \in \mathfrak{X}^*$. If $\mathbb{E}\|V\| < \infty$, then (see, e.g., Taylor (1978, p. 40)) V has an expected value.

Let $\{R_n,\ n \geqslant 1\}$ be a *Rademacher sequence*; that is, $\{R_n,\ n \geqslant 1\}$ is a sequence of independent and identically distributed random variables with $\mathbb{P}(R_1 = 1) = \mathbb{P}(R_1 = -1) = 1/2$. Let $\mathfrak{X}^\infty = \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \times \cdots$ and define

$$\mathscr{C}(\mathfrak{X}) = \left\{ (v_1, v_2, \ldots) \in \mathfrak{X}^\infty : \ \sum_{n=1}^{\infty} R_n v_n \ \text{ converges in probability} \right\}.$$

Let $1 \leqslant p \leqslant 2$. Then $\mathfrak{X}$ is said to be of *Rademacher type* p if there exists a constant

$0 < C < \infty$ such that

$$\mathbb{E}\left\|\sum_{n=1}^{\infty} R_n v_n\right\|^p \leqslant C \sum_{n=1}^{\infty} \|v_n\|^p \text{ for all } (v_1, v_2, \ldots) \in \mathscr{C}(\mathfrak{X}).$$

Rosalsky and Volodin (2007) pointed out that the condition that $\mathfrak{X}$ is of Rademacher type p is indeed equivalent to the structurally simpler condition that there exists a constant $0 < C < \infty$ such that

$$\mathbb{E}\left\|\sum_{n=1}^{N} R_n v_n\right\|^p \leqslant C \sum_{n=1}^{N} \|v_n\|^p \text{ for all } N \geqslant 1 \text{ and } v_n \in \mathfrak{X},\ 1 \leqslant n \leqslant N.$$

Moreover, Hoffmann-Jørgensen and Pisier (1976) proved for $1 \leqslant p \leqslant 2$ that $\mathfrak{X}$ is of Rademacher type p if and only if there exists a constant $0 < C < \infty$ such that

$$\mathbb{E}\left\|\sum_{j=1}^{n} V_j\right\|^p \leqslant C \sum_{j=1}^{n} \mathbb{E}\left\|V_j\right\|^p$$

for every finite collection $\{V_1, \ldots, V_n\}$ of independent 0 mean, $\mathfrak{X}$-valued random elements.

If $\mathfrak{X}$ is of Rademacher type p for some $p \in (1,2]$, then it is of Rademacher type q for all $q \in [1,p)$. For $1 \leqslant p < \infty$, the $\mathscr{L}_p$-spaces and l_p-spaces are of Rademacher type $p \wedge 2$. Every real separable Banach space is of Rademacher type (at least) 1. Every real separable Hilbert space and real separable finite-dimensional Banach space is of Rademacher type 2.

3. Mainstream

With the preliminaries accounted for, the main results may be presented.

THEOREM 1. *Let $\{V_{n,j}, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ be an array of rowwise independent random elements in a real separable Rademacher type p ($1 < p \leqslant 2$) Banach space $\mathfrak{X}$ and suppose that $\mathbb{E}\left\|V_{n,j}\right\|^p < \infty$, $1 \leqslant j \leqslant k_n$, $n \geqslant 1$. Let $h: [0,\infty) \to [0,\infty)$ be a continuous function with*

$$h(0) = 0,\ h(x) = O(x),\ \text{and}\ \frac{h^p(x)}{x} \uparrow \quad \text{as } 0 < x \uparrow \infty. \tag{1}$$

Set $X_{n,j} = h^{-1}(\|V_{n,j}\|)$, $1 \leqslant j \leqslant k_n$, $n \geqslant 1$. Let $\{b_n, n \geqslant 1\}$, $\{c_n, n \geqslant 1\}$, and $\{d_n, n \geqslant 1\}$ be sequences of positive constants with $c_n < b_n$, $n \geqslant 1$ such that

$$\frac{1}{d_n^p} \sum_{j=1}^{k_n} \mathbb{E}\left(X_{n,j}^p I(X_{n,j} > b_n)\right) \to 0, \tag{2}$$

$$\frac{h^p(b_n)}{d_n^p b_n} \sum_{j=1}^{k_n} \mathbb{E}\left(X_{n,j} I(X_{n,j} > c_n)\right) \to 0, \tag{3}$$

$$\frac{h^p(b_n)}{d_n^p b_n} \sum_{j=1}^{k_n} \mathbb{E} X_{n,j} = O(1), \tag{4}$$

and

$$\frac{h^p(c_n)}{c_n} = o\left(\frac{h^p(b_n)}{b_n}\right). \tag{5}$$

Then

$$\mathbb{E}\left\|\frac{S_n - \mathbb{E}S_n}{d_n}\right\|^p \to 0 \tag{6}$$

where $S_n = \sum_{j=1}^n V_{n,j}$, $n \geqslant 1$ *and, a fortiori,*

$$\frac{S_n - \mathbb{E}S_n}{d_n} \xrightarrow{\mathbb{P}} 0.$$

Proof. For $1 \leqslant j \leqslant k_n$ and $n \geqslant 1$, set $U_{n,j} = V_{n,j} I(\|V_{n,j}\| \leqslant h(b_n))$ and $W_{n,j} = V_{n,j} I(\|V_{n,j}\| > h(b_n))$. Then $U_{n,j} + W_{n,j} = V_{n,j}$, $1 \leqslant j \leqslant k_n$, $n \geqslant 1$. Set $T_n = \sum_{j=1}^{k_n} U_{n,j}$, $n \geqslant 1$. We will show that

$$\frac{\sum_{j=1}^{k_n} \mathbb{E}\left\|W_{n,j}\right\|^p}{d_n^p} \to 0 \tag{7}$$

and

$$\mathbb{E}\left\|\frac{T_n - \mathbb{E}T_n}{d_n}\right\|^p \to 0. \tag{8}$$

To prove (7), note that for $1 \leqslant j \leqslant k_n$ and $n \geqslant 1$,

$$\begin{aligned} \left\|W_{n,j}\right\| &= \|V_{n,j}\| I(\|V_{n,j}\| > h(b_n)) \\ &= h(X_{n,j}) I(h(X_{n,j}) > h(b_n)) \\ &= h(X_{n,j}) I(X_{n,j} > b_n) \\ &\leqslant C X_{n,j} I(X_{n,j} > b_n) \ \text{(by (1))} \end{aligned} \tag{9}$$

and hence

$$\frac{\sum_{j=1}^{k_n} \mathbb{E}\left\|W_{n,j}\right\|^p}{d_n^p} \leqslant \frac{C\sum_{j=1}^{k_n} \mathbb{E}\left(X_{n,j}^p I(X_{n,j} > b_n)\right)}{d_n^p} \to 0$$

by (2) proving (7).

To prove (8), note that for $1 \leqslant j \leqslant k_n$ and $n \geqslant 1$,

$$\begin{aligned} \left\|U_{n,j}\right\|^p &= \left\|V_{n,j}\right\|^p I(\|V_{n,j}\| \leqslant h(b_n)) \\ &= h^p(X_{n,j}) I(h(X_{n,j}) \leqslant h(b_n)) \\ &= h^p(X_{n,j}) I(X_{n,j} \leqslant b_n) \\ &= \frac{h^p(X_{n,j})}{X_{n,j}} \cdot X_{n,j} I(0 < X_{n,j} \leqslant c_n) + \frac{h^p(X_{n,j})}{X_{n,j}} \cdot X_{n,j} I(c_n < X_{n,j} \leqslant b_n). \end{aligned}$$

Then for $n \geqslant 1$,

$$\begin{aligned}
&\frac{\mathbb{E}\,\|T_n - \mathbb{E}T_n\|^p}{d_n^p} \\
&\leqslant \frac{C\sum_{j=1}^{k_n}\mathbb{E}\left\|U_{n,j} - \mathbb{E}U_{n,j}\right\|^p}{d_n^p} \quad \text{(since } \mathfrak{X} \text{ is of Rademacher type } p\text{)} \\
&\leqslant \frac{C\sum_{j=1}^{k_n}\mathbb{E}\left\|U_{n,j}\right\|^p}{d_n^p} \quad \text{(by Jensen's inequality)} \\
&= \frac{C}{d_n^p}\left(\sum_{j=1}^{k_n}\mathbb{E}\left(\frac{h^p(X_{n,j})}{X_{n,j}} \cdot X_{n,j}\left(I(0 < X_{n,j} \leqslant c_n) + I(c_n < X_{n,j} \leqslant b_n)\right)\right)\right) \\
&\leqslant \frac{C}{d_n^p}\left(\sum_{j=1}^{k_n}\frac{h^p(c_n)}{c_n} \cdot \mathbb{E}X_{n,j} + \frac{h^p(b_n)}{b_n}\sum_{j=1}^{k_n}\mathbb{E}\left(X_{n,j}I(X_{n,j} > c_n)\right)\right) \\
&= \frac{C}{d_n^p}\left(\sum_{j=1}^{k_n}\frac{h^p(b_n)}{b_n} \cdot o(1) \cdot \mathbb{E}X_{n,j} + \frac{h^p(b_n)}{b_n}\sum_{j=1}^{k_n}\mathbb{E}\left(X_{n,j}I(X_{n,j} > c_n)\right)\right) \quad \text{(by (5))} \\
&= \frac{Ch^p(b_n)}{d_n^p b_n}\left(\sum_{j=1}^{k_n}\mathbb{E}X_{n,j}\right)o(1) + \frac{Ch^p(b_n)}{d_n^p b_n}\sum_{j=1}^{k_n}\mathbb{E}\left(X_{n,j}I(X_{n,j} > c_n)\right) \\
&= o(1) \quad \text{(by (4) and (3))}
\end{aligned}$$

proving (8).

Next, note that for $n \geqslant 1$,

$$\begin{aligned}
&\mathbb{E}\left\|\frac{\sum_{j=1}^{k_n}W_{n,j} - \sum_{j=1}^{k_n}\mathbb{E}W_{n,j}}{d_n}\right\|^p \\
&= \mathbb{E}\left\|\frac{\sum_{j=1}^{k_n}(W_{n,j} - \mathbb{E}W_{n,j})}{d_n}\right\|^p \\
&\leqslant \frac{C\sum_{j=1}^{k_n}\mathbb{E}\left\|W_{n,j} - \mathbb{E}W_{n,j}\right\|^p}{d_n^p} \quad \text{(since } \mathfrak{X} \text{ is of Rademacher type } p\text{)} \qquad (10) \\
&\leqslant \frac{C\sum_{j=1}^{k_n}\mathbb{E}\left\|W_{n,j}\right\|^p}{d_n^p} \quad \text{(by Jensen's inequality)} \\
&\leqslant \frac{C\sum_{j=1}^{k_n}\mathbb{E}\left(X_{n,j}^p I(X_{n,j} > b_n)\right)}{d_n^p} \quad \text{(by (9))} \\
&\to 0 \quad \text{(by (2))}.
\end{aligned}$$

Finally, note that for $n \geqslant 1$,

$$\frac{S_n - \mathbb{E}S_n}{d_n} = \frac{\sum_{j=1}^{k_n} U_{n,j} + \sum_{j=1}^{k_n} W_{n,j} - \sum_{j=1}^{k_n} \mathbb{E}U_{n,j} - \sum_{j=1}^{k_n} \mathbb{E}W_{n,j}}{d_n}$$

$$= \frac{\sum_{j=1}^{k_n} W_{n,j} - \sum_{j=1}^{k_n} \mathbb{E}W_{n,j}}{d_n} + \frac{T_n - \mathbb{E}T_n}{d_n}$$

and the conclusion (6) follows from (10) and (8). □

COROLLARY 1. *Let $\{V_{n,j}, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ be a uniformly bounded array of rowwise independent random elements in a real separable Rademacher type p ($1 < p \leqslant 2$) Banach space $\mathfrak{X}$ and let $\{b_n,\ n \geqslant 1\}$ be a sequence of constants with $1 < b_n \to \infty$. Then*

$$\frac{\mathbb{E}\|S_n - \mathbb{E}S_n\|^p}{k_n b_n^{p-1}} = \mathbb{E}\left\|\frac{S_n - \mathbb{E}S_n}{k_n^{\frac{1}{p}} b_n^{\frac{p-1}{p}}}\right\|^p \to 0 \tag{11}$$

and, a fortiori,

$$\frac{S_n - \mathbb{E}S_n}{k_n^{\frac{1}{p}} b_n^{\frac{p-1}{p}}} \xrightarrow{\mathbb{P}} 0$$

where $S_n = \sum_{j=1}^{n} V_{n,j}$, $n \geqslant 1$.

Proof. Let $d_n = k_n^{\frac{1}{p}} b_n^{\frac{p-1}{p}}$, $n \geqslant 1$ and $c_n = \sqrt{b_n}$, $n \geqslant 1$. Let $h(x) = x$, $x \geqslant 0$. Set

$$X_{n,j} = h^{-1}(\|V_{n,j}\|) = \|V_{n,j}\|,\ 1 \leqslant j \leqslant k_n,\ n \geqslant 1.$$

Since the array is comprised of uniformly bounded random elements, conditions (2), (3), and (4) hold. Moreover, (5) holds since $p > 1$ and $b_n \to \infty$ ensures that $c_n = o(b_n)$. The conclusion (11) follows from Theorem 1. □

The following example, which was inspired by an example of Kuczmaszewska and Szynal (1994), shows that Theorem 1 and Corollary 1 can fail if the Banach space $\mathfrak{X}$ is not of Rademacher type p where $p \in (1,2]$.

EXAMPLE 1. Consider the real separable Banach space l_1 of absolutely summable real sequences $\mathbf{v} = \{v_k, k \geqslant 1\}$ with norm $\|\mathbf{v}\| = \sum_{k=1}^{\infty} |v_k|$. It is well known that l_1 is not of Rademacher type p for every $p \in (1,2]$. Let $\mathbf{v}^{(j)}$ denote the j-th element of the standard basis in l_1, $j \geqslant 1$; that is, $\mathbf{v}^{(j)}$ in the element in l_1 having 1 for its j-th coordinate and 0 for the other coordinates. Let $p \in (1,2]$. Define an array $\{V_{n,j}, 1 \leqslant j \leqslant n, n \geqslant 1\}$ of random elements in l_1 by requiring $\{V_{n,j}, 1 \leqslant j \leqslant n, n \geqslant 1\}$ to be a rowwise independent array with

$$\mathbb{P}\left(V_{n,j} = \mathbf{v}^{(j)}\right) = \mathbb{P}\left(V_{n,j} = -\mathbf{v}^{(j)}\right) = \frac{1}{2},\ 1 \leqslant j \leqslant n,\ n \geqslant 1.$$

Set $S_n = \sum_{j=1}^{n} V_{n,j}$, $n \geqslant 1$. Let $0 < c < 1$, $b_1 = 2^{c/2}$, and $b_n = n^c$, $n \geqslant 2$. Then for $n \geqslant 2$,

$$\frac{\mathbb{E}\|S_n - \mathbb{E}S_n\|^p}{nb_n^{p-1}} = \mathbb{E}\left\|\frac{S_n - \mathbb{E}S_n}{n^{\frac{1}{p}} b_n^{\frac{p-1}{p}}}\right\|^p = \left(\frac{n}{n^{\frac{1}{p}+c\left(\frac{p-1}{p}\right)}}\right)^p = n^{(1-\alpha)p} \to \infty$$

where $\alpha = \frac{1}{p} + c\left(\frac{p-1}{p}\right) < 1$. We have shown that the conclusions of Corollary 1 and Theorem 1 fail.

REMARK 1. Example 3.2 of Chandra, Li, and Rosalsky (2018) demonstrates that the hypotheses of Corollary 1 (hence of Theorem 1) do not necessarily ensure that

$$\frac{S_n - \mathbb{E}S_n}{d_n} = \frac{S_n - \mathbb{E}S_n}{k_n^{\frac{1}{p}} b_n^{\frac{p-1}{p}}} \to 0 \text{ a.s.}$$

THEOREM 2. *Let $\{W_n, n \geqslant 1\}$ be a sequence of random elements in a real separable Banach space $\mathfrak{X}$. Let g and h be continuous strictly increasing functions defined on $[0,\infty)$ with*

$$g(0) = h(0) = 0, \quad \lim_{x\to\infty} g(x) = \infty, \quad \lim_{x\to\infty} \frac{g(x)}{h(x)} = 0. \tag{12}$$

If

$$W_n \xrightarrow{\mathbb{P}} 0 \tag{13}$$

and

$$\sup_{n\geqslant 1} \mathbb{E}\left(h(2\|W_n - w_n\|)\right) < \infty \tag{14}$$

for some sequence $\{w_n, n \geqslant 1\}$ in $\mathfrak{X}$, then

$$\mathbb{E}\left(g(\|W_n\|)\right) \to 0. \tag{15}$$

Proof. It follows from (12) that $\lim_{x\to\infty} h(x) = \infty$ and hence by (14), a number $A > 0$ can be chosen so that

$$h(2A) > 2\sup_{n\geqslant 1} \mathbb{E}\left(h(2\|W_n - w_n\|)\right).$$

Then by the Markov inequality,

$$\begin{aligned}\inf_{n\geqslant 1} \mathbb{P}\left(\|W_n - w_n\| \leqslant A\right) &= 1 - \sup_{n\geqslant 1} \mathbb{P}\left(2\|W_n - w_n\| > 2A\right) \\ &= 1 - \sup_{n\geqslant 1} \mathbb{P}\left(h(2\|W_n - w_n\|) > h(2A)\right) \\ &\geqslant 1 - \frac{1}{h(2A)} \sup_{n\geqslant 1} \mathbb{E}\left(h(2\|W_n - w_n\|)\right) \\ &> \frac{1}{2}.\end{aligned} \tag{16}$$

Now by (13), there exists a positive integer N such that

$$\mathbb{P}(\|W_n\| \leqslant 1) > \frac{1}{2} \text{ for all } n \geqslant N. \tag{17}$$

It follows from (16) and (17) that for all $n \geqslant N$,

$$\{\|W_n - w_n\| \leqslant A\} \cap \{\|W_n\| \leqslant 1\} \neq \emptyset.$$

Thus for all $n \geqslant N$, there exists a sample point ω_n such that

$$\|W_n(\omega_n) - w_n\| \leqslant A \text{ and } \|W_n(\omega_n)\| \leqslant 1$$

implying

$$\|w_n\| \leqslant \|w_n - W_n(\omega_n)\| + \|W_n(\omega_n)\| \leqslant A + 1.$$

Thus

$$\sup_{n \geqslant N} h(2\|w_n\|) \leqslant h(2(A+1)).$$

Hence the sequence $\{h(2\|w_n\|), n \geqslant 1\}$ is bounded which together with (14) ensures that

$$\begin{aligned}
\sup_{n\geqslant 1} \mathbb{E}(h(\|W_n\|)) &\leqslant \sup_{n\geqslant 1} \mathbb{E}(h(\|W_n - w_n\| + \|w_n\|)) \\
&\leqslant \sup_{n\geqslant 1} \mathbb{E}(h(2\max\{\|W_n - w_n\|, \|w_n\|\})) \\
&= \sup_{n\geqslant 1} \mathbb{E}(h(\max\{2\|W_n - w_n\|, 2\|w_n\|\})) \qquad (18) \\
&\leqslant \sup_{n\geqslant 1} \mathbb{E}(h(2\|W_n - w_n\|)) + \sup_{n\geqslant 1} h(2\|w_n\|) \\
&< \infty.
\end{aligned}$$

It will now be shown that the sequence of random variables $\{g(\|W_n\|), n \geqslant 1\}$ is uniformly integrable. Let $\varepsilon > 0$ be arbitrary and let $B = \sup_{n\geqslant 1} \mathbb{E}(h(\|W_n\|))$. Then $B < \infty$ by (18). By (12), a number $A_0 > 0$ can be chosen so that

$$\frac{g(x)}{h(x)} \leqslant \frac{\varepsilon}{B} \text{ for all } x \geqslant A_0.$$

Then for all $a \geqslant g(A_0)$,

$$\begin{aligned}
&\sup_{n\geqslant 1} \mathbb{E}\left(g(\|W_n\|)I\left(g(\|W_n\|) \geqslant a\right)\right) \\
&\leqslant \sup_{n\geqslant 1} \mathbb{E}\left(g(\|W_n\|)I\left(g(\|W_n\|) \geqslant g(A_0)\right)\right) \\
&= \sup_{n\geqslant 1} \mathbb{E}\left(\frac{g(\|W_n\|)}{h(\|W_n\|)} \cdot h(\|W_n\|)I(\|W_n\| \geqslant A_0)\right) \\
&= \sup_{n\geqslant 1} \mathbb{E}\left(\frac{\varepsilon}{B} h(\|W_n\|)\right) \\
&= \varepsilon
\end{aligned}$$

proving that the sequence of random variables $\{g(\|W_n\|), n \geqslant 1\}$ is uniformly integrable. Since g is continuous and $g(0) = 0$, it follows from (13) that

$$g\left(\|W_n\|\right) \xrightarrow{\mathbb{P}} g(0) = 0. \tag{19}$$

The conclusion (15) follows from (19) and the uniform integrability of $\{g(\|W_n\|), n \geqslant 1\}$ by the mean convergence criterion (see, e.g., Chow and Teicher (1997, p. 99)). $\square$

The following corollary establishes degenerate mean convergence of order $p > 0$ for $\{\|W_n\|, n \geqslant 1\}$.

COROLLARY 2. *Let $\{W_n, n \geqslant 1\}$ be a sequence of random elements in a real separable Banach space $\mathfrak{X}$. If*

$$W_n \xrightarrow{\mathbb{P}} 0$$

and

$$\sup_{n\geqslant 1} \mathbb{E}\|W_n - w_n\|^q < \infty$$

for some $q > 0$ and some sequence $\{w_n, n \geqslant 1\}$ in $\mathfrak{X}$, then

$$\mathbb{E}\|W_n\|^p \to 0 \text{ for all } 0 < p < q.$$

Proof. The corollary follows immediately from Theorem 2 by taking $h(x) = x^q$, $x \geqslant 0$ and $g(x) = x^p$, $x \geqslant 0$. $\square$

COROLLARY 3. *Let $\{W_n, n \geqslant 1\}$ be a sequence of random elements in a real separable Banach space $\mathfrak{X}$ and let $\{w_n, n \geqslant 1\}$ be an unbounded sequence in $\mathfrak{X}$. If $W_n \xrightarrow{\mathbb{P}} 0$, then*

$$\sup_{n\geqslant 1} \mathbb{E}\|W_n - w_n\|^q = \infty \text{ for all } q > 0.$$

Proof. If $\sup_{n\geqslant 1}\mathbb{E}\|W_n - w_n\|^q < \infty$ for some $q > 0$, then by taking $h(x) = x^q$, $x \geqslant 0$, necessarily the sequence $\{w_n, n \geqslant 1\}$ is bounded as was shown in the proof of Theorem 2, a contradiction. □

The following example shows that under the hypotheses of Corollary 2,

$$\mathbb{E}\|W_n\|^q \to 0$$

does not necessarily hold.

EXAMPLE 2. Let the Banach space $(\mathfrak{X}, \|\cdot\|)$ be $(\mathbb{R}, |\cdot|)$ and let $\{W_n, n \geqslant 1\}$ be a sequence of random variables where $\mathbb{P}(W_1 = 0) = 1$ and for $n \geqslant 2$,

$$\mathbb{P}(W_n = n) = \mathbb{P}(W_n = -n) = \frac{1}{n}, \ \ \mathbb{P}(W_n = 0) = 1 - \frac{2}{n}.$$

Let $q = 1$ and $w_n = 0$, $n \geqslant 1$. It is clear that

$$W_n \xrightarrow{\mathbb{P}} 0 \ \text{ and } \ \sup_{n\geqslant 1}\mathbb{E}|W_n - w_n|^q = 2 < \infty.$$

But for $n \geqslant 2$, $\mathbb{E}|W_n|^q = 2 \nrightarrow 0$.

REFERENCES

[1] T. K. CHANDRA, D. LI AND A. ROSALSKY, *Some mean convergence theorems for arrays of rowwise pairwise negative quadrant dependent random variables*, J. Inequal. Appl., **2018:221** (2018).

[2] Y. S. CHOW AND H. TEICHER, *Probability Theory: Independence, Interchangeability, Martingales*, Third edition, Springer-Verlag, New York, 1997.

[3] B. FRISTEDT AND L. GRAY, *A Modern Approach to Probability Theory*, Birkhäuser, Boston, 1997.

[4] J. HOFFMANN-JØRGENSEN AND G. PISIER, *The law of large numbers and the central limit theorem in Banach spaces*, Ann. Probab. **4** (1976), 587–599.

[5] A. KUCZMASZEWSKA AND D. SZYNAL, *On complete convergence in a Banach space*, Internat. J. Math. Math. Sci. **17** (1994), 1–14.

[6] A. ROSALSKY AND A. VOLODIN, *On the weak law with random indices for arrays of Banach space valued random elements*, Sankhyā **69** (2007), 330–343.

[7] R. L. TAYLOR, *Convergence of Weighted Sums of Random Elements in Linear Spaces*, Lecture Notes in Mathematics, vol. **672**, Springer-Verlag, Berlin, 1978.

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