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SHARP INEQUALITIES FOR THE TOADER MEAN OF ORDER -1 IN TERMS OF OTHER BIVARIATE MEANS

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Abstract. In the article, we present the best possible parameters α_1, α_2, α_3, α_4, β_1, β_2, β_3, $\beta_4 \in \mathbb{R}$ such that the double inequalities

$$\frac{\alpha_1}{H(a,b)}+\frac{1-\alpha_1}{G(a,b)}<\frac{1}{T_{-1}(a,b)}<\frac{\beta_1}{H(a,b)}+\frac{1-\beta_1}{G(a,b)},$$
$$\frac{\alpha_2}{H(a,b)}+\frac{1-\alpha_2}{A(a,b)}<\frac{1}{T_{-1}(a,b)}<\frac{\beta_2}{H(a,b)}+\frac{1-\beta_2}{A(a,b)},$$
$$\frac{\alpha_3}{H(a,b)}+\frac{1-\alpha_3}{L(a,b)}<\frac{1}{T_{-1}(a,b)}<\frac{\beta_3}{H(a,b)}+\frac{1-\beta_3}{L(a,b)},$$
$$\frac{\alpha_4}{H(a,b)}+\frac{1-\alpha_4}{P(a,b)}<\frac{1}{T_{-1}(a,b)}<\frac{\beta_4}{H(a,b)}+\frac{1-\beta_4}{P(a,b)}$$

hold for all $a,b>0$ with $a\neq b$, and provide several new bounds for the complete elliptic integral of the second kind, where $T_{-1}(a,b)=\left(\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^{-1}\cos^2\theta+b^{-1}\sin^2\theta}d\theta\right)^2$ is the Toader mean of order -1, and $H(a,b)=2ab/(a+b)$, $G(a,b)=\sqrt{ab}$, $L(a,b)=(a-b)/(\log a-\log b)$, $P(a,b)=(a-b)/[2\arcsin((a-b)/(a+b))]$ and $A(a,b)=(a+b)/2$ are the harmonic, geometric, logarithmic, Seiffert and arithmetic means, respectively.

1. Introduction

For $r\in(0,1)$, Legendre's complete elliptic integrals of the first kind $\mathscr{K}(r)$ and second kind $\mathscr{E}(r)$ [1, 2, 9, 11, 16, 17, 25, 28, 31, 34, 35, 36, 40, 43, 44, 45, 47, 52, 57, 58] are defined by

$$\mathscr{K}=\mathscr{K}(r)=\int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta,\quad \mathscr{E}=\mathscr{E}(r)=\int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta. \tag{1.1}$$

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It is well-known that $\mathscr{K}(r)$ and $\mathscr{E}(r)$ are the special cases of the Gaussian hypergeometric function [21, 22, 23, 29, 30, 32, 33, 37, 38, 51, 59]

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}\quad(|x|<1),$$

where $(a)_0=1$ for $a\neq 0$ and $(a)_n=a(a+1)(a+2)\cdots(a+n-1)=\Gamma(a+n)/\Gamma(a)$ is the shifted factorial function and $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dx$ is the gamma function [24, 46, 50]. Indeed,

$$\mathscr{K}(r)=\frac{\pi}{2}F(1/2,1/2;1;r^2),\quad \mathscr{E}(r)=\frac{\pi}{2}F(-1/2,1/2;1;r^2).$$

We clearly see that the function $r\mapsto\mathscr{K}(r)$ is strictly increasing from $(0,1)$ onto $(\pi/2,\infty)$ and the function $r\mapsto\mathscr{E}(r)$ is strictly decreasing from $(0,1)$ onto $(1,\pi/2)$. $\mathscr{K}(r)$ and $\mathscr{E}(r)$ satisfy the derivative formulas and Landen identities (See [2, Appendix E. pp. 474–475])

$$\frac{d\mathscr{K}(r)}{dr}=\frac{\mathscr{E}(r)-r'^2\mathscr{K}(r)}{rr'^2},\qquad \frac{d\mathscr{E}(r)}{dr}=\frac{\mathscr{E}(r)-\mathscr{K}(r)}{r},\tag{1.2}$$

$$\mathscr{K}\left(\frac{2\sqrt{r}}{1+r}\right)=(1+r)\mathscr{K}(r),\quad \mathscr{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2\mathscr{E}(r)-r'^2\mathscr{K}(r)}{1+r},\tag{1.3}$$

where and what follows we denote by $r'=\sqrt{1-r^2}$ for $r\in(0,1)$.

There is a close relationship between complete elliptic integrals and bivariate means. Many mathematicians established various asymptotic bounds for $\mathscr{K}(r)$ and $\mathscr{E}(r)$ by studying their related bivariate means in the past few years [6, 7, 12, 13, 14, 19, 20, 27, 39, 41, 42, 48, 49, 53, 54, 55, 56].

Let $a,b>0$. Then the harmonic mean $H(a,b)$, geometric mean $G(a,b)$, arithmetic mean $A(a,b)$, logarithmic mean $L(a,b)$, Seiffert mean $P(a,b)$ and arithmetic-geometric mean $AGM(a,b)$ are defined by

$$H(a,b)=\frac{2ab}{a+b},\quad G(a,b)=\sqrt{ab},\quad A(a,b)=\frac{a+b}{2},$$

$$L(a,b)=\begin{cases}\dfrac{a-b}{\log a-\log b}, & a\neq b\\ a, & a=b,\end{cases}\qquad P(a,b)=\begin{cases}\dfrac{a-b}{2\arcsin[(a-b)/(a+b)]}, & a\neq b\\ a, & a=b,\end{cases}$$

and

$$AGM(a,b)=\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n,$$

respectively, where the sequences $\{a_n\}$ and $\{b_n\}$ are defined by

$$a_0=a,\quad b_0=b,$$

$$a_{n+1}=A(a_n,b_n)=\frac{a_n+b_n}{2},\quad b_{n+1}=G(a_n,b_n)=\sqrt{a_nb_n}.$$

It is well-known that the inequalities

$$H(a,b) < G(a,b) < L(a,b) < AGM(a,b) < P(a,b) < A(a,b)$$

hold for all $a,b > 0$ with $a \neq b$, and the arithmetic-geometric mean $AGM(a,b)$ can be expressed by

$$AGM(a,b) = \frac{\pi/2}{\int_0^{\pi/2} (a^2\cos^2\theta + b^2\sin^2\theta)^{-1/2} d\theta} = \begin{cases} \pi a \Big/ \left[2\mathscr{K}\left(\sqrt{1-(b/a)^2}\right)\right], & a \geqslant b, \\ \pi b \Big/ \left[2\mathscr{K}\left(\sqrt{1-(a/b)^2}\right)\right], & a < b. \end{cases}$$

In particular the Gauss identity (See [2, Theorem 4.4]) shows that

$$AGM(1,r)\mathscr{K}(r') = \frac{\pi}{2}.$$

More properties for $AGM(a,b)$ and its application to the circumference rate calculation can be found in the literatures [5, 15].

Chu and Wang [8, 10] gave the bounds for the perimeter of an ellipse by use of the Toader mean [26]

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\, d\theta = \begin{cases} 2a\mathscr{E}\left(\sqrt{1-(b/a)^2}\right)/\pi, & a \geqslant b, \\ 2b\mathscr{E}\left(\sqrt{1-(a/b)^2}\right)/\pi, & a < b, \end{cases} \tag{1.4}$$

and established several sharp inequalities for $T(a,b)$ in terms of other classical means. As applications, numerous new asymptotic upper and lower bounds for the perimeter of an ellipse and the complete elliptic integral of the second kind $\mathscr{E}(r)$ were derived, which improve some previous well-known results in [3, 4]. Very recently, Chu et al. [12] extended $T(a,b)$ to the one-parameter Toader mean $T_n(a,b)$ for $n \in \mathbb{Z}$, called Toader mean of order n, given by

$$T_n(a,b) = \begin{cases} \left(\frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^n\cos^2\theta + b^n\sin^2\theta}\, d\theta\right)^{2/n}, & n \neq 0, \\ \sqrt{ab}, & n = 0. \end{cases}$$

According to (1.4), T_n can be rewritten by

$$T_n(a,b) = \begin{cases} a\left[\frac{2}{\pi}\mathscr{E}\left(\sqrt{1-(b/a)^n}\right)\right]^{2/n}, & a^n \geqslant b^n, \\ b\left[\frac{2}{\pi}\mathscr{E}\left(\sqrt{1-(a/b)^n}\right)\right]^{2/n}, & a^n < b^n. \end{cases} \tag{1.5}$$

In particular, $T_2(a,b)=T(a,b)$, and $T_1(a,b)=E(a,b)$ is the quasi-arithmetic mean which has been investigated in [18, 60].

In this paper, we focus on studying the Toader mean of order -1 and establishing several double inequalities for $T_{-1}(a,b)$ in terms of other bivariate means. Specifically, we would like to find the optimal constants α_1, α_2, α_3, α_4, β_1, β_2, β_3, $\beta_4\in\mathbb{R}$ such that

$$\frac{\alpha_1}{H(a,b)}+\frac{1-\alpha_1}{G(a,b)}<\frac{1}{T_{-1}(a,b)}<\frac{\beta_1}{H(a,b)}+\frac{1-\beta_1}{G(a,b)},$$

$$\frac{\alpha_2}{H(a,b)}+\frac{1-\alpha_2}{A(a,b)}<\frac{1}{T_{-1}(a,b)}<\frac{\beta_2}{H(a,b)}+\frac{1-\beta_2}{A(a,b)},$$

$$\frac{\alpha_3}{H(a,b)}+\frac{1-\alpha_3}{L(a,b)}<\frac{1}{T_{-1}(a,b)}<\frac{\beta_3}{H(a,b)}+\frac{1-\beta_3}{L(a,b)},$$

$$\frac{\alpha_4}{H(a,b)}+\frac{1-\alpha_4}{P(a,b)}<\frac{1}{T_{-1}(a,b)}<\frac{\beta_4}{H(a,b)}+\frac{1-\beta_4}{P(a,b)}$$

hold for all $a,b>0$ with $a\neq b$. From these results, some new bounds for the complete elliptic integral of the second kind can be discovered.

2. Lemmas

LEMMA 2.1. [2, Theorem 1.25] Suppose that $a,b\in(-\infty,\infty)$, $f,g:[a,b]\to\mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b), and $g'\neq 0$ on (a,b). If f'/g' is increasing (decreasing) on (a,b), then so are the functions

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If f'/g' is strictly monotone, then the monotonicity in the conclusion is also strict.

The following Lemma 2.2 can be found in [2, Theorem 3.21(8) and Exercise 3.43(11), (16), (29)] and [40, Lemma 2.3].

LEMMA 2.2. *Let* $r\in(0,1)$. *Then the following statements are true:*

(1) *The function* $r\mapsto r'^{c}\mathscr{E}(r)$ *is strictly increasing from* $(0,1)$ *onto* $(0,\pi/2)$ *if* $c\leqslant -1/2$;

(2) *The function* $r\mapsto[\mathscr{K}(r)-\mathscr{E}(r)]/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4,\infty)$;

(3) *The function* $r\mapsto[(1+r'^2)\mathscr{K}(r)-2\mathscr{E}(r)]/r^4$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/16,\infty)$;

(4) *The function* $r\mapsto[(\mathscr{K}(r)-\mathscr{E}(r))^2-2\mathscr{E}(r)\left((1+r'^2)\mathscr{K}(r)-2\mathscr{E}(r)\right)]/r^6$ *is strictly increasing from* $(0,1)$ *onto* $(\pi^2/64,\infty)$,

(5) *The function* $r\mapsto[\mathscr{E}^2(r)-r'^2\mathscr{K}^2(r)]/r^4$ *is strictly increasing from* $(0,1)$ *onto* $(\pi^2/32,1)$.

LEMMA 2.3. *The function*

$$\varphi(r)=[\mathscr{K}(r)-\mathscr{E}(r)][\mathscr{E}(r)-r'^2\mathscr{K}(r)]-2[\mathscr{E}^2(r)-r'^2\mathscr{K}^2(r)]$$

is strictly increasing from $(0,1)$ *onto* $(0,\infty)$.

Proof. Clearly $\varphi(0^+)=0$ and $\varphi(1^-)=\infty$. Elaborated computations lead to

$$\begin{aligned}\frac{d\varphi(r)}{dr}&=\frac{r\mathscr{E}}{r'^2}(\mathscr{E}-r'^2\mathscr{K})+r\mathscr{K}(\mathscr{K}-\mathscr{E})\\&\quad-2\left[\frac{2\mathscr{E}(\mathscr{E}-\mathscr{K})}{r}+2r\mathscr{K}^2-\frac{2\mathscr{K}\left(\mathscr{E}-r'^2\mathscr{K}\right)}{r}\right]\\&=\frac{\varphi_1(r)}{rr'^2},\end{aligned}\tag{2.1}$$

where

$$\varphi_1(r)=r^4\mathscr{E}^2-(3+r'^2)r'^2(\mathscr{K}-\mathscr{E})^2.$$

Note that

$$\varphi_1(0)=0,\tag{2.2}$$

$$\begin{aligned}\frac{d\varphi_1(r)}{dr}&=2r[9\mathscr{E}^2-(10+4r'^2)\mathscr{K}\mathscr{E}+(3+2r'^2)\mathscr{K}^2]\\&=2r^7\left\{\frac{2(\mathscr{K}-\mathscr{E})}{r^2}\frac{[(1+r'^2)\mathscr{K}-2\mathscr{E}]}{r^4}+\frac{(\mathscr{K}-\mathscr{E})^2-2\mathscr{E}[(1+r'^2)\mathscr{K}-2\mathscr{E}]}{r^6}\right\}.\end{aligned}\tag{2.3}$$

Making use of Lemma 2.2(2), (3) and (4), we conclude that $\varphi_1(r)$ is strictly increasing on $(0,1)$. Finally, combining with (2.1) and (2.2), the monotonicity of $\varphi(r)$ on $(0,1)$ is obtained. □

LEMMA 2.4. *The function*

$$\gamma(r)=\frac{\mathscr{E}(r)[(1+r'^2)\mathscr{E}(r)-2r'^2\mathscr{K}(r)]}{r^4}$$

is strictly increasing from $(0,1)$ *onto* $(3\pi^2/32,1)$.

Proof. Clearly $\gamma(1^-)=1$. Since

$$\frac{d[(1+r'^2)\mathscr{E}-2r'^2\mathscr{K}]}{dr}=3r(\mathscr{K}-\mathscr{E}),$$

differentiating $\gamma(r)$ yields

$$\begin{aligned}\frac{d\gamma(r)}{dr} &= \frac{\left\{-(\mathscr{K}-\mathscr{E})[(1+r'^2)\mathscr{E}-2r'^2\mathscr{K}]/r+3\mathscr{E}r(\mathscr{K}-\mathscr{E})\right\}r^4}{r^8} \\ &\quad -\frac{\mathscr{E}[(1+r'^2)\mathscr{E}-2r'^2\mathscr{K}]4r^3}{r^8} \\ &= \frac{3r^2\mathscr{E}(\mathscr{K}-\mathscr{E})-(\mathscr{K}+3\mathscr{E})[(1+r'^2)\mathscr{E}-2r'^2\mathscr{K}]}{r^5} \\ &= \frac{2[(1-r^2)\mathscr{K}^2+(2-r^2)\mathscr{E}\mathscr{K}-3\mathscr{E}^2]}{r^5} \\ &= \frac{2}{r^5}[(\mathscr{K}-\mathscr{E})(\mathscr{E}-r'^2\mathscr{K})-2(\mathscr{E}^2-r'^2\mathscr{K}^2)] = \frac{2}{r^5}\varphi(r), \end{aligned} \tag{2.4}$$

where $\varphi(r)$ is defined as in Lemma 2.3. Thus the monotonicity of $\gamma(r)$ follows easily from (2.4) and Lemma 2.3. By Lemma 2.2(3), we get $\gamma(0^+) = 3\pi^2/32$. □

LEMMA 2.5. *The function*

$$\eta(r) = \frac{\mathscr{E}^2(r)[\mathscr{E}^2(r)-r'^2\mathscr{K}^2(r)]}{r^4}$$

is strictly increasing from $(0,1)$ *onto* $(\pi^4/128, 1)$.

Proof. Clearly $\eta(1^-) = 1$. Since

$$\frac{d(\mathscr{E}^2-r'^2\mathscr{K}^2)}{dr} = 2\frac{(\mathscr{K}-\mathscr{E})^2}{r},$$

differentiating $\eta(r)$ yields

$$\begin{aligned}\frac{d\eta(r)}{dr} &= \frac{\left\{2\mathscr{E}(\mathscr{E}-\mathscr{K})[\mathscr{E}^2-r'^2\mathscr{K}^2]/r+2\mathscr{E}^2(\mathscr{K}-\mathscr{E})^2/r\right\}r^4-\mathscr{E}^2[\mathscr{E}^2-r'^2\mathscr{K}^2]4r^3}{r^8} \\ &= \frac{2\mathscr{E}}{r^5}\left[-(\mathscr{K}-\mathscr{E})(\mathscr{E}^2-r'^2\mathscr{K}^2)+\mathscr{E}(\mathscr{K}-\mathscr{E})^2-2\mathscr{E}(\mathscr{E}^2-r'^2\mathscr{K}^2)\right] \\ &= \frac{2\mathscr{K}\mathscr{E}}{r^5}\varphi(r), \end{aligned} \tag{2.5}$$

where $\varphi(r)$ is defined as in Lemma 2.3. It follows from Lemma 2.3 and (2.5) that $\eta(r)$ is strictly increasing on $(0,1)$. Moreover, by Lemma 2.2(5), one has $\eta(0^+) = \pi^4/128$. □

LEMMA 2.6. *The function*

$$\lambda(r) = \frac{(1+r'^2)^3}{r'^2\mathscr{E}^2(r)}$$

is strictly increasing from $(0,1)$ *onto* $(32/\pi^2, \infty)$.

Proof. Clearly $\lambda(0^+) = 32/\pi^2$ and $\lambda(1^-) = \infty$. Differentiating $\lambda(r)$ yields

$$\begin{aligned}
\frac{d\lambda(r)}{dr} &= \frac{-6r\left(1+r'^2\right)^2 r'^2\mathscr{E}^2 - \left(1+r'^2\right)^3[-2r\mathscr{E}^2 + 2r'^2\mathscr{E}(\mathscr{E}-\mathscr{K})/r]}{r'^4\mathscr{E}^4} \\
&= \frac{2\left(1+r'^2\right)^2}{rr'^4\mathscr{E}^3}[-3r^2r'^2\mathscr{E} + r^2(1+r'^2)\mathscr{E} + (1+r'^2)r'^2(\mathscr{K}-\mathscr{E})] \\
&= \frac{2\left(1+r'^2\right)^2}{rr'^4\mathscr{E}^3}[r^2\mathscr{E}(1-2r'^2) + (1+r'^2)r'^2(\mathscr{K}-\mathscr{E})] \\
&= \frac{2\left(1+r'^2\right)^2}{rr'^4\mathscr{E}^3}\{r'^2[(1+r'^2)\mathscr{K} - 2\mathscr{E}] + r^4\mathscr{E}\} \\
&= \frac{2r^3\left(1+r'^2\right)^2}{r'^2\mathscr{E}^3}\left[\frac{(1+r'^2)\mathscr{K} - 2\mathscr{E}}{r^4} + \frac{\mathscr{E}}{r'^2}\right]. \qquad (2.6)
\end{aligned}$$

Therefore, Lemma 2.6 follows easily from (2.6) together with Lemma 2.2(1) and (3). □

LEMMA 2.7. *The function*

$$\mu(r) = \frac{8\mathscr{E}(r)[\mathscr{K}(r) - \mathscr{E}(r)]/(\pi^2 r^2) - 1}{4/(1+r'^2)^2 - 1}$$

is strictly increasing from $(0,1)$ *onto* $(1/8, \infty)$.

Proof. Let $\mu_1(r) = 8\mathscr{E}(r)[\mathscr{K}(r) - \mathscr{E}(r)]/(\pi^2 r^2) - 1$ and $\mu_2(r) = 4/\left(1+r'^2\right)^2 - 1$. Then $\mu(r) = \mu_1(r)/\mu_2(r)$, $\mu_1(0) = \mu_2(0) = 0$, and

$$\mu_1{}'(r) = \frac{8}{\pi^2}\frac{\left[-(\mathscr{K}-\mathscr{E})^2/r + r\mathscr{E}^2/r'^2\right]r^2 - \mathscr{E}(\mathscr{K}-\mathscr{E})2r}{r^4} = \frac{8}{\pi^2}\frac{\mathscr{E}^2 - r'^2\mathscr{K}^2}{r'^2r^3},$$

$$\mu_2{}'(r) = \frac{16r}{\left(1+r'^2\right)^3},$$

and consequently

$$\frac{\mu_1'(r)}{\mu_2'(r)} = \frac{1}{2\pi^2}\frac{\left(1+r'^2\right)^3(\mathscr{E}^2 - r'^2\mathscr{K}^2)}{r'^2r^4} = \frac{1}{2\pi^2}\eta(r)\lambda(r), \qquad (2.7)$$

where $\eta(r)$ and $\lambda(r)$ are defined as in Lemma 2.5 and 2.6.

It follows from Lemmas 2.5 and 2.6 together with equation (2.7) that $\mu_1'(r)/\mu_2'(r)$ is strictly increasing on $(0,1)$, so that $\mu(r)$ is also strictly increasing on $(0,1)$ by Lemma 2.1. Clearly $\mu(1^-)=\infty$, and by l'Hôptial's rule we obtain

$$\lim_{r\to0^+}\mu(r)=\lim_{r\to0^+}\mu_1'(r)/\mu_2'(r)=1/8. \quad \square$$

LEMMA 2.8. *The function*

$$\zeta(r)=\frac{4\mathscr{E}(r)[(1+r'^2)\mathscr{E}(r)-r'^2\mathscr{K}(r)]/\pi^2-r'^2}{r^4}$$

is strictly decreasing from $(0,1)$ *onto* $(4/\pi^2,13/32)$.

Proof. Clearly $\zeta(1^-)=4/\pi^2$. Since

$$\frac{d[(1+r'^2)\mathscr{E}-r'^2\mathscr{K}]}{dr}=\frac{\mathscr{E}-3r^2\mathscr{E}-\mathscr{K}+2r^2\mathscr{K}}{r},$$

differentiating $\zeta(r)$ yields

$$\frac{d\zeta(r)}{d(r)}=-\frac{2}{\pi^2r^5}\zeta_1(r), \tag{2.8}$$

where

$$\zeta_1(r)=10\mathscr{E}^2-2r'^2\mathscr{K}^2-\pi^2(1+r'^2).$$

Simple computations lead to

$$\zeta_1(0)=0, \tag{2.9}$$

$$\frac{d\zeta_1(r)}{d(r)}=\frac{4[\mathscr{K}(r)-\mathscr{E}(r)][\mathscr{K}(r)-5\mathscr{E}(r)]}{r}+2\pi^2r=:2\pi^2r\zeta_2(r), \tag{2.10}$$

where

$$\zeta_2(r)=\frac{2[\mathscr{K}(r)-\mathscr{E}(r)][\mathscr{K}(r)-5\mathscr{E}(r)]}{\pi^2r^2}+1,$$

$$\zeta_2(0)=0, \tag{2.11}$$

$$\frac{d\zeta_2(r)}{dr}=\frac{4}{\pi^2r'^2r^3}\varphi(r), \tag{2.12}$$

where $\varphi(r)$ is defined as in Lemma 2.3 and is positive on $(0,1)$. Hence the monotonicity of $\zeta(r)$ on $(0,1)$ can be derived by (2.8)–(2.12). Finally, using Maclaurin expansions of $\mathscr{K}(r)$ and $\mathscr{E}(r)$ we get

$$\begin{aligned}&\lim_{r\to0^+}\frac{4\mathscr{E}[(1+r'^2)\mathscr{E}-r'^2\mathscr{K}]/\pi^2-r'^2}{r^4}\\&=\lim_{r\to0^+}\frac{4/\pi^2\cdot(\pi^2/4)\left[1-1/4\cdot r^2-3/64\cdot r^4+o(r^4)\right]\left[1-3/4\cdot r^2+17/64\cdot r^4+o(r^4)\right]-1+r^2}{r^4}\\&=\frac{13}{32}. \quad \square\end{aligned}$$

3. Main results

THEOREM 3.1. *The double inequality*

$$\frac{\alpha_1}{H(a,b)}+\frac{1-\alpha_1}{G(a,b)}<\frac{1}{T_{-1}(a,b)}<\frac{\beta_1}{H(a,b)}+\frac{1-\beta_1}{G(a,b)}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_1\leqslant 3/4$ *and* $\beta_1\geqslant 8/\pi^2=0.8105\cdots$.

Proof. Since $H(a,b)$, $G(a,b)$ and $T_{-1}(a,b)$ are symmetric and homogeneous of degree one with respect to a and b, without loss of generality, we assume that $a=1>b=r'^2$ $(0<r<1)$. Then simple computations lead to

$$H(a,b)=A(a,b)\frac{4r'^2}{(1+r'^2)^2},\quad G(a,b)=A(a,b)\frac{2r'}{1+r'^2}, \tag{3.1}$$

$$T_{-1}(a,b)=A(a,b)\frac{2b}{(a+b)\left[2\mathscr{E}(\sqrt{1-b/a})/\pi\right]^2}=A(a,b)\frac{\pi^2r'^2}{2(1+r'^2)\mathscr{E}^2(r)}. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\frac{1/T_{-1}(a,b)-1/G(a,b)}{1/H(a,b)-1/G(a,b)}=\frac{4\mathscr{E}^2(r)/\pi^2-r'}{(1+r'^2)/2-r'}=\frac{4\mathscr{E}^2(r)/(\pi^2r')-1}{(1+r'^2)/(2r')-1}=:f(r). \tag{3.3}$$

Let $f_1(r)=4\mathscr{E}^2(r)/(\pi^2r')-1$ and $f_2(r)=(1+r'^2)/(2r')-1$. Then $f_1(0)=f_2(0)=0$, $f(r)=f_1(r)/f_2(r)$,

$$f_1'(r)=\frac{4}{\pi}\frac{2r'^2\mathscr{E}(\mathscr{E}-\mathscr{K})+r^2\mathscr{E}}{rr'^3},$$

$$f_2'(r)=\frac{1}{2}\left(\frac{r}{r'}\right)^3,$$

and thereby

$$\frac{f_1{}'(r)}{f_2{}'(r)}=\frac{8}{\pi^2}\frac{\mathscr{E}(r)[(1+r'^2)\mathscr{E}(r)-2r'^2\mathscr{K}(r)]}{r^4}=\frac{8}{\pi^2}\gamma(r), \tag{3.4}$$

where $\gamma(r)$ is defined as in Lemma 2.4.

Lemma 2.4 and (3.4) imply that $f_1{}'(r)/f_2{}'(r)$ is strictly increasing on $(0,1)$, so is $f(r)$ by Lemma 2.1. Note that

$$\lim_{r\to0^+}f(r)=\lim_{r\to0^+}\frac{f_1{}'(r)}{f_2{}'(r)}=\frac{3}{4},\quad \lim_{r\to1^-}f(r)=\frac{8}{\pi^2}=0.8105\cdots. \tag{3.5}$$

Therefore, Theorem 3.1 follows easily from (3.3) and (3.5) together with the monotonicity of $f(r)$ on the interval $(0,1)$. $\square$

THEOREM 3.2. *The double inequality*

$$\frac{\alpha_2}{H(a,b)} + \frac{1-\alpha_2}{A(a,b)} < \frac{1}{T_{-1}(a,b)} < \frac{\beta_2}{H(a,b)} + \frac{1-\beta_2}{A(a,b)}$$

holds for all $a,b>0$ *with* $a \neq b$ *if and only if* $\alpha_2 \leqslant 8/\pi^2$ *and* $\beta_2 \geqslant 7/8$.

Proof. Without loss of generality, we suppose that $a=1>b=r'^2$ for $r \in (0,1)$. Then from (3.1) and (3.2) we have

$$\frac{1/T_{-1}(a,b)-1/A(a,b)}{1/H(a,b)-1/A(a,b)} = \frac{4\mathscr{E}^2(r)/\pi^2 - 2r'^2/(1+r'^2)}{(1+r'^2)/2-2r'^2/(1+r'^2)} =: g(r). \tag{3.6}$$

Let $g_1(r)=4\mathscr{E}^2(r)/\pi^2-2r'^2/(1+r'^2)$, $g_2(r)=(1+r'^2)/2-2r'^2/(1+r'^2)$. Then $g_1(0)=g_2(0)=0$, $g(r)=g_1(r)/g_2(r)$ and

$$g_1{}'(r) = \frac{4}{\pi^2}\frac{2\mathscr{E}(\mathscr{E}-\mathscr{K})}{r} + \frac{4r}{(1+r'^2)^2}, \quad g_2{}'(r) = \frac{4r}{(1+r'^2)^2} - r,$$

and thereby

$$\frac{g_1{}'(r)}{g_2{}'(r)} = 1-\mu(r), \tag{3.7}$$

where $\mu(r)$ is defined as Lemma 2.7 and it is strictly increasing on $(0,1)$. By (3.7) we conclude that $g_1{}'(r)/g_2{}'(r)$ is strictly decreasing on $(0,1)$. Applying Lemma 2.1, the same monotonicity property of $g(r)$ is obtained. Moreover,

$$\lim_{r\to 0^+} g(r) = \lim_{r\to o^+} \frac{g_1{}'(r)}{g_2{}'(r)} = \frac{7}{8}, \quad \lim_{r\to 1^-} g(r) = \frac{8}{\pi^2} = 0.8105\cdots. \tag{3.8}$$

Therefore, Theorem 3.2 follows easily from (3.6) and (3.8) together with the monotonicity of $g(r)$ on the interval $(0,1)$. $\square$

THEOREM 3.3. *The double inequality*

$$\frac{\alpha_3}{H(a,b)} + \frac{1-\alpha_3}{L(a,b)} < \frac{1}{T_{-1}(a,b)} < \frac{\beta_3}{H(a,b)} + \frac{1-\beta_3}{L(a,b)}$$

holds for all $a,b>0$ *with* $a \neq b$ *if and only if* $\alpha_3 \leqslant 8/\pi^2$ *and* $\beta_3 \geqslant 13/16$.

Proof. Suppose that $a=1>b=r'^2 \in (0,1)$. Then simple computations leads to

$$L(a,b) = \frac{A(a,b)r^2}{(1+r'^2)\log(1/r')}, \tag{3.9}$$

$$\frac{1/T_{-1}(a,b)-1/L(a,b)}{1/H(a,b)-1/L(a,b)} = \frac{2r^2\mathscr{E}^2(r)/(\pi^2 r'^2)-\log(1/r')}{r^2(1+r'^2)/(4r'^2)-\log(1/r')} =: h(r). \tag{3.10}$$

Let $h_1(r)=2r^2\mathscr{E}^2(r)/(\pi^2r'^2)-\log(1/r')$, $\quad h_2(r)=r^2(1+r'^2)/(4r'^2)-\log(1/r')$. Then $h_1(0)=h_2(0)=0$, $h(r)=h_1(r)/h_2(r)$, $h_2{}'(r)=r^5/(2r'^4)$,

$$\begin{aligned}h_1{}'(r)&=\frac{2}{\pi^2}\left(\frac{2r\mathscr{E}}{r'}\right)\frac{[\mathscr{E}+(\mathscr{E}-\mathscr{K})]r'+r^2\mathscr{E}/r'}{r'^2}-\frac{r}{r'^2}\\&=\frac{4}{\pi^2}\frac{r\mathscr{E}}{r'^4}[(1+r'^2)\mathscr{E}-r'^2\mathscr{K}]-\frac{r}{r'^2},\end{aligned}$$

and thereby

$$\frac{h_1{}'(r)}{h_2{}'(r)}=2\zeta(r), \tag{3.11}$$

where $\zeta(r)$ is defined by Lemma 2.8.

It follows from Lemma 2.8 and (3.11) that $h_1{}'(r)/h_2{}'(r)$ is strictly decreasing on $(0,1)$. So is $h(r)$ by application of Lemma 2.1. Note that

$$\lim_{r\to0^+}h(r)=\lim_{r\to0^+}\frac{h_1{}'(r)}{h_2{}'(r)}=\frac{13}{16},\quad \lim_{r\to1^-}h(r)=\frac{8}{\pi^2}=0.8105\cdots. \tag{3.12}$$

Therefore, Theorem 3.3 follows easily from (3.10) and (3.12) together with the monotonicity of $h(r)$ on the interval $(0,1)$. $\square$

THEOREM 3.4. *The double inequality*

$$\frac{\alpha_4}{H(a,b)}+\frac{1-\alpha_4}{P(a,b)}<\frac{1}{T_{-1}(a,b)}<\frac{\beta_4}{H(a,b)}+\frac{1-\beta_4}{P(a,b)}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_4\leqslant 8/\pi^2$ *and* $\beta_4\geqslant 17/20$.

Proof. Let $r\in(0,1)$, $a=1>b=r'^2\in(0,1)$. Then by simple computations one has

$$P(a,b)=\frac{A(a,b)r^2}{(1+r'^2)\arcsin[r^2/(1+r'^2)]} \tag{3.13}$$

and

$$\frac{1/T_{-1}(a,b)-1/P(a,b)}{1/H(a,b)-1/P(a,b)}=\frac{2r^2\mathscr{E}^2(r)/(\pi^2r'^2)-\arcsin\left[r^2/(1+r'^2)\right]}{r^2(1+r'^2)/(4r'^2)-\arcsin[r^2/(1+r'^2)]}=:1-k(r), \tag{3.14}$$

where

$$k(r)=\frac{r^2(1+r'^2)/(4r'^2)-2r^2\mathscr{E}^2(r)/(\pi^2r'^2)}{r^2(1+r'^2)/(4r'^2)-\arcsin[r^2/(1+r'^2)]}.$$

Let $k_1(r)=r^2(1+r'^2)/(4r'^2)-2r^2\mathscr{E}^2(r)/(\pi^2r'^2)$, $\quad k_2(r)=r^2(1+r'^2)/(4r'^2)-\arcsin[r^2/(1+r'^2)]$. Then $k(r)=k_1(r)/k_2(r)$, $k_1(0)=k_2(0)=0$,

$$k_1{}'(r)=\frac{1}{2}\frac{r(1+r'^4)}{r'^4}-\frac{4}{\pi^2}\frac{r\mathscr{E}[(1+r'^2)\mathscr{E}-r'^2\mathscr{K}]}{r'^4}$$

$$k_2{}'(r) = \frac{1}{2}\frac{r(1+r'^4)}{r'^4} - \frac{2r}{r'(1+r'^2)},$$

and thereby

$$\begin{aligned}\frac{k_1{}'(r)}{k_2{}'(r)} &= \frac{(1+r'^2)(1+r')^2}{r'^4+2r'^3+4r'^2+2r'+1} \cdot \frac{(1+r'^4) - 8\mathscr{E}(r)[(1+r'^2)\mathscr{E}(r) - r'^2\mathscr{K}(r)]}{r^4} \\ &= \omega(r')[1-2\zeta(r)], \end{aligned} \tag{3.15}$$

where $\zeta(r)$ is defined as in Lemma 2.8 and it is strictly decreasing from $(0,1)$ onto $(4/\pi^2, 13/32)$, and

$$\omega(r) = \frac{(1+r^2)(1+r)^2}{r^4+2r^3+4r^2+2r+1}$$

is positive and strictly decreasing on $(0,1)$ since

$$\omega'(r) = -\frac{4r(1+r)(1-r^3)}{(r^4+2r^3+4r^2+2r+1)^2} < 0$$

for all $r \in (0,1)$. Hence by (3.15) we know that $k_1{}'(r)/k_2{}'(r)$ is positive and strictly increasing on $(0,1)$, so is $k(r)$ by Lemma 2.1. Moreover,

$$\lim_{r\to 0^+} k(r) = \lim_{r\to 0^+} \frac{k_1{}'(r)}{k_2{}'(r)} = \frac{3}{20}, \quad \lim_{r\to 1^-} k(r) = 1 - \frac{8}{\pi^2} = 0.1894\cdots. \tag{3.16}$$

Therefore, Theorem 3.4 follows easily from (3.14) and (3.16) together with the monotonicity of $k(r)$ on the interval $(0,1)$. □

According to Theorems 3.1–3.4, we get the following Corollary 3.5 immediately, in which some new upper and lower bounds for the complete elliptic integral of the second kind $\mathscr{E}$ are given.

COROLLARY 3.5. *The double inequalities*

$$\frac{\pi\sqrt{6(1+r'^2)+4r'}}{8} < \mathscr{E}(r) < \sqrt{(1+r'^2) + \left(\frac{\pi^2}{2}-4\right)r'},$$

$$\sqrt{(1+r'^2) + \frac{(\pi^2/2-4)r'^2}{1+r'^2}} < \mathscr{E}(r) < \frac{\pi\sqrt{7(1+r'^2)+4r'^2/(1+r'^2)}}{8},$$

$$\sqrt{(1+r'^2) + \frac{(\pi^2/2-4)r'^2\log(1/r')}{r^2}} < \mathscr{E}(r) < \frac{\pi\sqrt{13(1+r'^2)/2+6r'^2\log(1/r')/r^2}}{8},$$

$$\sqrt{(1+r'^2) + \frac{(\pi^2/2-4)r'^2\sin^{-1}(r^*)}{r^2}} < \mathscr{E}(r) < \frac{\pi\sqrt{17(1+r'^2)/10+6r'^2\sin^{-1}(r^*)/5r^2}}{4}$$

hold for $r \in (0,1)$, *where* $r^* = r^2/(1+r'^2)$.

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