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ON STEVIĆ–SHARMA OPERATORS FROM THE MINIMAL MÖBIUS INVARIANT SPACE INTO ZYGMUND–TYPE SPACES

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Abstract. The boundedness, essential norm and compactness of Stević-Sharma operators from the minimal Möbius invariant space into Zygmund-type spaces are investigated in this paper.

1. Introduction

Let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$, where $\mathbb{D}$ is the open unit disk in the complex plane $\mathbb{C}$, and $S(\mathbb{D})$ the family of all analytic self-maps of $\mathbb{D}$. Denote by $\mathbb{N}$ the set of positive integers.

An $f \in H(\mathbb{D})$ is said to belong to the Zygmund-type space, which is denoted by $\mathscr{Z}_\mu = \mathscr{Z}_\mu(\mathbb{D})$, if

$$\|f\| = \sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty,$$

where μ is a weight, namely a strictly positive continuous function on $\mathbb{D}$. We also assume that μ is radial: $\mu(z) = \mu(|z|)$ for any $z \in \mathbb{D}$. Under the norm $\|f\|_{\mathscr{Z}_\mu} = |f(0)| + |f'(0)| + \|f\|$, $\mathscr{Z}_\mu$ becomes a Banach space. When $\mu(z) = 1-|z|^2$, the induced space $\mathscr{Z}_\mu$ reduce to the classical Zygmund space. The little Zygmund-type space $\mathscr{Z}_{\mu,0}$ consists of those functions f in $\mathscr{Z}_\mu$ satisfying $\lim_{|z|\to 1} \mu(z)|f''(z)| = 0$, and it is easily seen that $\mathscr{Z}_{\mu,0}$ is a closed subspace of $\mathscr{Z}_\mu$. For some results on the Zygmund-type spaces and operators on them see for instance [2, 6, 9, 10, 12, 13, 14, 15, 17, 19, 26, 36, 40].

For $w \in \mathbb{D}$, let σ_w be the automorphism of $\mathbb{D}$ exchanging points w and 0, that is,

$$\sigma_w(z) = \frac{w-z}{1-\overline{w}z}, \quad z \in \mathbb{D}.$$

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The analytic Besov space $\mathscr{B}_1$ consists of all $f \in H(\mathbb{D})$ which can be written as

$$f(z) = \sum_{n=1}^{\infty} a_n \sigma_{\lambda_n}(z)$$

for some sequences $\{a_n\}_{n\in\mathbb{N}} \subset l^1$ and $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{D}$. For $f \in \mathscr{B}_1$, the norm is defined by

$$\|f\|_{\mathscr{B}_1} = \inf\left\{ \sum_{n=1}^{\infty} |a_n| : f(z) = \sum_{n=1}^{\infty} a_n \sigma_{\lambda_n}(z) \right\}.$$

The space $\mathscr{B}_1$ was extensively studied in [3], where it was shown that if one defines appropriately the notion of a "Möbius invariant space", then $\mathscr{B}_1$ is the smallest one. For this reason, $\mathscr{B}_1$ is also called the minimal Möbius invariant space. It is evident that $\mathscr{B}_1 \subset H^\infty$. In fact, functions in $\mathscr{B}_1$ can extend continuously to the boundary, hence $\mathscr{B}_1$ is also a "boundary regular" space (see [5]). Furthermore, there exists a positive constant C such that for each $f \in \mathscr{B}_1$ (see [3, 38]),

$$\begin{aligned} C^{-1} \int_{\mathbb{D}} |f''(z)| dA(z) &\leqslant \|f - f(0) - f'(0)z\|_{\mathscr{B}_1} \\ &\leqslant C \int_{\mathbb{D}} |f''(z)| dA(z), \end{aligned}$$

where dA denotes the normalized area measure, i.e., $A(\mathbb{D}) = 1$. See [4, 12, 18, 23, 39] for some more related results about $\mathscr{B}_1$ space.

Let $\varphi \in S(\mathbb{D})$, the composition operator C_φ is defined by

$$C_\varphi f(z) = f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

For $\psi \in H(\mathbb{D})$ the multiplication operator M_ψ is defined by

$$M_\psi f(z) = \psi(z) f(z), \quad f \in H(\mathbb{D}).$$

The product $W_{\psi,\varphi} := M_\psi C_\varphi$ of these two operators is known as the weighted composition operator, which has been extensively studied. For more research about the (weighted) composition operators acting on several spaces of analytic functions, we refer to [5]. The differentiation operator D, which is defined by $Df(z) = f'(z)$, where $f \in H(\mathbb{D})$, plays an important role in operator theory and dynamical system.

In [32, 33], Stević et al. introduced the following so-called Stević-Sharma operator:

$$T_{\psi_1,\psi_2,\varphi} f(z) = \psi_1(z) f(\varphi(z)) + \psi_2(z) f'(\varphi(z)), \quad f \in H(\mathbb{D}),$$

where $\psi_1, \psi_2 \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Meanwhile, the boundedness, compactness and essential norm of $T_{\psi_1,\psi_2,\varphi}$ on the weighted Bergman space were characterized. By taking some specific choices of the involving symbols, we can easily get the general product-type operators:

$$M_\psi C_\varphi = T_{\psi,0,\varphi},\quad C_\varphi M_\psi = T_{\psi\circ\varphi,0,\varphi},\quad M_\psi D = T_{0,\psi,id},\quad DM_\psi = T_{\psi',\psi,id},$$
$$C_\varphi D = T_{0,1,\varphi},\quad DC_\varphi = T_{0,\varphi',\varphi},\quad M_\psi C_\varphi D = T_{0,\psi,\varphi},\quad M_\psi DC_\varphi = T_{0,\psi\varphi',\varphi},$$
$$C_\varphi M_\psi D = T_{0,\psi\circ\varphi,\varphi},\quad DM_\psi C_\varphi = T_{\psi',\psi\varphi',\varphi},$$
$$C_\varphi DM_\psi = T_{\psi'\circ\varphi,\psi\circ\varphi,\varphi},\quad DC_\varphi M_\psi = T_{\varphi'(\psi'\circ\varphi),\varphi'(\psi\circ\varphi),\varphi}.$$

Recently, the research of Stević-Sharma operator between analytic function spaces has aroused the interest of experts. For instance, Zhu in [41] characterized the boundedness and compactness of $T_{\psi_1,\psi_2,\varphi}$ from the Besov space $\mathscr{B}_p$ into Bloch space. Wang et al. in [35] considered the difference of two Stević-Sharma operators and investigated its boundedness, compactness and order boundedness between Banach spaces of analytic functions. Abbasi et al. in [1] generalized the Stević-Sharma operator and studied its boundedness, compactness and essential norm from Hardy space into the nth weighted-type space. Some more related results can be found (see, e.g., [2, 4, 6, 8, 9, 10, 11, 12, 16, 19, 21, 24, 27, 28, 29, 30, 31, 34, 36, 39, 40] and the references therein). Motivated by these, in this paper, we investigate the boundedness, compactness and essential norm of Stević-Sharma operator from the minimal Möbius invariant space into Zygmund-type space.

Recall that the essential norm of a bounded linear operator $T: X \to Y$ is the distance from T to the compact operators $K: X \to Y$, namely

$$\|T\|_{e,X\to Y} = \inf\big\{\|T-K\|_{X\to Y} : K \text{ is compact}\big\}.$$

Here X and Y are Banach spaces. Notice that $\|T\|_{e,X\to Y} = 0$ if and only if $T: X\to Y$ is compact.

Throughout this paper, for nonnegative quantities X and Y, we use the abbreviation $X \lesssim Y$ or $Y \gtrsim X$ if there exists a positive constant C independent of X and Y such that $X \leqslant CY$. Moreover, we write $X \approx Y$ if $X \lesssim Y \lesssim X$.

2. Preliminaries

In this section, we state several lemmas which will be used in the proofs of the main results. The first one is folklore (see, for instance, [37]).

LEMMA 1. *Let $k\in\mathbb{N}$, then*

$$\|f\|_\infty \lesssim \|f\|_{\mathscr{B}_1} \quad \textit{and} \quad (1-|z|^2)^k |f^{(k)}(z)| \lesssim \|f\|_{\mathscr{B}_1}$$

for any $f \in \mathscr{B}_1$.

For any $w\in\mathbb{D}$ and $j\in\mathbb{N}$, set

$$f_{j,w}(z) = \frac{(1-|w|^2)^j}{(1-\overline{w}z)^j}, \quad z\in\mathbb{D}. \tag{1}$$

It is known that $f_{j,w}\in\mathscr{B}_1$, and for each $j\in\mathbb{N}$, $\|f_{j,w}\|_{\mathscr{B}_1}\lesssim 1$. Moreover, $f_{j,w}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|w|\to 1$.

LEMMA 2. *For any* $0 \neq w \in \mathbb{D}$ *and* $i,k \in \{0,1,2,3\}$, *there exists a function* $g_{i,w} \in \mathscr{B}_1$ *such that*

$$g_{i,w}^{(k)}(w) = \frac{\overline{w}^k \delta_{ik}}{(1-|w|^2)^k},$$

where δ_{ik} *is Kronecker delta.*

Proof. For any $0 \neq w \in \mathbb{D}$ and constants c_1, c_2, c_3, c_4, we set

$$g_w(z) = \sum_{j=1}^{4} c_j f_{j,w}(z),$$

where $f_{j,w}$ is defined in (1). For each $i \in \{0,1,2,3\}$, the system of linear equations

$$\begin{cases} g_w(w) = c_1 + c_2 + c_3 + c_4 = \delta_{i0} \\ g_w'(w) = (c_1 + 2c_2 + 3c_3 + 4c_4)\frac{\overline{w}}{1-|w|^2} = \frac{\overline{w}\delta_{i1}}{1-|w|^2} \\ g_w''(w) = \big(2c_1 + 6c_2 + 12c_3 + 20c_4\big)\frac{\overline{w}^2}{(1-|w|^2)^2} = \frac{\overline{w}^2\delta_{i2}}{(1-|w|^2)^2} \\ g_w'''(w) = \big(6c_1 + 24c_2 + 60c_3 + 120c_4\big)\frac{\overline{w}^3}{(1-|w|^2)^3} = \frac{\overline{w}^3\delta_{i3}}{(1-|w|^2)^3} \end{cases}$$

has a unique solution $c_j^i, j \in \{1,2,3,4\}$ that is independent of w, since the determinant of coefficient matrix is not equal to zero. For such chosen numbers $c_j^i, j \in \{1,2,3,4\}$ the function

$$g_{i,w}(z) := \sum_{j=1}^{4} c_j^i f_{j,w}(z)$$

satisfies the desired conditions. □

By a standard arguments in [5, Proposition 3.11], we can get the following lemma and we omit the details.

LEMMA 3. *Let* $\psi_1, \psi_2 \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$. *Then the operator* $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_1 \to \mathscr{Z}_\mu$ *is compact if and only if for each bounded sequence* $\{f_n\}_{n\in\mathbb{N}}$ *in* $\mathscr{B}_1$ *and converges to zero uniformly on compact subsets of* $\mathbb{D}$ *as* $n \to \infty$, *we have* $\|T_{\psi_1,\psi_2,\varphi} f_n\|_{\mathscr{Z}_\mu} \to 0$ *as* $n \to \infty$.

LEMMA 4. [39] *Every sequence in* $\mathscr{B}_1$ *bounded in norm has a subsequence which converges uniformly in* $\overline{\mathbb{D}}$ *to a function in* $\mathscr{B}_1$.

LEMMA 5. [22] *A closed set* Q *in* $\mathscr{Z}_{\mu,0}$ *is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in Q} \mu(z)|f''(z)| = 0.$$

3. Boundedness

In this section, we give some necessary and sufficient conditions for Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_1\to\mathscr{Z}_\mu(\mathscr{Z}_{\mu,0})$ to be bounded. To simplify notation of this paper, we set

$$\begin{aligned}
A_0(z) &= \psi_1''(z),\\
A_1(z) &= 2\psi_1'(z)\varphi'(z)+\psi_1(z)\varphi''(z)+\psi_2''(z),\\
A_2(z) &= \psi_1(z)\varphi'(z)^2+2\psi_2'(z)\varphi'(z)+\psi_2(z)\varphi''(z),\\
A_3(z) &= \psi_2(z)\varphi'(z)^2.
\end{aligned}$$

THEOREM 1. *Let $\psi_1,\psi_2\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$ and μ be a radial weight. Then the following statements are equivalent.*

(i) *The operator $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_1\to\mathscr{Z}_\mu$ is bounded.*

(ii) *For each $i\in\{0,1,2,3\}$,*

$$\sup_{w\in\mathbb{D}}\|T_{\psi_1,\psi_2,\varphi}f_{i+1,w}\|_{\mathscr{Z}_\mu}<\infty \quad \text{and} \quad \sup_{z\in\mathbb{D}}\mu(z)|A_i(z)|<\infty,$$

where $f_{i+1,w}$ is defined in (1)*.*

(iii) *For each $i\in\{0,1,2,3\}$,*

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^i}<\infty. \tag{2}$$

Proof. (i)$\Rightarrow$(ii). Suppose that $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_1\to\mathscr{Z}_\mu$ is bounded. For any $w\in\mathbb{D}$ and $i\in\{0,1,2,3\}$, we have $\sup_{w\in\mathbb{D}}\|f_{i+1,w}\|_{\mathscr{B}_1}\lesssim 1$. Therefore,

$$\sup_{w\in\mathbb{D}}\|T_{\psi_1,\psi_2,\varphi}f_{i+1,w}\|_{\mathscr{Z}_\mu}\leqslant\|T_{\psi_1,\psi_2,\varphi}\|_{\mathscr{B}_1\to\mathscr{Z}_\mu}\sup_{w\in\mathbb{D}}\|f_{i+1,w}\|_{\mathscr{B}_1}<\infty.$$

Taking $f_0(z)=1\in\mathscr{B}_1$ (see [37], where it was proved that each polynomial belongs to $\mathscr{B}_1$), by the boundedness of $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_1\to\mathscr{Z}_\mu$ we get

$$\sup_{z\in\mathbb{D}}\mu(z)|A_0(z)|\leqslant\|T_{\psi_1,\psi_2,\varphi}f_0\|_{\mathscr{Z}_\mu}<\infty. \tag{3}$$

Applying the operator $T_{\psi_1,\psi_2,\varphi}$ to $f_1(z)=z\in\mathscr{B}_1$, we obtain

$$\begin{aligned}
\infty>\|T_{\psi_1,\psi_2,\varphi}f_1\|_{\mathscr{Z}_\mu} &\geqslant \sup_{z\in\mathbb{D}}\mu(z)|(T_{\psi_1,\psi_2,\varphi}f_1)''(z)|\\
&=\sup_{z\in\mathbb{D}}\mu(z)|A_0(z)\varphi(z)+A_1(z)|\\
&\geqslant\sup_{z\in\mathbb{D}}\mu(z)|A_1(z)|-\sup_{z\in\mathbb{D}}\mu(z)|A_0(z)\varphi(z)|,
\end{aligned}$$

which along with (3) and the fact that $|\varphi(z)|<1$ implies

$$\sup_{z\in\mathbb{D}}\mu(z)|A_1(z)|<\infty. \tag{4}$$

Taking $f_2(z) = z^2 \in \mathscr{B}_1$, then we have

$$\infty > \|T_{\psi_1,\psi_2,\varphi} f_2\|_{\mathscr{Z}_\mu} \geqslant \sup_{z\in\mathbb{D}} \mu(z)|(T_{\psi_1,\psi_2,\varphi} f_2)''(z)| = \sup_{z\in\mathbb{D}} \mu(z)|A_0(z)\varphi(z)^2 + 2A_1(z)\varphi(z) + 2A_2(z)|,$$

which along with (3), (4), the triangle inequality and the fact that $|\varphi(z)| < 1$ we get

$$\sup_{z\in\mathbb{D}} \mu(z)|A_2(z)| < \infty. \tag{5}$$

By using the function $f_3(z) = z^3 \in \mathscr{B}_1$, we obtain

$$\infty > \|T_{\psi_1,\psi_2,\varphi} f_3\|_{\mathscr{Z}_\mu} \geqslant \sup_{z\in\mathbb{D}} \mu(z)|(T_{\psi_1,\psi_2,\varphi} f_3)''(z)| = \sup_{z\in\mathbb{D}} \mu(z)|A_0(z)\varphi(z)^3 + 3A_1(z)\varphi(z)^2 + 6A_2(z)\varphi(z) + 6A_3(z)|,$$

which along with (3), (4), (5), the triangle inequality and the fact that $|\varphi(z)| < 1$ yields

$$\sup_{z\in\mathbb{D}} \mu(z)|A_3(z)| < \infty.$$

(ii)$\Rightarrow$(iii). Note that we only need to show that for $i \in \{1,2,3\}$, (2) holds. By Lemma 2, for each $i \in \{1,2,3\}$ and $\varphi(w) \neq 0$, there exist constants $c_1^i, c_2^i, c_3^i, c_4^i$ such that

$$g_{i,\varphi(w)}(z) = \sum_{j=1}^{4} c_j^i f_{j,\varphi(w)}(z) \in \mathscr{B}_1, \tag{6}$$

and

$$g_{i,\varphi(w)}^{(k)}(z) = \frac{\overline{\varphi(w)}^k \delta_{ik}}{(1-|\varphi(w)|^2)^k},$$

where $k \in \{0,1,2,3\}$. Then we have

$$\sum_{j=1}^{4} |c_j^i| \sup_{w\in\mathbb{D}} \|T_{\psi_1,\psi_2,\varphi} f_{j,\varphi(w)}\|_{\mathscr{Z}_\mu} \geqslant \sup_{w\in\mathbb{D}} \|T_{\psi_1,\psi_2,\varphi} g_{i,\varphi(w)}\|_{\mathscr{Z}_\mu} \geqslant \mu(w)\big|(T_{\psi_1,\psi_2,\varphi} g_{i,\varphi(w)})''(w)\big| = \frac{\mu(w)|A_i(w)||\varphi(w)|^i}{(1-|\varphi(w)|^2)^i}. \tag{7}$$

From (7) and (ii), for each $i \in \{1,2,3\}$, we have

$$\sup_{|\varphi(w)|>\frac{1}{2}} \frac{\mu(w)|A_i(w)|}{(1-|\varphi(w)|^2)^i} \lesssim \sum_{j=1}^{4} \sup_{w\in\mathbb{D}} \|T_{\psi_1,\psi_2,\varphi} f_{j,\varphi(w)}\|_{\mathscr{Z}_\mu} < \infty,$$

and

$$\sup_{|\varphi(w)|\leqslant\frac{1}{2}} \frac{\mu(w)|A_i(w)|}{(1-|\varphi(w)|^2)^i} \lesssim \sup_{w\in\mathbb{D}} \mu(w)|A_i(w)| < \infty.$$

Therefore,

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^i} < \infty.$$

(iii)$\Rightarrow$(i). Assume that (iii) holds. For any $f \in \mathscr{B}_1$, by Lemma 1 we have

$$\begin{aligned} \mu(z)|(T_{\psi_1,\psi_2,\varphi}f)''(z)| &\leqslant \sum_{i=0}^{3} \mu(z)|A_i(z)||f^{(i)}(\varphi(z))| \\ &\lesssim \|f\|_{\mathscr{B}_1} \sum_{i=0}^{3} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^i}. \end{aligned} \tag{8}$$

Moreover,

$$\begin{aligned} &|(T_{\psi_1,\psi_2,\varphi}f)(0)| + |(T_{\psi_1,\psi_2,\varphi}f)'(0)| \\ \leqslant&(|\psi_1(0)| + |\psi_1'(0)|)|f(\varphi(0))| + (|\psi_2(0)| + |\psi_2'(0)| + |\psi_1(0)||\varphi'(0)|)|f'(\varphi(0))| \\ &+ |\psi_2(0)||\varphi'(0)||f''(\varphi(0))| \\ \lesssim&\left(|\psi_1(0)| + |\psi_1'(0)| + \frac{|\psi_2(0)| + |\psi_2'(0)| + |\psi_1(0)||\varphi'(0)|}{1-|\varphi(0)|^2} + \frac{|\psi_2(0)||\varphi'(0)|}{(1-|\varphi(0)|^2)^2}\right)\|f\|_{\mathscr{B}_1}. \end{aligned}$$

Thus $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_1 \to \mathscr{Z}_\mu$ is bounded. The proof is completed. □

THEOREM 2. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and μ be a radial weight. Then the operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_1 \to \mathscr{Z}_{\mu,0}$ is bounded if and only if $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_1 \to \mathscr{Z}_\mu$ is bounded and for each $i \in \{0,1,2,3\}$,*

$$\lim_{|z|\to 1} \mu(z)|A_i(z)| = 0. \tag{9}$$

Proof. Assume that $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_1 \to \mathscr{Z}_{\mu,0}$ is bounded. It is immediate that $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_1 \to \mathscr{Z}_\mu$ is bounded, and for every $f \in \mathscr{B}_1$, we have $T_{\psi_1,\psi_2,\varphi}f \in \mathscr{Z}_{\mu,0}$. Taking $f_0(z) = 1 \in \mathscr{B}_1$, we get

$$\lim_{|z|\to 1} \mu(z)|A_0(z)| = \lim_{|z|\to 1} \mu(z)|(T_{\psi_1,\psi_2,\varphi}f_0)''(z)| = 0. \tag{10}$$

Taking $f_1(z) = z \in \mathscr{B}_1$, we obtain

$$\lim_{|z|\to 1} \mu(z)|A_0(z)\varphi(z) + A_1(z)| = 0,$$

which along with (10), the triangle inequality, and the fact that $|\varphi(z)| < 1$ implies that

$$\lim_{|z|\to 1} \mu(z)|A_1(z)| = 0.$$

By using the functions $f_2(z) = z^2$ and $f_3(z) = z^3 \in \mathscr{B}_1$, we conclude similarly that (9) holds for $i = 2,3$.

On the contrary, if $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_1 \to \mathscr{Z}_\mu$ is bounded and (9) holds, then for any polynomial p, we have

$$\mu(z)|(T_{\psi_1,\psi_2,\varphi}p)''(z)| \lesssim \sum_{i=0}^{3} \mu(z)|A_i(z)|\|p^{(i)}\|_\infty.$$

Letting $|z| \to 1$ in the last inequality yields $T_{\psi_1,\psi_2,\varphi}p \in \mathscr{Z}_{\mu,0}$. Since the set of all polynomials is dense in $\mathscr{B}_1$ (see [3]), and consequently for each $f \in \mathscr{B}_1$, there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \|f - p_n\|_{\mathscr{B}_1} = 0$. It follows that

$$\|T_{\psi_1,\psi_2,\varphi}f - T_{\psi_1,\psi_2,\varphi}p_n\|_{\mathscr{Z}_\mu} \leqslant \|T_{\psi_1,\psi_2,\varphi}\| \cdot \|f - p_n\|_{\mathscr{B}_1} \to 0$$

as $n \to \infty$. Thus $T_{\psi_1,\psi_2,\varphi}f \in \mathscr{Z}_{\mu,0}$ and $T_{\psi_1,\psi_2,\varphi}(\mathscr{B}_1) \subseteq \mathscr{Z}_{\mu,0}$. As a consequence, $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_1 \to \mathscr{Z}_{\mu,0}$ is bounded. □

4. Essential norm

In this section, we mainly estimate the essential norm of Stević-Sharma operator acting from the minimal Möbius invariant space to Zygmund-type space. For some results on essential norm of operators, see, e.g., [6, 7, 20, 25, 30, 31, 32, 39, 40].

THEOREM 3. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and μ be a radial weight such that $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_1 \to \mathscr{Z}_\mu$ is bounded. Then*

$$\|T_{\psi_1,\psi_2,\varphi}\|_{e,\mathscr{B}_1\to\mathscr{Z}_\mu} \approx \max\{\rho_i\}_{i=0}^{3} \approx \max\{\tau_l\}_{l=1}^{3},$$

where

$$\rho_i = \limsup_{|w|\to 1} \|T_{\psi_1,\psi_2,\varphi}f_{i+1,w}\|_{\mathscr{Z}_\mu}, \quad \tau_l = \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_l(z)|}{(1-|\varphi(z)|^2)^l}.$$

Proof. First we prove that

$$\|T_{\psi_1,\psi_2,\varphi}\|_{e,\mathscr{B}_1\to\mathscr{Z}_\mu} \gtrsim \max\{\rho_i\}_{i=0}^{3}.$$

It is obvious that for each $i \in \{0,1,2,3\}$, $\sup_{w\in\mathbb{D}} \|f_{i+1,w}\|_{\mathscr{B}_1} \lesssim 1$ and $f_{i+1,w}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|w| \to 1$. For any compact operator K from $\mathscr{B}_1$ into $\mathscr{Z}_\mu$, analysis similar to [20, Theorem 3.6] (see also [7, 25]) shows that

$$\lim_{|w|\to 1} \|Kf_{i+1,w}\|_{\mathscr{Z}_\mu} = 0.$$

Hence, for each $i \in \{0,1,2,3\}$,

$$\|T_{\psi_1,\psi_2,\varphi} - K\|_{\mathscr{B}_1 \to \mathscr{Z}_\mu} \gtrsim \limsup_{|w|\to 1} \|(T_{\psi_1,\psi_2,\varphi} - K) f_{i+1,w}\|_{\mathscr{Z}_\mu}$$
$$\geqslant \limsup_{|w|\to 1} \|T_{\psi_1,\psi_2,\varphi} f_{i+1,w}\|_{\mathscr{Z}_\mu} - \limsup_{|w|\to 1} \|K f_{i+1,w}\|_{\mathscr{Z}_\mu} = \rho_i.$$

Therefore

$$\|T_{\psi_1,\psi_2,\varphi}\|_{e,\mathscr{B}_1 \to \mathscr{Z}_\mu} = \inf_K \|T_{\psi_1,\psi_2,\varphi} - K\|_{\mathscr{B}_1 \to \mathscr{Z}_\mu} \gtrsim \max\{\rho_i\}_{i=0}^3. \tag{11}$$

Next, we show that

$$\|T_{\psi_1,\psi_2,\varphi}\|_{e,\mathscr{B}_1 \to \mathscr{Z}_\mu} \gtrsim \max\{\tau_l\}_{l=1}^3. \tag{12}$$

Let $\{z_j\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Since $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_1 \to \mathscr{Z}_\mu$ is bounded, using Lemma 3 and (7) for any compact operator $K : \mathscr{B}_1 \to \mathscr{Z}_\mu$ and $l \in \{1,2,3\}$, we obtain

$$\|T_{\psi_1,\psi_2,\varphi} - K\|_{\mathscr{B}_1 \to \mathscr{Z}_\mu} \gtrsim \limsup_{j\to\infty} \|T_{\psi_1,\psi_2,\varphi} g_{l,\varphi(z_j)}\|_{\mathscr{Z}_\mu} - \limsup_{j\to\infty} \|K g_{l,\varphi(z_j)}\|_{\mathscr{Z}_\mu}$$
$$\gtrsim \limsup_{j\to\infty} \frac{\mu(z_j)|A_l(z_j)||\varphi(z_j)|^l}{(1-|\varphi(z_j)|^2)^l},$$

where $g_{l,\varphi(z_j)}$ is defined in (8). Thus we have

$$\|T_{\psi_1,\psi_2,\varphi}\|_{e,\mathscr{B}_1 \to \mathscr{Z}_\mu} \gtrsim \limsup_{j\to\infty} \frac{\mu(z_j)|A_l(z_j)||\varphi(z_j)|^l}{(1-|\varphi(z_j)|^2)^l}$$
$$= \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_l(z)|}{(1-|\varphi(z)|^2)^l} = \tau_l,$$

and consequently (12) holds.

It will thus be sufficient to prove that

$$\|T_{\psi_1,\psi_2,\varphi}\|_{e,\mathscr{B}_1 \to \mathscr{Z}_\mu} \lesssim \min\left\{ \max\{\rho_i\}_{i=0}^3, \max\{\tau_l\}_{l=1}^3 \right\}.$$

Define

$$K_r f(z) = f_r(z) = f(rz), \quad 0 \leqslant r < 1.$$

Then $K_r : \mathscr{B}_1 \to \mathscr{B}_1$ is a compact operator with $\|K_r\| \leqslant 1$. Moreover, it is easily seen that $f_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Let $\{r_j\} \subset (0,1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Consequently, for any $j \in \mathbb{N}$, $T_{\psi_1,\psi_2,\varphi} K_{r_j} : \mathscr{B}_1 \to \mathscr{Z}_\mu$ is compact, and so

$$\|T_{\psi_1,\psi_2,\varphi}\|_{e,\mathscr{B}_1 \to \mathscr{Z}_\mu} \leqslant \limsup_{j\to\infty} \|T_{\psi_1,\psi_2,\varphi} - T_{\psi_1,\psi_2,\varphi} K_{r_j}\|_{\mathscr{B}_1 \to \mathscr{Z}_\mu}.$$

Hence, we only need to show that

$$\limsup_{j\to\infty}\|T_{\psi_1,\psi_2,\varphi}-T_{\psi_1,\psi_2,\varphi}K_{r_j}\|_{\mathscr{B}_1\to\mathscr{Z}_\mu}\lesssim\min\left\{\max\{\rho_i\}_{i=0}^3,\max\{\tau_l\}_{l=1}^3\right\}. \tag{13}$$

For any $f\in\mathscr{B}_1$ such that $\|f\|_{\mathscr{B}_1}\leqslant 1$, we have

$$\begin{aligned}
&\|(T_{\psi_1,\psi_2,\varphi}-T_{\psi_1,\psi_2,\varphi}K_{r_j})f\|_{\mathscr{Z}_\mu}\\
=&|(T_{\psi_1,\psi_2,\varphi}f-T_{\psi_1,\psi_2,\varphi}f_{r_j})(0)|+|(T_{\psi_1,\psi_2,\varphi}f-T_{\psi_1,\psi_2,\varphi}f_{r_j})'(0)|\\
&+\sup_{z\in\mathbb{D}}\mu(z)|(T_{\psi_1,\psi_2,\varphi}f-T_{\psi_1,\psi_2,\varphi}f_{r_j})''(z)|\\
\lesssim&\underbrace{|(f-f_{r_j})(\varphi(0))|+|(f-f_{r_j})'(\varphi(0))|+|(f-f_{r_j})''(\varphi(0))|}_{E_0}\\
&+\underbrace{\sup_{z\in\mathbb{D}}\mu(z)|(f-f_{r_j})(\varphi(z))A_0(z)|}_{E_1}+\underbrace{\sup_{|\varphi(z)|\leqslant r_N}\mu(z)\sum_{l=1}^3|(f-f_{r_j})^{(l)}(\varphi(z))A_l(z)|}_{E_2}\\
&+\underbrace{\sup_{|\varphi(z)|>r_N}\mu(z)\sum_{l=1}^3|(f-f_{r_j})^{(l)}(\varphi(z))A_l(z)|}_{E_3},
\end{aligned} \tag{14}$$

where $N\in\mathbb{N}$ such that $r_j\geqslant\frac{2}{3}$ for all $j\geqslant N$. Moreover, for any nonnegative integer s, $(f-f_{r_j})^{(s)}\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$. Theorem 1 now implies

$$\limsup_{j\to\infty}E_0=\limsup_{j\to\infty}E_2=0. \tag{15}$$

From Lemma 4,

$$\lim_{j\to\infty}E_1\lesssim\lim_{j\to\infty}\sup_{z\in\mathbb{D}}|(f-f_{r_j})(z)|=0, \tag{16}$$

where we used the condition (2). Finally, we estimate E_3.

$$E_3\leqslant\sum_{l=1}^3\underbrace{\sup_{|\varphi(z)|>r_N}\mu(z)|f^{(l)}(\varphi(z))A_l(z)|}_{F_l}+\sum_{l=1}^3\underbrace{\sup_{|\varphi(z)|>r_N}\mu(z)|r_j^lf^{(l)}(r_j\varphi(z))A_l(z)|}_{G_l}. \tag{17}$$

For each $l\in\{1,2,3\}$, from Lemma 1, (7) and (8) it follows that

$$\begin{aligned}
F_l=&\sup_{|\varphi(z)|>r_N}\frac{(1-|\varphi(z)|^2)^l|f^{(l)}(\varphi(z))|}{|\varphi(z)|^l}\frac{\mu(z)|A_l(z)||\varphi(z)|^l}{(1-|\varphi(z)|^2)^l}\\
\lesssim&\|f\|_{\mathscr{B}_1}\sup_{|\varphi(z)|>r_N}\|T_{\psi_1,\psi_2,\varphi}g_{l,\varphi(z)}\|_{\mathscr{Z}_\mu}\\
\lesssim&\sum_{j=0}^3\sup_{|w|>r_N}\|T_{\psi_1,\psi_2,\varphi}f_{j+1,w}\|_{\mathscr{Z}_\mu}.
\end{aligned} \tag{18}$$

On the other hand,

$$F_l = \sup_{|\varphi(z)|>r_N} (1-|\varphi(z)|^2)^l |f^{(l)}(\varphi(z))| \frac{\mu(z)|A_l(z)|}{(1-|\varphi(z)|^2)^l}$$
$$\lesssim \|f\|_{\mathscr{B}_1} \sup_{|\varphi(z)|>r_N} \frac{\mu(z)|A_l(z)|}{(1-|\varphi(z)|^2)^l}. \tag{19}$$

Taking the limits as $N \to \infty$ in (18) and (19), we obtain

$$\limsup_{j\to\infty} F_l \lesssim \sum_{j=0}^{3} \limsup_{|w|\to 1} \|T_{\psi_1,\psi_2,\varphi} f_{j+1,w}\|_{\mathscr{Z}_\mu} \lesssim \max\{\rho_i\}_{i=0}^3, \tag{20}$$

and

$$\limsup_{j\to\infty} F_l \lesssim \max\{\tau_l\}_{l=1}^3. \tag{21}$$

Similarly, we have

$$\limsup_{j\to\infty} G_l \lesssim \max\{\rho_i\}_{i=0}^3 \quad \text{and} \quad \limsup_{j\to\infty} G_l \lesssim \max\{\tau_l\}_{l=1}^3. \tag{22}$$

Therefore, by (14)–(17) and (20)–(22), we get

$$\limsup_{j\to\infty} \|T_{\psi_1,\psi_2,\varphi} - T_{\psi_1,\psi_2,\varphi}K_{r_j}\|_{\mathscr{B}_1\to\mathscr{Z}_\mu} = \limsup_{j\to\infty} \sup_{\|f\|_{\mathscr{B}_1}\leqslant 1} \|(T_{\psi_1,\psi_2,\varphi} - T_{\psi_1,\psi_2,\varphi}K_{r_j})f\|_{\mathscr{Z}_\mu}$$
$$\lesssim \min\left\{ \max\{\rho_i\}_{i=0}^3, \max\{\tau_l\}_{l=1}^3 \right\}.$$

That is, (13) holds. The proof is completed. □

From Theorem 3, we immediately obtain the following corollary, which characterizes the compactness of $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_1 \to \mathscr{Z}_\mu$.

COROLLARY 1. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and μ be a radial weight such that $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_1 \to \mathscr{Z}_\mu$ is bounded. Then the following statements are equivalent.*

(i) *The operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_1 \to \mathscr{Z}_\mu$ is compact.*

(ii) *For each $i \in \{0,1,2,3\}$,*

$$\limsup_{|w|\to 1} \|T_{\psi_1,\psi_2,\varphi} f_{i+1,w}\|_{\mathscr{Z}_\mu} = 0.$$

(iii) *For each $l \in \{1,2,3\}$,*

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|A_l(z)|}{(1-|\varphi(z)|^2)^l} = 0.$$

THEOREM 4. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and μ be a radial weight. Then the operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_1 \to \mathscr{Z}_{\mu,0}$ is compact if and only if for each $i \in \{0,1,2,3\}$,*

$$\limsup_{|z|\to 1} \frac{\mu(z)|A_i(z)|}{(1-|\varphi(z)|^2)^i} = 0. \tag{23}$$

Proof. Suppose that $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_1\to\mathscr{Z}_{\mu,0}$ is compact, then $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_1\to\mathscr{Z}_{\mu}$ is compact and $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_1\to\mathscr{Z}_{\mu,0}$ is bounded, which yields that $\lim_{|z|\to 1}\mu(z)|A_0(z)|=0$ by Theorem 2, that is, (23) holds for $i=0$. Moreover, for any $\varepsilon>0$, there exists $\eta\in(0,1)$ such that

$$\mu(z)|A_l(z)|<\varepsilon,\quad l\in\{1,2,3\}, \tag{24}$$

for $\eta<|z|<1$. From Corollary 1, for any $\varepsilon>0$, there exists $\delta\in(0,1)$ such that

$$\frac{\mu(z)|A_l(z)|}{(1-|\varphi(z)|^2)^l}<\varepsilon,\quad l\in\{1,2,3\}, \tag{25}$$

for $\delta<|\varphi(z)|<1$. Therefore, when $\delta<|\varphi(z)|<1$ and $\eta<|z|<1$, we have (25) holds. When $|\varphi(z)|\leqslant\delta$ and $\eta<|z|<1$, by using (24) we obtain

$$\frac{\mu(z)|A_l(z)|}{(1-|\varphi(z)|^2)^l}\leqslant\frac{\varepsilon}{(1-\delta^2)^l}\lesssim\varepsilon,\quad l\in\{1,2,3\}. \tag{26}$$

Along with (25) and (26), by the arbitrariness of ε, we can see that (23) holds.

Conversely, assume that (23) holds. It is evident that $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_1\to\mathscr{Z}_{\mu,0}$ is bounded by Theorem 1. Taking the supremum in (8) over all $f\in\mathscr{B}_1$ such that $\|f\|_{\mathscr{B}_1}\leqslant 1$ and letting $|z|\to 1$, we get

$$\lim_{|z|\to 1}\sup_{\|f\|_{\mathscr{B}_1}\leqslant 1}\mu(z)|(T_{\psi_1,\psi_2,\varphi}f)''(z)|=0.$$

From Lemma 5 it follows that $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_1\to\mathscr{Z}_{\mu,0}$ is compact. □

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