Journal of Mathematical Inequalities
Volume 16, Number 1 (2022), 157–168 doi:10.7153/jmi-2022-16-12

OPTIMAL LEHMER MEAN BOUNDS FOR THE nTH POWER–TYPE TOADER MEANS OF $n=-1,1,3$

TIE-HONG ZHAO, HONG-HU CHU AND YU-MING CHU*

(*Communicated by J. Matkowski*)

Abstract. In the article, we prove that $\lambda_1=0$, $\mu_1=5/8$, $\lambda_2=-1/8$, $\mu_2=0$, $\lambda_3=-1$ and $\mu_3=-7/8$ are the best possible parameters such that the double inequalities

$$L_{\lambda_1}(a,b)<T_3(a,b)<L_{\mu_1}(a,b),$$
$$L_{\lambda_2}(a,b)<T_1(a,b)<L_{\mu_2}(a,b),$$
$$L_{\lambda_3}(a,b)<T_{-1}(a,b)<L_{\mu_3}(a,b)$$

hold for $a,b>0$ with $a\neq b$, and provide new bounds for the complete elliptic integral of the second kind $\mathscr{E}(r)=\int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta$ on the interval $(0,1)$, where $L_p(a,b)=(a^{p+1}+b^{p+1})/(a^p+b^p)$ is the p-th Lehmer mean and $T_n(a,b)=\left(\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^n\cos^2\theta+b^n\sin^2\theta}d\theta\right)^{2/n}$ is the nth power-type Toader mean.

1. Introduction

Let $p\in\mathbb{R}$ and $a,b>0$. Then the p-th Lehmer mean [1] is defined by

$$L_p(a,b)=\frac{a^{p+1}+b^{p+1}}{a^p+b^p}, \tag{1.1}$$

and it is also called Gini means $G_{p+1,p}(a,b)$ [2], where

$$G_{p,q}(a,b)=\begin{cases}\left(\frac{a^p+b^p}{a^q+b^q}\right)^{1/(p-q)}, & p\neq q,\\ \exp\left(\frac{a^p\ln a+b^p\ln b}{a^p+b^p}\right), & p=q.\end{cases}$$

As special case of Gini means, it is easy to see that $L_p(a,b)$ is continuous and strictly increasing with respect to $p\in\mathbb{R}$ for fixed $a,b>0$ with $a\neq b$.

Another one-parameter mean considered in this paper is the nth power-type Toader mean, which is defined as

$$T_n(a,b):=\left[T\left(a^{n/2},b^{n/2}\right)\right]^{2/n}=\left(\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^n\cos^2\theta+b^n\sin^2\theta}d\theta\right)^{2/n}$$

Mathematics subject classification (2020): 26E60, 33E05.
Keywords and phrases: Lehmer mean, Gini mean, Toader mean, complete elliptic integrals.
This research was supported by the National Natural Science Foundation of China (Grant No. 11971142) and the Natural Science Foundation of Zhejiang Province (Grant No. LY19A010012).
* Corresponding author.

© ELEMENT, Zagreb

for $n \in \mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$, where $T(a,b)$ is the Toader mean [3] defined by

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta.$$

This is also called $n/2$-modification of Toader mean introduced in [4], in which the authors study the t-modification of arithmetic-geometric mean $AGM(a,b)$ [5, 6]. In particular, some well-known means are the p-modification of the classical means, such as the power mean [7, 8, 9], which is given by

$$M_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.2}$$

Many classical two variable means are the special means of Lehmer mean and power mean, for example, $L_0(a,b) = M_1(a,b) = A(a,b)$ is the arithmetic mean, $L_{-1/2}(a,b) = M_0(a,b) = G(a,b)$ is the geometric mean, $L_{-1}(a,b) = M_{-1}(a,b) = H(a,b)$ is the Harmonic mean, and $L_1(a,b) = C(a,b)$ is the contraharmonic mean.

Clearly, $T_n(a,b)$ can be rewritten by

$$T_n(a,b) = \begin{cases} a\left[\frac{2}{\pi}\mathscr{E}\left(\sqrt{1-(b/a)^n}\right)\right]^{2/n}, & a^n \geqslant b^n, \\ b\left[\frac{2}{\pi}\mathscr{E}\left(\sqrt{1-(a/b)^n}\right)\right]^{2/n}, & a^n < b^n, \end{cases} \tag{1.3}$$

where

$$\begin{cases} \mathscr{E}(r) = \int_0^{\pi/2}\left(1-r^2\sin^2\theta\right)^{1/2}d\theta = \frac{\pi}{2}{}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;r^2\right), \\ \mathscr{E}(0^+) = \pi/2, \quad \mathscr{E}(1^-) = 1 \end{cases}$$

is the complete elliptic integral of the second kind [10, 11, 12, 13] and the *Gaussian hypergeometric function* [14, 15, 16, 17, 18] is given by

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}.$$

Here $(a)_n = \Gamma(a+n)/\Gamma(a)$ and $\Gamma(u) = \int_0^\infty x^{u-1}e^{-x}dx$ is the Euler gamma function [19, 20, 21].

Throughout this paper, we denote $s = \sqrt{1-r^2}$ for $r \in (0,1)$ and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Recall the *Gauss identity* [6]

$$AGM(1,r)\mathscr{K}(s) = \frac{\pi}{2} \tag{1.4}$$

for $r \in (0,1)$, where $\mathscr{K}(r)$ is the complete elliptic integral of the first kind [22, 23, 24, 25, 26] defined by

$$\begin{cases} \mathscr{K}(r) = \int_0^{\pi/2}\left(1-r^2\sin^2\theta\right)^{-1/2}d\theta = \frac{\pi}{2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;r^2\right) = \sum_{n=0}^{\infty}\frac{[(\frac{1}{2})_n]^2}{(n!)^2}r^{2n}, \\ \mathscr{K}(0^+) = \frac{\pi}{2}, \quad \mathscr{K}(1^-) = \infty. \end{cases}$$

Legendre complete elliptic integrals and Gaussian hypergeometric function play very important roles in many branches of modern mathematics such as classical analysis, number theory, geometric function theory, and conformal and quasi-conformal mappings [27, 28, 29, 30, 31, 32, 33].

Barnard, Pearce and Richards [34], and Alzer and Qiu [35] proved that $\lambda = 3/2$ and $\mu = \log 2/(\log\pi - \log 2) = 1.5349\cdots$ are the best possible parameters such that the double inequality

$$M_\lambda(a,b) < T_2(a,b) < M_\mu(a,b) \tag{1.5}$$

holds for all $a,b>0$ with $a\neq b$.

Chu and Wang [36] proved that the double inequality

$$L_p(a,b) < T_2(a,b) < L_q(a,b) \tag{1.6}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $p\leqslant 0$ and $q\geqslant 1/4$.

In [37], Qian and Chu proved that $\lambda = \left(1-\sqrt{1-(2\sqrt{2}/\pi)^{4/p}}\right)/2$ and $\mu = 1/2-\sqrt{p}/(4p)$ are the best possible parameters such that the double inequality

$$\begin{aligned} G^p\big(\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a\big)A^{1-p}(a,b) &< T_1(a,b) \\ &< G^p\big(\mu a+(1-\mu)b,\mu b+(1-\mu)a\big)A^{1-p}(a,b) \end{aligned} \tag{1.7}$$

holds for all $p\in[1,\infty)$ and $a,b>0$ with $a\neq b$.

Recently, Chu et al. [12] showed that the double inequality

$$C[\alpha a+(1-p\alpha)b,\alpha b+(1-\alpha)a] < T_3(a,b) < C[\beta a+(1-\beta)b,\beta b+(1-\beta)a] \tag{1.8}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leqslant\left(\sqrt{2\sqrt[3]{4/\pi^2}-1}+1\right)/2$ and $\beta\geqslant 1/2+\sqrt{10}/8$. Moreover, Li and Zhao [38] gave the sharp generalized Seiffert mean bounds for $T_4(a,b)$, that is the two-sided inequality

$$S_{p_0}(a,b) < T_4(a,b) < S_{\sqrt{3}/2}(a,b) \tag{1.9}$$

holds for all $a>b>0$ with the best possible constants p_0 and $\sqrt{3}/2$, where $p_0 = 0.81366\cdots$ is the unique positive solution of the equation $p/\arctan(2p) = \sqrt{2/\pi}$ and $S_p(a,b)$ is defined by

$$S_p(a,b) = \frac{p(a-b)}{\arctan[2p(a-b)/(a+b)]} \quad (p\neq 0), \quad S_0(a,b) = \frac{a+b}{2}.$$

Motivated by inequalities (1.5)–(1.9), the main purpose of the article is to provide the sharp Lehmer mean bounds for $T_{-1}(a,b)$, $T_1(a,b)$ and $T_3(a,b)$, specifically, to find the best possible parameters λ_i, μ_i $(i=1,2,3)$ such that the double inequalities

$$L_{\lambda_1}(a,b) < T_3(a,b) < L_{\mu_1}(a,b),$$

$$L_{\lambda_2}(a,b) < T_1(a,b) < L_{\mu_2}(a,b),$$

$$L_{\lambda_3}(a,b) < T_{-1}(a,b) < L_{\mu_3}(a,b)$$

hold for all $a,b>0$ with $a\neq b$. As consequences, we also provide new bounds for complete elliptic integral of the second kind.

2. Lemmas

In order to prove our main results, we need several lemmas which we present in this section.

By substituting $a = 1$ and $b = s = \sqrt{1-r^2}$ into (1.2) and (1.3), the following lemma can be derived immediately from (1.5).

LEMMA 2.1. *The double inequality*

$$\left(\frac{1+s^{\lambda}}{2}\right)^{1/\lambda} < \frac{2}{\pi}\mathscr{E}(r) < \left(\frac{1+s^{\mu}}{2}\right)^{1/\mu}$$

holds for all $r \in (0,1)$ *if and only if* $\lambda \leqslant 3/2$ *and* $\mu \geqslant \log 2/(\log\pi - \log 2) = 1.5349\cdots$.

LEMMA 2.2. (See [33, Lemmas 2.1 and 2.4]). *Let* $f_1(r) = (1-r)^2 + 4\sqrt{r}(1+r)$ *and* $f_2(r) = (1+\sqrt{r})^2 + 2\sqrt{2(1+r)\sqrt{r}}$. *Then*

$$\frac{2}{\pi}\mathscr{E}(s) < \frac{f_1(r)}{f_2(r)} < \frac{1+r^{5/4}}{1+r^{1/4}}$$

for all $r \in (0,1)$.

LEMMA 2.3. (See [6, (3.22)]). *Let* $a_0 = 1$, $b_0 = r \in (0,1)$, $d_0 = 2$ *and* $a_{n+1} = (a_n + b_n)/2$, $b_{n+1} = \sqrt{a_n b_n}$, $d_{n+1} = d_n - 2^n(a_n^2 - b_n^2)$ *for* $n \in \mathbb{N}_0$. *Then*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = AGM(1,r) \quad \text{and} \quad \lim_{n\to\infty} d_n = \frac{2\mathscr{E}(s)}{\mathscr{K}(s)}.$$

LEMMA 2.4. *Let the sequences* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ *and* $\{d_n\}_{n=0}^{\infty}$ *be defined as in Lemma 2.3. Then the following statements are true:*

(1) $d_n > 2^n(a_n - b_n)(2a_n + b_n)$ *for* $n \in \mathbb{N}_0$;

(2) The sequence $\{d_{n+1}/a_n\}_{n=0}^{\infty}$ *is positive, strictly increasing and*

$$\lim_{n\to\infty} \frac{d_{n+1}}{a_n} = \frac{4}{\pi}\mathscr{E}(s). \tag{2.1}$$

Proof. (1) Let $e_n = 2^n(a_n - b_n)(2a_n + b_n)$. Then we can prove that $d_n > e_n$ for $n = 0,1,2,\cdots$ by mathematical induction. Firstly, it is easy to verify that $d_0 = 2 > 2 - r - r^2 = e_0$ and $d_1 = 1 + r^2 > 1 + r^2 - \sqrt{r}(1+r) = e_1$.

We assume that $d_n > e_n$ for $n = 0,1,\cdots,k$ $(k \geqslant 1)$, then

$$\begin{aligned}
&d_{k+1} - e_{k+1}\\
&= d_k - 2^k(a_k^2 - b_k^2) - 2^{k+1}(a_{k+1} - b_{k+1})(2a_{k+1} + b_{k+1})\\
&> 2^k(a_k - b_k)(2a_k + b_k) - 2^k(a_k^2 - b_k^2) - 2^k(a_k + b_k - 2\sqrt{a_k b_k})(a_k + b_k + \sqrt{a_k b_k})\\
&= 2^k\sqrt{b_k}(a_k + b_k)(\sqrt{a_k} - \sqrt{b_k}) > 0.
\end{aligned}$$

(2) From the definitions of $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$, we clearly see that $a_n > b_n > 0$ for $n \in \mathbb{N}_0$. This in conjunction with part (1) implies that $d_n > 0$ for $n \in \mathbb{N}_0$. Therefore, $d_{n+1}/a_n > 0$ for all $n \in \mathbb{N}_0$.

Moreover, it follows from part (1) that

$$\begin{aligned}\frac{d_{n+2}}{a_{n+1}} - \frac{d_{n+1}}{a_n} &= \frac{a_n d_{n+2} - a_{n+1} d_{n+1}}{a_n a_{n+1}} \\ &= \frac{a_n \left[d_{n+1} - 2^{n+1}(a_{n+1}^2 - b_{n+1}^2)\right] - d_{n+1}(a_n + b_n)/2}{a_n a_{n+1}} \\ &= \frac{d_{n+1}(a_n - b_n)/2 - 2^{n+1} a_n (a_{n+1}^2 - b_{n+1}^2)}{a_n a_{n+1}} \\ &= \frac{(a_n - b_n)\left[d_n - 2^n (a_n - b_n)(2a_n + b_n)\right]}{2 a_n a_{n+1}} > 0. \end{aligned} \tag{2.2}$$

Therefore, the increasingness of the sequence $\{d_{n+1}/a_n\}_{n=0}^{\infty}$ follows from (2.2) and equation (2.1) can be derived from (1.4) and Lemma 2.3. □

LEMMA 2.5. *Let $f_2(r)$ be defined as in Lemma 2.2, $g_1(r) = (1+\sqrt{r})^2(1 - 6\sqrt{r} + r + 4\sqrt{2(1+r)\sqrt{r}})$ and $g_2(r) = 2f_2(r)$. Then*

$$\frac{2}{\pi}\mathscr{E}(s) > \frac{g_1(r)}{g_2(r)}$$

for all $r \in (0,1)$.

Proof. Let the sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=0}^{\infty}$ be defined as in Lemma 2.3. Then it is easy to verify that

$$d_4 = \frac{1}{8} g_1(r) \tag{2.3}$$

and

$$a_3 = \frac{1}{16} g_2(r). \tag{2.4}$$

Therefore, Lemma 2.5 follows from (2.3) and (2.4) together with Lemma 2.4(2). □

LEMMA 2.6. *Let $f_1(r)$ and $f_2(r)$ be defined as in Lemma 2.2. Then*

$$\left[\frac{f_1(r^{3/2})}{f_2(r^{3/2})}\right]^{2/3} < \frac{1+r^{13/8}}{1+r^{5/8}} \tag{2.5}$$

for all $r \in (0,1)$.

Proof. According to the second inequality of Lemma 2.2, it suffices to prove the inequality

$$\left(\frac{1+r^{15/8}}{1+r^{3/8}}\right)^{2/3} < \frac{1+r^{13/8}}{1+r^{5/8}} \tag{2.6}$$

for all $r \in (0,1)$ instead of (2.5).

Let $u = r^{1/8} \in (0,1)$. Then it is easy to see that inequality (2.6) is equivalent to

$$\begin{aligned}&(1+u^{15})^2(1+u^5)^3-(1+u^{13})^3(1+u^3)^2\\&=-u^3(1-u)^4(1+u)^5(1+u^2)(1-u+u^2)^2(1+u+u^2)(1+u^3+u^6)\sigma(u)\\&<0,\end{aligned} \tag{2.7}$$

where

$$\sigma(u)=2+u^2-3u^3+2u^4+4u^6-3u^7-u^8-3u^9+4u^{10}+2u^{12}-3u^{13}+u^{14}+2u^{16}.$$

More precisely,

$$\begin{aligned}\sigma(u)&=2+u^2-3u^3+2u^4+4u^6-3u^7-u^8-3u^9+3u^{10}\\&\qquad+u^{10}\left[(1-u^3)+2u^2(1-u)\right]+u^{14}+2u^{16}\\&>2+u^2-3u^3+2u^4+4u^6-3u^7-u^8-3u^9+3u^{10}\\&=1+u^2(1-2u+2u^2)+(1-u)(1+u+u^2-3u^9)+u^6(1-u)(u+4)>0\end{aligned}$$

for all $u \in (0,1)$. This gives the positive answer of inequality (2.7). □

LEMMA 2.7. *Let $g_1(r)$ and $g_2(r)$ be defined as in Lemma 2.5. Then*

$$\left[\frac{g_1(\sqrt{r})}{g_2(\sqrt{r})}\right]^2 > \frac{1+r^{7/8}}{1+r^{-1/8}} \tag{2.8}$$

for all $r \in (0,1)$.

Proof. Let $u = r^{1/8} \in (0,1)$. Then it is easy to see that inequality (2.8) is equivalent to

$$(1+u)g_1^2(u^4)-(u+u^8)g_2^2(u^4)>0 \tag{2.9}$$

for all $u \in (0,1)$. Elaborated computations lead to

$$(1+u)g_1^2(u^4)-(u+u^8)g_2^2(u^4)=a(u)+8\sqrt{2}u(1+u)(1+u^2)^2b(u)\sqrt{1+u^4}, \tag{2.10}$$

where

$$\begin{aligned}a(u)&=1-3u+24u^2-24u^3+124u^4+100u^5+296u^6+248u^7+386u^8+386u^9\\&\qquad+248u^{10}+296u^{11}+100u^{12}+124u^{13}-24u^{14}+24u^{15}-3u^{16}+u^{17},\\b(u)&=1-2u-2u^2-2u^3-8u^4-2u^5-2u^6-2u^7+u^8.\end{aligned}$$

Moreover, we have

$$\begin{aligned}a(u)&>3\left(\frac{1}{3}-u\right)+24u^2(1-u)+18u^4\\&=2+\frac{23}{3}\left(u-\frac{1}{3}\right)+12\left(u-\frac{1}{3}\right)^2+18\left(u-\frac{1}{3}\right)^4,\end{aligned} \tag{2.11}$$

$$b'(u)=-2[1+2u+3u^2+16u^3+5u^4+6u^5+3u^6+4u^6(1-u)<0. \tag{2.12}$$

It follows easily from (2.11) that $a(u)>0$ for $u \in (0,1/3)$ and $a(u)>0$ for $u \in [1/3,1)$.

Inequality (2.12) together with $b(3/10) = 9451021/100000000 > 0$ and $b(1) = -18 < 0$ leads to the conclusion that there exists $u_0 \in (3/10,1)$ such that $b(u) > 0$ for $u \in (0,u_0)$ and $b(u) < 0$ for $u \in (u_0,1)$.

Combining (2.10) with the sign of $a(u)$ and $b(u)$ on the interval $(0,1)$, it remains to prove that the inequality (2.9) holds for $u \in (u_0,1)$. For $u \in (u_0,1)$, elaborated computations give

$$\begin{aligned}
&a^2(u) - 128u^2(1+u)^2(1+u^2)^4(1+u^4)b^2(u)\\
&= (1-u)^4(1+u)^2\Big(1-4u-78u^2-100u^3+1091u^4+1720u^5+13540u^6+19448u^7\\
&\quad +59969u^8+75364u^9+156494u^{10}+169988u^{11}+270459u^{12}+249680u^{13}\\
&\quad +324152u^{14}+249680u^{15}+270459u^{16}+169988u^{17}+156494u^{18}+75364u^{19}\\
&\quad +59969u^{20}+19448u^{21}+13540u^{22}+1720u^{23}\\
&\quad +1091u^{24}-100u^{25}-78u^{26}-4u^{27}+u^{28}\Big)\\
&> (1-u)^4(1+u)^2(1-4u-78u^2-100u^3+1091u^4+1720u^5)\\
&= (1-u)^4(1+u)^2\left[\frac{30967}{10000}+\frac{13711}{125}\left(u-\frac{3}{10}\right)+\frac{44277}{50}\left(u-\frac{3}{10}\right)^2\right.\\
&\quad \left.+\frac{13786}{5}\left(u-\frac{3}{10}\right)^3+3671\left(u-\frac{3}{10}\right)^4+1720\left(u-\frac{3}{10}\right)^5\right] > 0. \quad \square
\end{aligned}$$

3. Main results

THEOREM 3.1. *The inequality $L_0(a,b) < T_3(a,b) < L_{5/8}(a,b)$ holds for all $a,b > 0$ with $a \neq b$, where $L_0(a,b)$ and $L_{5/8}(a,b)$ are the best possible lower and upper Lehmer mean bounds for $T_3(a,b)$.*

Proof. By homogeneity it suffices to prove that the inequality in Theorem 3.1 holds only for $a=1$ and $0<b<1$.

Let $a=1$ and $b=s^{2/3} \in (0,1)$. Then Lemma 2.1 and (1.3) enable us to give

$$T_3(a,b) = \left[\frac{2}{\pi}\mathscr{E}(r)\right]^{2/3} > \frac{1+s^{2/3}}{2} > \frac{1+s}{2} = L_0(a,b).$$

Next we prove that

$$T_3(a,b) < L_{5/8}(a,b) \tag{3.1}$$

for all $a,b>0$ with $a \neq b$.

We now assume that $a=1$ and $b=r \in (0,1)$. Then it follows form (1.1) and (1.3) that

$$\begin{aligned}
T_3(a,b) - L_{5/8}(a,b) &= \left[\frac{2}{\pi}\mathscr{E}(\sqrt{1-r^3})\right]^{2/3} - \frac{1+r^{13/8}}{1+r^{5/8}}\\
&= \left[\frac{2}{\pi}\mathscr{E}(\sqrt{1-r^3})\right]^{2/3} - \left[\frac{f_1(r^{3/2})}{f_2(r^{3/2})}\right]^{2/3} + \left[\frac{f_1(r^{3/2})}{f_2(r^{3/2})}\right]^{2/3} - \frac{1+r^{13/8}}{1+r^{5/8}},
\end{aligned} \tag{3.2}$$

where $f_1(r)$ and $f_2(r)$ are defined as in Lemma 2.2.

Therefore, inequality (3.1) follows from Lemmas 2.2 and 2.6 together with (3.2).

It remains to show that $L_0(a,b)$ and $L_{5/8}(a,b)$ are the best possible lower and upper Lehmer mean bounds for $T_3(a,b)$.

Let $0<p<5/8$. Then (1.1) and (1.3) lead to

$$\lim_{r\to 0^+}\left[T_3(1,r)-L_p(1,r)\right]=\left(\frac{2}{\pi}\right)^{2/3}-1<0, \tag{3.3}$$

$$\lim_{r\to 1^-}\frac{T_3(1,r)-L_p(1,r)}{(r-1)^2}=\frac{1}{4}\left(\frac{5}{8}-p\right)>0. \tag{3.4}$$

Inequalities (3.3) and (3.4) lead to the conclusion that there exist small enough $\tau_1,\tau_2\in(0,1)$ such that $T_3(1,r)<L_p(1,r)$ for $r\in(0,\tau_1)$ and $T_3(1,r)>L_p(1,r)$ for $r\in(1-\tau_2,1)$. □

THEOREM 3.2. *The inequality $L_{-1/8}(a,b)<T_1(a,b)<L_0(a,b)$ holds for all $a,b>0$ with $a\neq b$, where $L_{-1/8}(a,b)$ and $L_0(a,b)$ are the best possible lower and upper Lehmer mean bounds for $T_1(a,b)$.*

Proof. Since $T_1(a,b)$ and $L_p(a,b)$ are symmetric and homogeneous of degree 1 with respect to a and b, it suffices to prove that the inequality $T_1(a,b)<L_0(a,b)$ holds only for $a=1$ and $b=s^2\in(0,1)$. It follows from (1.3) and Lemma 2.1 that

$$T_1(a,b)=\left[\frac{2}{\pi}\mathscr{E}(r)\right]^2<\left(\frac{1+s^\mu}{2}\right)^{2/\mu}<\frac{1+s^2}{2}=L_0(a,b),$$

where the last inequality follows from the monotonicity of power mean with respect to p and μ is defined as in Lemma 2.1.

Next we prove that

$$T_1(a,b)>L_{-1/8}(a,b) \tag{3.5}$$

for all $a,b>0$ with $a\neq b$. We now assume that $a=1$ and $b=r\in(0,1)$. Then it follows from (1.1) and (1.3) that

$$\begin{aligned} T_1(a,b)-L_{-1/8}(a,b)&=\left[\frac{2}{\pi}\mathscr{E}(\sqrt{1-r})\right]^2-\frac{1+r^{7/8}}{1+r^{-1/8}}\\ &=\left[\frac{2}{\pi}\mathscr{E}(\sqrt{1-r})\right]^2-\left[\frac{g_1(\sqrt{r})}{g_2(\sqrt{r})}\right]^2+\left[\frac{g_1(\sqrt{r})}{g_2(\sqrt{r})}\right]^2-\frac{1+r^{7/8}}{1+r^{-1/8}}, \end{aligned} \tag{3.6}$$

where $g_1(r)$ and $g_2(r)$ are defined as in Lemma 2.5.

Therefore, inequality (3.5) follows from Lemmas 2.5 and 2.7 together with (3.6).

In the end, we prove that $L_{-1/8}(a,b)$ and $L_0(a,b)$ are the best possible lower and upper Lehmer mean bounds for $T_1(a,b)$.

Let $-1/8<p<0$ and $0<r<1$. Then from (1.1) and (1.3) we get

$$\lim_{r\to 0^+}\left[T_1(1,r)-L_p(1,r)\right]=\frac{4}{\pi^2}>0, \tag{3.7}$$

$$\lim_{r\to 1^-}\frac{T_1(1,r)-L_p(1,r)}{(r-1)^2}=-\frac{1}{4}\left(\frac{1}{8}+p\right)<0. \tag{3.8}$$

Inequalities (3.7) and (3.8) lead to the conclusion that there exist small enough $\tau_3,\tau_4\in(0,1)$ such that $T_1(1,r)>L_p(1,r)$ for $r\in(0,\tau_3)$ and $T_1(1,r)<L_p(1,r)$ for $r\in(1-\tau_4,1)$. □

THEOREM 3.3. *The inequality $L_{-1}(a,b)<T_{-1}(a,b)<L_{-7/8}(a,b)$ holds for all $a,b>0$ with $a\neq b$, where $L_{-1}(a,b)$ and $L_{-7/8}(a,b)$ are the best possible lower and upper Lehmer mean bounds for $T_{-1}(a,b)$.*

Proof. By the same argument as above, we only need to prove the inequality $L_{-1}(a,b)<T_{-1}(a,b)$ holds for $a=1$ and $b=s^2\in(0,1)$. From (1.3) and Lemma 2.1 we clearly see that

$$T_{-1}(a,b)=s^2\left[\frac{2}{\pi}\mathscr{E}(r)\right]^{-2}>s^2\left(\frac{1+s^{\mu}}{2}\right)^{-2/\mu}>s^2\left(\frac{1+s^2}{2}\right)^{-1}=L_{-1}(a,b),$$

where the last inequality follows from the monotonicity of power mean with respect to p and μ is defined as in Lemma 2.1.

Next we prove that

$$T_{-1}(a,b)<L_{-7/8}(a,b) \tag{3.9}$$

for all $a,b>0$ with $a\neq b$. We now assume that $a=1$ and $b=r\in(0,1)$. Then it follows from (1.1) and (1.3) that

$$\begin{aligned}T_{-1}(a,b)-L_{-7/8}(a,b)&=r\left[\frac{2}{\pi}\mathscr{E}(\sqrt{1-r})\right]^{-2}-\frac{1+r^{1/8}}{1+r^{-7/8}}\\&=r\left[\frac{2}{\pi}\mathscr{E}(\sqrt{1-r})\right]^{-2}-r\left[\frac{g_1(\sqrt{r})}{g_2(\sqrt{r})}\right]^{-2}+r\left[\frac{g_1(\sqrt{r})}{g_2(\sqrt{r})}\right]^{-2}-\frac{1+r^{1/8}}{1+r^{-7/8}},\end{aligned} \tag{3.10}$$

where $g_1(r)$ and $g_2(r)$ are defined as in Lemma 2.5.

From Lemma 2.7, we clearly see that

$$r\left[\frac{g_1(\sqrt{r})}{g_2(\sqrt{r})}\right]^{-2}<r\cdot\frac{1+r^{-1/8}}{1+r^{7/8}}=\frac{1+r^{1/8}}{1+r^{-7/8}} \tag{3.11}$$

for $r\in(0,1)$.

Therefore, inequality (3.9) follows from Lemma 2.5, (3.10) and (3.11).

Finally, we prove that $L_{-1}(a,b)$ and $L_{-7/8}(a,b)$ are the best possible lower and upper Lehmer mean bounds for $T_{-1}(a,b)$.

For $-1<p<-7/8$ and $0<r<1$, we clearly see from (1.1) and (1.3) that

$$\begin{aligned}&\lim_{r\to0^+}r^p\Big[T_{-1}(1,r)-L_p(1,r)\Big]\\&\quad=\lim_{r\to0^+}\left(r^{p+1}\left[\frac{2}{\pi}\mathscr{E}(\sqrt{1-r})\right]^{-2}-\frac{1+r^{p+1}}{1+r^{-p}}\right)=-1<0,\end{aligned} \tag{3.12}$$

$$\lim_{r\to1^-}\frac{T_{-1}(1,r)-L_p(1,r)}{(r-1)^2}=-\frac{1}{4}\left(\frac{7}{8}+p\right)>0. \tag{3.13}$$

Inequalities (3.12) and (3.13) lead to the conclusion that there exist small enough $\tau_5,\tau_6\in(0,1)$ such that $T_{-1}(1,r)<L_p(1,r)$ for $r\in(0,\tau_5)$ and $T_{-1}(1,r)>L_p(1,r)$ for $r\in(1-\tau_6,1)$. □

Table 1: *Comparison of $\mathscr{E}(r)$ with $L(r)$ and $U(r)$ for some $r \in (0,1)$*

r	$L(r)$	$\mathscr{E}(r)$	$U(r)$
0.05	$1.56981411841636\cdots$	$1.56981411841639\cdots$	$1.56981411841641\cdots$
0.1	$1.566861942013\cdots$	$1.566861942021\cdots$	$1.566861942027\cdots$
0.2	$1.554968544\cdots$	$1.554968546\cdots$	$1.554968547\cdots$
0.3	$1.534833405\cdots$	$1.534833464\cdots$	$1.534833511\cdots$
0.4	$1.5059409\cdots$	$1.5059416\cdots$	$1.5059421\cdots$
0.5	$1.467457\cdots$	$1.467462\cdots$	$1.467466\cdots$
0.6	$1.418056\cdots$	$1.418083\cdots$	$1.418104\cdots$
0.7	$1.355530\cdots$	$1.355661\cdots$	$1.355764\cdots$
0.8	$1.27572\cdots$	$1.27634\cdots$	$1.27684\cdots$

By the virtue of Theorem 3.1 and Theorem 3.2, new lower and upper bounds for the complete elliptic integral $\mathscr{E}(r)$ of the second kind are given as follows.

COROLLARY 3.4. *The double inequality*

$$\frac{\pi}{2}\left[\frac{1+(1-r^2)^{7/8}}{1+(1-r^2)^{-1/8}}\right]^{1/2} < \mathscr{E}(r) < \frac{\pi}{2}\left[\frac{1+(1-r^2)^{13/24}}{1+(1-r^2)^{5/24}}\right]^{3/2}$$

holds for all $r \in (0,1)$.

REMARK 3.5. Let

$$L(r) = \frac{\pi}{2}\left[\frac{1+(1-r^2)^{7/8}}{1+(1-r^2)^{-1/8}}\right]^{1/2} \quad \text{and} \quad U(r) = \frac{\pi}{2}\left[\frac{1+(1-r^2)^{13/24}}{1+(1-r^2)^{5/24}}\right]^{3/2}.$$

Computational and numerical experiments show that the lower bound $L(r)$ and upper bound $U(r)$ are very accurate for some $r \in (0,1)$, refer to Table 1.

REFERENCES

[1] H. ALZER, *Über Lehmers Mittelwertfamilie*, Elem. Math., 1988, **43** (2), 50–54.

[2] C. GINI, *Di una formula compresive delle medie*, Metron, 1938, **13**, 3–22.

[3] GH. TOADER, *Some mean values related to the arithmetic-geometric mean*, J. Math. Anal. Appl., 1998, **218** (2), 358–368.

[4] M. K. VAMANAMURTHY AND M. VUORINEN, *Inequalities for means*, J. Math. Anal. Appl., 1994, **183** (1), 155–166.

[5] T.-H. ZHAO, M.-K. WANG AND Y.-M. CHU, *On the bounds of the perimeter of an ellipse*, Acta Math. Sci., 2022, **42B** (2), 491–501.

[6] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.

[7] T.-H. ZHAO, W.-M. QIAN AND Y.-M. CHU, *Sharp power mean bounds for the tangent and hyperbolic sine means*, J. Math. Inequal., 2021, **15** (4), 1459–1472.

[8] Y.-M. CHU AND T.-H. ZHAO, *Convavity of the error function with respect to Hölder means*, Math. Inequal. Appl., 2016, **19** (2), 589–595.

[9] T.-H. ZHAO, Z.-H. SHEN AND Y.-M. CHU, *Sharp power mean bounds for the lemniscate type means*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A. Mat. RACSAM, 2021, **115** (4), paper no. 175, 16 pages.

[10] Z.-H. YANG, Y.-M. CHU AND W. ZHANG, *High accuracy asymptotic bounds for the complete elliptic integral of the second kind*, Appl. Math. Comput., 2019, **348**, 552–564.

[11] Z.-Y. HE, M.-K. WANG, Y.-P. JIANG AND Y.-M. CHU, *Bounds for the perimeter of an ellipse in terms of power means*, J. Math. Inequal., 2020, **14** (3), 887–899.

[12] H.-H. CHU, T.-H. ZHAO AND Y.-M. CHU, *Sharp bounds for the Toader mean of order 3 in terms of arithmetic, quadratic and contraharmonic means*, Math. Slovaca, 2020, **70** (5), 1097–1112.

[13] W.-M. QIAN, M.-K. WANG, H.-Z. XU AND Y.-M. CHU, *Approximations for the complete elliptic integral of the second kind*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2021, **115** (2), paper no. 88, 11 pages.

[14] T.-H. ZHAO, M.-K. WANG, W. ZHANG AND Y.-M. CHU, *Quadratic transformation inequalities for Gaussian hypergeometric function*, J. Inequal. Appl., 2018, **2018**, paper no. 251, 15 pages.

[15] T.-H. ZHAO, Z.-Y. HE AND Y.-M. CHU, *On some refinements for inequalities involving zero-balanced hypergeometric function*, AIMS Math., 2020, **5** (6), 6479–6495.

[16] T.-H. ZHAO, Z.-Y. HE AND Y.-M. CHU, *Sharp bounds for the weighted Hölder mean of the zero-balanced generalized complete elliptic integrals*, Comput. Methods Funct. Theory, 2021, **21** (3), 413–426.

[17] T.-H. ZHAO, W.-M. QIAN AND Y.-M. CHU, *On approximating the arc lemniscate functions*, Indian J. Pure Appl. Math., 2021, https://doi.org/10.1007/s13226-021-00016-9.

[18] H.-Z. XU, W.-M. QIAN AND Y.-M. CHU, *Sharp bounds for the lemniscatic mean by the one-parameter geometric and quadratic means*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2022, **116** (1), paper no. 21, 15 pages.

[19] F. QI, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl., 2010, **2010**, Article ID 493058, 84 pages.

[20] T.-H. ZHAO, M.-K. WANG, G.-J. HAI AND Y.-M. CHU, *Landen inequalities for Gaussian hypergeometric function*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2022, **116** (1), paper no. 53, 23 pages.

[21] T.-H. ZHAO, L. SHI AND Y.-M. CHU, *Convexity and concavity of the modified Bessel functions of the first kind with respect to Hölder means*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A. Mat. RACSAM, 2020, **114** (2), paper no. 96, 14 pages.

[22] Z.-H. YANG AND Y.-M. CHU, *A monotonicity property involving the generalized elliptic integral of the first kind*, Math. Inequal. Appl., 2017, **20** (3), 729–735.

[23] T.-H. ZHAO, M.-K. WANG AND Y.-M. CHU, *A sharp double inequality involving generalized complete elliptic integral of the first kind*, AIMS Math., 2020, **5** (5), 4512–4528.

[24] W.-M. QIAN, Z.-Y. HE AND Y.-M. CHU, *Approximation for the complete elliptic integral of the first kind*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2020, **114**(2), paper no. 57, 12 pages.

[25] T.-H. ZHAO, M.-K. WANG AND Y.-M. CHU, *Concavity and bounds involving generalized elliptic integral of the first kind*, J. Math. Inequal., 2021, **15** (2), 701–724.

[26] T.-H. ZHAO, M.-K. WANG AND Y.-M. CHU, *Monotonicity and convexity involving generalized elliptic integral of the first kind*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A. Mat. RACSAM, 2021, **115** (2), paper no. 46, 13 pages.

[27] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Special functions of quasiconformal theory*, Exposition. Math., 1989, **7** (2), 97–136.

[28] G. D. ANDERSON, S.-L. QIU AND M. K. VAMANAMURTHY, *Grötzsch ring and quasiconformal distortion functions*, Hokkaido Math. J., 1995, **24** (3), 551–566.

[29] G. D. ANDERSON AND M. K. VAMANAMURTHY, *Some properties of quasiconformal distortion functions*, New Zealand J. Math., 1995, **24** (1), 1–15.

[30] S.-L. QIU, M. K. VAMANAMURTHY AND M. VUORINEN, *Bounds for quasiconformal distortion functions*, J. Math. Anal. Appl., 1997, **205** (1), 43–64.

[31] G. D. ANDERSON, S.-L. QIU AND M. K. VAMANAMURTHY, *Inequalities for distortion functions and plane quasiconformal mappings*, New Zealand J. Math., 1997, **26** (1), 1–19.

[32] M.-K. WANG, M.-Y. HONG, Y.-F. XU, Z.-H. SHEN AND Y.-M. CHU, *Inequalities for generalized trigonometric and hyperbolic functions with one parameter*, J. Math. Inequal., 2020, **14** (1), 1–21.

[33] T.-H. ZHAO, B. A. BHAYO AND Y.-M. CHU, *Inequalities for generalized Grötzsch ring function*, Comput. Methods Funct. Theory, 2021, https://doi.org/10.1007/s40315-021-00415-3.
[34] R. W. BARNARD, K. PEARCE AND K. C. RICHARDS, *An inequality involving the generalized hypergeometric function and the arc length of an ellipse*, SIAM J. Math. Anal., 2000, **31** (3), 693–699.
[35] H. ALZER AND S.-L. QIU, *Monotonicity theorems and inequalities for the complete elliptic integrals*, J. Comput. Appl. Math., 2004, **172** (2), 289–312.
[36] Y.-M. CHU AND M.-K. WANG, *Optimal Lehmer mean bounds for the Toader mean*, Results Math., 2012, **61** (3–4), 223–229.
[37] W.-M. QIAN AND Y.-M. CHU, *Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two-parameters*, J. Inequal. Appl., 2017, **2017**, Article 274, 10 pages.
[38] Y.-L. LI AND T.-H. ZHAO, *Sharp generalized Seiffert mean bounds for the Toader mean of order 4*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2021, **115** (3), paper no. 106, 15 pages.

(Received May 10, 2020)

Tie-Hong Zhao
Department of Mathematics
Hangzhou Normal University
Hangzhou 311121, P. R. China
e-mail: tiehong.zhao@hznu.edu.cn

Hong-Hu Chu
College of Civil Engineering
Hunan University
Changsha 410082, P. R. China
e-mail: chuhonghu2005@126.com

Yu-Ming Chu
Department of Mathematics
Huzhou University
Huzhou 313000, P. R. China
e-mail: chuyuming2005@126.com