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A FRACTIONAL MAGNETIC HARDY–SOBOLEV INEQUALITY WITH TWO VARIABLES

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Abstract. A fractional magnetic Hardy-Sobolev inequality with two variables and critical exponents is considered in this paper, and the achievement to the best constant correspongding to this inequality is obtained.

1. Introduction

The magnetic relativistic Schrödinger operators corresponding to the classical relativistic Hamiltonian symbol with magnetic vector potential and electric scalar potential

$$\sqrt{(\xi - A(x))^2 + m^2} + V(x), \quad (\xi, x) \in \mathbb{R}^N \times \mathbb{R}^N$$

have been discussed by Ichinose [9]. Here $A(x) \in \mathbb{R}^N$ is a magnetic vector potential, m is the mass of a relativistic spinless particle, and $V(x) \in \mathbb{R}$ is an electric scalar potential. Dependent on how to quantize the kinetic energy term $\sqrt{(\xi - A(x))^2 + m^2}$, Ichinose considered three magnetic relativistic Schrödinger operators with shape $H := H_A + V$. The first two quantized operators are defined by mean formulas, i.e., for any function $f \in C_0^\infty(\mathbb{R}^N, \mathbb{C})$,

$$(H_A^{(1)} f)(x) := \frac{1}{(2\pi)^N} \int_{\mathbb{R}^{2N}} e^{i(x-y)\cdot\xi} \sqrt{\left(\xi - A\left(\frac{x+y}{2}\right)\right)^2 + m^2} f(y) \mathrm{d}y \mathrm{d}\xi,$$

$$(H_A^{(2)} f)(x) := \frac{1}{(2\pi)^N} \int_{\mathbb{R}^{2N}} e^{i(x-y)\cdot\xi} \sqrt{\left(\xi - \int_0^1 A((1-\theta)x + \theta y)\mathrm{d}\theta\right)^2 + m^2} f(y) \mathrm{d}y \mathrm{d}\xi.$$

The third one is defined as the square root of the nonnegative selfadjoint operator $(-i\nabla - A(x))^2 + m^2$ in $L^2(\mathbb{R}^N)$:

$$H_A^{(3)} := \sqrt{(-i\nabla - A(x))^2 + m^2}.$$

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The operator $H_A^{(1)}$ is called the Weyl pseudo-differential operator defined with *mid-point prescription* and well compatible with path integrals, but in general, it is not covariant under gauge transformations, that is, there exists a real-valued function $\phi(x)$ for which it fails to hold that $H_{A+\nabla\phi}^{(1)} = e^{i\phi}H_A^{(1)}e^{-i\phi}$. $H_A^{(2)}$ is a gauge-covariant modification of $H_A^{(1)}$. $H_A^{(3)}$ is gauge-covariant and used in the description of the 'stability of matter' in relativistic quantum mechanics.

Proposition 2.6 in [9] told us that $H_A^{(1)}$, $H_A^{(2)}$ and $H_A^{(3)}$ are in general different. About $H_A^{(1)}$, d'Avenia and Squassina [4] investigated the existence of ground states and established other useful estimates. For $H_A^{(2)}$, Guo and Melgaard [8] studied a fractional magnetic Sobolev inequality with two variables and critical exponents. About $H_A^{(3)}$, Cingolani and Secchi [3] proved the existence of infinitely many intertwining solutions by means of a new local realization of the square root of the magnetic laplacian to a local elliptic operator with Neumann boundary condition on a half-space. Moreover, they derived an existence result of a ground state intertwining solution for bounded vector potentials.

Nonlocal magnetic problems have been investigated recently, such as [1, 2, 4, 5, 9]. The fractional magnetic Laplacian related to magnetic relativistic Schrödinger operator $H_A^{(1)}$ is defned by

$$(-\Delta)_A^s u(x) = c_{N,s} \lim_{\varepsilon\to 0^+} \int_{B_\varepsilon^c(x)} \frac{u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^{N+2s}} \mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where $0 < s < 1, N > 4s$, $A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic vector potential and

$$c_{N,s} := \left(\int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}} \mathrm{d}\zeta \right)^{-1}.$$

$(-\Delta)_A^s$ can be considered as a fractional counterpart of the magnetic Laplacian $(\nabla - iA)^2$. For $N = 3$, the fractional magnetic Laplacian had been studied by d'Avenia and Squassina [4]. Via variational methods and Ljusternick-Schnirelmann category, Ambrosio and d'Avenia [1] studied a nonlinear fractional Schrödinger equation with magnetic field and a subcritical nonlinearity, and obtained existence and multiplicity of solutions for parameter small. Zhang, Squassina and Xia [2] studied a singularly perturbed fractional Schrödinger equations involving critical frequency and critical growth in the presence of a magnetic field. Using variational methods, they obtained the existence of mountain pass solutions u_ε which tend to the trivial solution as $\varepsilon \to 0$. Fiscella, Pinamonti and Vecchi [5] got the existence of multiple solutions for a boundary value problem driven by the fractional magnetic Laplacian with a subcritical nonlinear term. The motivations for this kind of problems rely essentially on the Lévy-Khintchine formula for the generator of a semigroup associated to a general Lévy process, which is more appropriate for some mathematical models in finance. For more details, we refer readers to [4, 9].

In the present paper, we study a fractional magnetic Hardy-Sobolev inequality with two variables:

$$\begin{aligned}\Lambda_{s,t,A}&\left(\int_{\mathbb{R}^N}\left(\mu_1\frac{|u|^{2_s^*(t)}}{|x|^t}+\mu_2\frac{|v|^{2_s^*(t)}}{|x|^t}+\lambda\frac{|u|^\alpha|v|^\beta}{|x|^t}\right)\mathrm{d}x\right)^{\frac{2}{2_s^*(t)}}\\&\leqslant\int_{\mathbb{R}^{2N}}\frac{\left|e^{-i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(x)-u(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\&\quad+\int_{\mathbb{R}^{2N}}\frac{\left|e^{-i(x-y)\cdot A\left(\frac{x+y}{2}\right)}v(x)-v(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y,\quad\forall u,v\in D_A^s(\mathbb{R}^N,\mathbb{C}),\end{aligned}\tag{1}$$

where $0<s<1$, $0<t<2s<N$, $2_s^*(t):=\frac{2(N-t)}{N-2s}$ is fractional Hardy-Sobolev critical exponent, $\mu_1,\mu_2,\alpha,\beta,\gamma>0$, $\alpha+\beta=2_s^*(t)$, $\Lambda_{s,t,A}$ is a constant, $A:\mathbb{R}^N\to\mathbb{R}^N$ is a magnetic vector potential and continuous function with locally bounded gradient, which ensures that $C_c^\infty(\mathbb{R}^N,\mathbb{C})$ is a subspace of $D_A^s(\mathbb{R}^N,\mathbb{C})$ (see Proposition of 2.2 in [4]). Define $D_A^s(\mathbb{R}^N,\mathbb{C})$ by the completion of $C_c^\infty(\mathbb{R}^N,\mathbb{C})$ with respect to magnetic Gagliardo seminorm $[\cdot]_{D_A^s}$ given by

$$[u]_{D_A^s}^2:=\frac{c_{N,s}}{2}\int_{\mathbb{R}^{2N}}\frac{\left|e^{-i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(x)-u(y)\right|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

The scalar product in $D_A^s(\mathbb{R}^N,\mathbb{C})$ is defined by

$$\begin{aligned}\langle u,v\rangle_{D_A^s}:=\frac{c_{N,s}}{2}\mathrm{Re}\int_{\mathbb{R}^{2N}}&\frac{\left(e^{-i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(x)-u(y)\right)}{|x-y|^{N+2s}}\\&\cdot\overline{\left(e^{-i(x-y)\cdot A\left(\frac{x+y}{2}\right)}v(x)-v(y)\right)}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

By fractional magnetic Sobolev embeddings (see Lemma 3.5 in [4]), the seminorm $[\cdot]_{D_A^s}$ can be viewed as a norm $\|\cdot\|_{D_A^s}:=[\cdot]_{D_A^s}$ of space $D_A^s(\mathbb{R}^N,\mathbb{C})$. Following Proposition 2.1 and 2.2 in [4], it can be verified that $D_A^s(\mathbb{R}^N,\mathbb{C})$ is a Hilbert space.

Denote $|u|_{2_s^*(t),t}:=\left(\int_{\mathbb{R}^N}\frac{|u|^{2_s^*(t)}}{|x|^t}\mathrm{d}x\right)^{1/2_s^*(t)}$ and $\mathscr{D}_A^s(\mathbb{R}^N,\mathbb{C}):=D_A^s(\mathbb{R}^N,\mathbb{C})\times D_A^s(\mathbb{R}^N,\mathbb{C})$ equiped with norm $\|(u,v)\|_{\mathscr{D}_A^s}^2:=\|u\|_{D_A^s}^2+\|v\|_{D_A^s}^2$. Letting $S_A:=c_{N,s}\Lambda_{s,t,A}/2$, then the inequality (1) is equivalent to the following minimization problem

$$S_A=\inf_{\substack{(u,v)\in\mathscr{D}_A^s(\mathbb{R}^N,\mathbb{C})\\(u,v)\neq(0,0)}}\frac{\|(u,v)\|_{\mathscr{D}_A^s}^2}{\left(\int_{\mathbb{R}^N}\left(\mu_1\frac{|u|^{2_s^*(t)}}{|x|^t}+\mu_2\frac{|v|^{2_s^*(t)}}{|x|^t}+\lambda\frac{|u|^\alpha|v|^\beta}{|x|^t}\right)\mathrm{d}x\right)^{\frac{2}{2_s^*(t)}}}.\tag{2}$$

Equivalently, we can characterize S_A as:

$$S_A=\inf_{u\in\mathscr{J}}\|(u,v)\|_{\mathscr{D}_A^s}^2,\tag{3}$$

where

$$\mathscr{J}=\left\{(u,v)\in\mathscr{D}_A^s(\mathbb{R}^N,\mathbb{C}):\int_{\mathbb{R}^N}\left(\mu_1\frac{|u|^{2_s^*(t)}}{|x|^t}+\mu_2\frac{|v|^{2_s^*(t)}}{|x|^t}+\lambda\frac{|u|^{\alpha}|v|^{\beta}}{|x|^t}\right)\mathrm{d}x=1\right\}. \tag{4}$$

For special case there is no magnetic field, that is, $A\equiv 0$, it has been proved in [7] that S_0 is attained by (U,V), which is radially symmetric decreasing.

To give our main result, we need the following lemmas.

LEMMA 1. (Diamagnetic inequality) (see Lemma 3.1 in [4]) *For any $u\in D_A^s(\mathbb{R}^N,\mathbb{C})$, we have*

$$\big||u(x)|-|u(y)|\big|\leqslant\left|e^{-i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(x)-u(y)\right|,\quad \textit{for a.e. } x,y\in\mathbb{R}^N \tag{5}$$

and

$$\big\||u|\big\|_{D_0^s}\leqslant\|u\|_{D_A^s}, \tag{6}$$

which means $|u|\in D_0^s(\mathbb{R}^N,\mathbb{R})$.

LEMMA 2. (the Fractional magnetic Hardy-Sobolev embedding) (by Diamagnetic inequality and Fractional Hardy-Sobolev embeddings, c.f. [6, 10]) *The embedding*

$$D_A^s(\mathbb{R}^N,\mathbb{C})\hookrightarrow L^{2_s^*(t)}\left(\mathbb{R}^N,\frac{dx}{|x|^t}\right)$$

is continuous. That is, for any $u\in D_A^s(\mathbb{R}^N,\mathbb{C})$, we have

$$\left(\int_{\mathbb{R}^N}\frac{|u|^{2_s^*(t)}}{|x|^t}dx\right)^{\frac{2}{2_s^*(t)}}\leqslant C\|u\|_{D_A^s}^2.$$

2. Main result

Now we give our main result and its proof.

THEOREM 1. *If $0<s<1$, $0<t<2s<N$, $\mu_1,\mu_2,\alpha,\beta,\gamma>0$, $\alpha+\beta=2_s^*(t)$, and $A:\mathbb{R}^N\to\mathbb{R}^N$ is a magnetic vector potential and a continuous function with locally bounded gradient, then S_A is achieved by some nontrivial element $(U_A,V_A)\in\mathscr{D}_A^s(\mathbb{R}^N,\mathbb{C})$.*

Proof. Since S_0 is achieved by nontrivial element $(U,V)\in\mathscr{D}_0^s(\mathbb{R}^N,\mathbb{R})$, which is proved in [7]. We only need to show that $S_A=S_0$, which is inspired by Lemma 4.6 in [4].

It follows from (3) and (4) that, for any $\varepsilon>0$, there exist $u,v\in C_c^\infty(\mathbb{R}^N,\mathbb{R})$ such that

$$\|(u,v)\|_{\mathscr{D}_0^s}^2\leqslant S_0+\varepsilon,\qquad \int_{\mathbb{R}^N}\left(\mu_1\frac{|u|^{2_s^*(t)}}{|x|^t}+\mu_2\frac{|v|^{2_s^*(t)}}{|x|^t}+\lambda\frac{|u|^{\alpha}|v|^{\beta}}{|x|^t}\right)\mathrm{d}x=1. \tag{7}$$

For any $\varepsilon > 0$, denote

$$u_\varepsilon(x) := \varepsilon^{(2s-N)/2} u\left(\frac{x}{\varepsilon}\right), \quad v_\varepsilon(x) := \varepsilon^{(2s-N)/2} v\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^N.$$

Direct computations yield that

$$\|u_\varepsilon\|_{D_A^s}^2 = \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{\left|e^{-i\varepsilon(x-y)\cdot A(\varepsilon\frac{x+y}{2})} u(x) - u(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

According to invariance of scaling, we have

$$\begin{aligned}
\|(u_\varepsilon, v_\varepsilon)\|_{\mathscr{D}_0^s} &= \|(u,v)\|_{\mathscr{D}_0^s}, \\
|u_\varepsilon|_{2_s^*(t),t} &= |u|_{2_s^*(t),t}, \\
|v_\varepsilon|_{2_s^*(t),t} &= |v|_{2_s^*(t),t}, \\
\int_{\mathbb{R}^N} \frac{|u_\varepsilon|^\alpha |v_\varepsilon|^\beta}{|x|^t} \mathrm{d}x &= \int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^\beta}{|x|^t} \mathrm{d}x.
\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^N} \left(\mu_1 \frac{|u_\varepsilon|^{2_s^*(t)}}{|x|^t} + \mu_2 \frac{|v_\varepsilon|^{2_s^*(t)}}{|x|^t} + \lambda \frac{|u_\varepsilon|^\alpha |v_\varepsilon|^\beta}{|x|^t} \right) \mathrm{d}x = 1.$$

Then we derive that

$$\begin{aligned}
&\|(u_\varepsilon, v_\varepsilon)\|_{\mathscr{D}_A^s}^2 - \|(u,v)\|_{\mathscr{D}_0^s}^2 \\
=&\frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{2\mathrm{Re}\left((1 - e^{-i\varepsilon(x-y)\cdot A(\varepsilon\frac{x+y}{2})}) u(x)u(y)\right)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
&+ \frac{c_{N,s}}{2} \int_{\mathbb{R}^{2N}} \frac{2\mathrm{Re}\left((1 - e^{-i\varepsilon(x-y)\cdot A(\varepsilon\frac{x+y}{2})}) v(x)v(y)\right)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
=&c_{N,s} \int_{\mathbb{R}^{2N}} \frac{\left[1 - \cos\left(\varepsilon(x-y)\cdot A(\varepsilon\frac{x+y}{2})\right)\right] \left(u(x)u(y) + v(x)v(y)\right)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
=:& \; c_{N,s} \int_{\mathbb{R}^{2N}} \Xi_\varepsilon(x,y) \mathrm{d}x\mathrm{d}y \\
=&c_{N,s} \int_{K\times K} \Xi_\varepsilon(x,y) \mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where K is a compact support of $|u| + |v|$. It is easy to see that $\Xi_\varepsilon(x,y) \to 0$ a.e. in $\mathbb{R}^{2N}$ as $\varepsilon \to 0$. Since A is locally bounded, for $x,y \in K$ and small $\varepsilon > 0$, we have

$$1 - \cos\left(\varepsilon(x-y)\cdot A\left(\varepsilon\frac{x+y}{2}\right)\right) \leqslant C|x-y|^2.$$

For $x,y \in K$, it follows from the boundedness of u and v that

$$|\Xi_\varepsilon(x,y)| \leqslant \begin{cases} \frac{C}{|x-y|^{N-2+2s}}, & \text{if } |x-y| < 1, \\ \frac{C}{|x-y|^{N+2s}}, & \text{if } |x-y| \geqslant 1. \end{cases}$$

That is, there exists a suitable constant $C > 0$ satisfying

$$|\Xi_\varepsilon(x,y)| \leqslant C \min\left\{\frac{1}{|x-y|^{N-2+2s}}, \frac{1}{|x-y|^{N+2s}}\right\} =: b(x,y), \quad x,y \in K.$$

Since

$$\begin{aligned}
&\int_{K\times K} b(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int_{(K\times K)\cap\{|x-y|<1\}} b(x,y)\mathrm{d}x\mathrm{d}y + \int_{(K\times K)\cap\{|x-y|\geqslant 1\}} b(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int_{\{|x-y|<1\}} \frac{C}{|x-y|^{N-2+2s}}\mathrm{d}x\mathrm{d}y + \int_{\{|x-y|\geqslant 1\}} \frac{C}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&\leqslant C\int_{\{|z|<1\}} \frac{1}{|z|^{N-2+2s}}\mathrm{d}z + C\int_{\{|z|\geqslant 1\}} \frac{1}{|z|^{N+2s}}\mathrm{d}z \\
&< +\infty,
\end{aligned}$$

we see that $b \in L^1(K\times K)$. By the Lebsgue Dominated Convergence Theorem, we have $\lim_{\varepsilon\to 0} \|(u_\varepsilon, v_\varepsilon)\|^2_{\mathscr{D}^s_A} = \|(u,v)\|^2_{\mathscr{D}^s_0}$. Then, it follows from (7) that

$$\begin{aligned}
S_A &\leqslant \lim_{\varepsilon\to 0} \|(u_\varepsilon, v_\varepsilon)\|^2_{\mathscr{D}^s_A} \\
&= \|(u,v)\|^2_{\mathscr{D}^s_0} \leqslant S_0 + \varepsilon,
\end{aligned}$$

which means that $S_A \leqslant S_0$. The opposite inequality $S_0 \leqslant S_A$ follows from Lemma 1. This completes the proof. $\square$

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