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LOWER DIMENSIONAL ELLIPSOIDS OF MAXIMAL VOLUME IN CONVEX BODIES

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Abstract. In this paper, we show that the volume of a k-dimensional ellipsoid in the convex body formed by centered isotropic measures on the unit sphere is no large than that of a k-dimensional Ball of radius $\sqrt{n(n+1)/k(k+1)}$. It generalizes the John theorem to the lower dimensional cases.

1. Introduction

Associated with each convex body in $\mathbb{R}^n$ is a unique ellipsoid of maximal volume contained in the body (or minimal volume containing the body). This ellipsoid is called the *John ellipsoid* (or *Löwner ellipsoid*) and plays an important role in convex geometric analysis. A well-known fact about these ellipsoids is the classical John theorem [13] stating that the John ellipsoid of a convex body K is the Euclidean unit ball B_2^n in $\mathbb{R}^n$ if and only if $B_2^n \subset K$ and there are Euclidean unit vectors $(u_i)_1^m$ on the boundary of K and positive numbers $(c_i)_1^m$ satisfying

$$\sum_{i=1}^{m} c_i u_i = 0 \tag{1}$$

and

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n, \tag{2}$$

where $u_i \otimes u_i$ is the rank-one orthogonal projection onto the space spanned by u_i and I_n is the identity map on $\mathbb{R}^n$. As a direct consequence of the John theorem, every n-dimensional normed space is isomorphic, with isomorphism constant at most $\sqrt{n}$, to n-dimensional Euclidean space. Ball [3] made an important observation that the above fact can be perfectly combined with the Brascamp-Lieb inequality and the constant in the Brascamp-Lieb inequality takes a surprisingly simple form. Using this geometric Brascamp-Lieb inequality, Ball [3, 2] established the well-known reverse isoperimetric inequalities and solved the Hensley's conjecture [12] (the maximal volume section of

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the cube). For more information about the John theorem, see, e.g., [1, 5, 6, 7, 8, 9, 10, 11, 14, 15, 16, 17].

As another proof to the John theorem, Ball [4] showed that the largest discs contained in a regular tetrahedron lie in its faces: For each $k < n$, the regular n-dimensional simplex circumscribing B_2^n contains the largest k-dimensional Euclidean ball whose radius is $\sqrt{\frac{n(n+1)}{k(k+1)}}$ in each of its k-dimensional faces. Let

$$K = \{x \in \mathbb{R}^n : x \cdot u_i \leqslant 1, 1 \leqslant i \leqslant m\},$$

and let $(u_i)_1^m$ satisfy (1) and (2) for some positive numbers $(c_i)_1^m$, where $x \cdot u_i$ is the standard inner product of x and u_i in $\mathbb{R}^n$. It was proved by Ball [4] that K does not contain k-dimensional ellipsoid whose volume is larger than of a k-dimensional ball of radius $\sqrt{\frac{n(n+1)}{k(k+1)}}$. In this paper, we are going to follow the lines of Ball [4] and show that it is true for the convex body formed by centered isotropic measures on the unit sphere S^{n-1} in $\mathbb{R}^n$.

It is easy to see that (2) is equivalent to

$$|x|^2 = \int_{S^{n-1}} |x \cdot u|^2 d\gamma_n(u),$$

where $\gamma_n = \frac{1}{2}\sum_{i=1}^m (c_i\delta_{u_i} + c_i\delta_{-u_i})$, and δ_{u_i} denotes the delta measure defined on S^{n-1} by having it concentrated exclusively on $u \in S^{n-1}$. Now (2) leads to the important concept of isotropy of measures, which may be viewed as an extension of the Pythagorean theorem. A nonnegative finite Borel measure ν on S^{n-1} is said to be *isotropic* if

$$\int_{S^{n-1}} u \otimes u d\nu(u) = I_n. \tag{3}$$

Obviously,

$$x \cdot y = \int_{S^{n-1}} (x \cdot u)(y \cdot u) d\nu(u), \quad x, y \in \mathbb{R}^n. \tag{4}$$

Taking the trace in (3) gives

$$\nu(S^{n-1}) = n. \tag{5}$$

If, in addition, ν is *centered*, that is to say, if

$$\int_{S^{n-1}} u d\nu(u) = 0,$$

then the origin 0 is an interior point of the convex hull of the support $\operatorname{supp}\nu$ of ν, and hence

$$Z(\nu) := \{x \in \mathbb{R}^n : x \cdot u \leqslant 1, \ u \in \operatorname{supp}\nu\}$$

is a convex body.

The main result of this note is the following.

THEOREM 1. *Let ν be a centered isotropic measure on S^{n-1}. If E is a k-dimensional ellipsoid in $Z(\nu)$, then the k-dimensional volume of E is no large than that of a k-dimensional Ball of radius $\sqrt{n(n+1)/k(k+1)}$.*

2. Proof of the main result

Note that the proof of Theorem 1 is closely related the classical John theorem [13] which can be stated in the following sense. See [4] for the discrete case and [7] for the symmetric case.

THEOREM 2. *If* ν *is a centered isotropic measure on* S^{n-1}*, then the Euclidean unit ball* B_2^n *is the maximal volume ellipsoid of* $Z(\nu)$*.*

Proof. Let E be the ellipsoid

$$E = \left\{x \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{((x-y)\cdot v_i)^2}{\alpha_i^2} \leqslant 1\right\}$$

for some $y \in \mathbb{R}^n$, orthogonal basis $(v_i)_1^n$ and positive numbers $(\alpha_i)_1^n$. It suffices to show that if $E \subset Z(\nu)$, then $\prod_{i=1}^n \alpha_i \leqslant 1$ with equality only if $\alpha_i = 1$ for all i, and $y = 0$.

For $u \in \operatorname{supp}\nu$, define $g : S^{n-1} \to \mathbb{R}^n$ by

$$g(u) = y + \left(\sum_{i=1}^{n} \alpha_i^2 (u\cdot v_i)^2\right)^{-\frac{1}{2}} \sum_{i=1}^{n} \alpha_i^2 (u\cdot v_i) v_i.$$

It is easy to check that $g(u) \in E$. From the definition of $Z(\nu)$ and $E \subset Z(\nu)$, we have $u \cdot g(u) \leqslant 1$ for each $u \in \operatorname{supp}\nu$. Hence,

$$u\cdot y + \left(\sum_{i=1}^{n} \alpha_i^2 (u\cdot v_i)^2\right)^{\frac{1}{2}} \leqslant 1, \tag{6}$$

for each $u \in \operatorname{supp}\nu$. Integrating both sides with respect to ν and using the fact that $\int_{S^{n-1}} u\, d\nu(u) = 0$ and (5), we get

$$\int_{S^{n-1}} \left(\sum_{i=1}^{n} \alpha_i^2 (u\cdot v_i)^2\right)^{\frac{1}{2}} d\nu(u) \leqslant \int_{S^{n-1}} d\nu(u) = n.$$

Since $\int_{S^{n-1}} (u\cdot v_i)^2 d\nu(u) = |v_i|^2 = 1$ for each i, by the Hölder inequality, we have

$$\begin{aligned}
\sum_{i=1}^{n} \alpha_i &= \sum_{i=1}^{n} \alpha_i \int_{S^{n-1}} (u\cdot v_i)^2 d\nu(u) = \int_{S^{n-1}} \sum_{i=1}^{n} \alpha_i (u\cdot v_i)^2 d\nu(u) \\
&\leqslant \int_{S^{n-1}} \left(\sum_{i=1}^{n} \alpha_i^2 (u\cdot v_i)^2\right)^{1/2} \left(\sum_{i=1}^{n} (u\cdot v_i)^2\right)^{1/2} d\nu(u) \\
&= \int_{S^{n-1}} \left(\sum_{i=1}^{n} \alpha_i^2 (u\cdot v_i)^2\right)^{1/2} d\nu(u) \leqslant n.
\end{aligned}$$

By the arithmetic geometric mean inequality we get $\prod_{i=1}^n \alpha_i \leqslant 1$. There is equality only if $\alpha_i = 1$ for all i. Thus, (6) implies that for each $u \in \operatorname{supp}\nu$

$$u\cdot y + \left(\sum_{i=1}^{n} (u\cdot v_i)^2\right)^{\frac{1}{2}} = u\cdot y + |u| \leqslant 1,$$

which is $u\cdot y \leqslant 0$. Since $\int_{S^{n-1}} u\cdot y d\nu(u) = 0$, this implies that $u\cdot y = 0$ for each $u \in \operatorname{supp}\nu$ and so $y = 0$. □

In order to prove Theorem 1 we need the following lemma.

LEMMA 2. *If x_u is a vectors in $\mathbb{R}^n$ associated with $u \in \operatorname{supp}\nu$ so that $\int_{S^{n-1}} x_u d\nu(u) = 0$, then*

$$\Big(\int_{S^{n-1}} (x_u\cdot u) d\nu(u)\Big)^2 \leqslant \int_{S^{n-1}}\int_{S^{n-1}} |x_u||x_v|(1-(u\cdot v)) d\nu(u) d\nu(v).$$

Proof. By homogeneity, we may assume that $\int_{S^{n-1}} |x_u| d\nu(u) = 1$. Let

$$w = \int_{S^{n-1}} |x_u| u d\nu(u).$$

Then

$$\begin{aligned}\Big(\int_{S^{n-1}} (x_u\cdot u) d\nu(u)\Big)^2 &= \Big(\int_{S^{n-1}} (x_u\cdot u - w) d\nu(u)\Big)^2\\ &\leqslant \Big(\int_{S^{n-1}} |x_u||u-w| d\nu(u)\Big)^2\\ &\leqslant \int_{S^{n-1}} |x_u||u-w|^2 d\nu(u)\\ &= \int_{S^{n-1}} |x_u|(1-2(u\cdot w)+|w|^2) d\nu(u)\\ &= 1-|w|^2\\ &= \int_{S^{n-1}}\int_{S^{n-1}} |x_u||x_v|(1-(u\cdot v)) d\nu(u) d\nu(v). \quad \square\end{aligned}$$

Proof of Theorem 1. Let E be a k-dimensional ellipoid

$$E = \Big\{x\in\mathbb{R}^n : \sum_{i=1}^{k} \frac{((x-y)\cdot v_i)^2}{\alpha_i^2} \leqslant 1,\ (x-y)\cdot v_i = 0,\ k+1 \leqslant i \leqslant n\Big\}$$

for some $y \in \mathbb{R}^n$, orthogonal basis $(v_i)_1^n$ and positive numbers $(\alpha_i)_1^k$. The problem is to show that

$$\Big(\prod_{i=1}^{k} \alpha_i\Big)^{\frac{1}{k}} \leqslant \sqrt{\frac{n(n+1)}{k(k+1)}}.$$

It certainly suffices to show that

$$\sum_{i=1}^{k} \alpha_k \leqslant \sqrt{\frac{kn(n+1)}{(k+1)}}.$$

For $u \in \operatorname{supp}\nu$, define $f : S^{n-1} \to \mathbb{R}^n$ by

$$f(u) = y + \Big(\sum_{i=1}^{k} \alpha_i^2 (u\cdot v_i)^2\Big)^{-\frac{1}{2}} \sum_{i=1}^{k} \alpha_i^2 (u\cdot v_i) v_i.$$

It is easy to check that $f(u) \in E$. From the definition of $Z(\nu)$ and $E \subset Z(\nu)$, we have $u \cdot f(u) \leqslant 1$ for each $u \in \operatorname{supp}\nu$. Hence,

$$u \cdot y + \Big(\sum_{i=1}^{k} \alpha_i^2 (u \cdot v_i)^2\Big)^{\frac{1}{2}} \leqslant 1,$$

for each $u \in \operatorname{supp}\nu$. Define $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Tx = \sum_{i=1}^{k} \alpha_i (x \cdot v_i) v_i.$$

Then

$$u \cdot y + |Tu| \leqslant 1 \tag{7}$$

for each $u \in \operatorname{supp}\nu$. Moreover, from (3) we have

$$\begin{aligned}\sum_{i=1}^{k} \alpha_i^2 &= \sum_{i=1}^{k} \alpha_i^2 \int_{S^{n-1}} (u \cdot v_i)^2 d\nu(u) = \int_{S^{n-1}} \sum_{i=1}^{k} \alpha_i^2 (u \cdot v_i)^2 d\nu(u) \\ &= \int_{S^{n-1}} |Tu|^2 d\nu(u),\end{aligned} \tag{8}$$

and

$$\begin{aligned}\sum_{i=1}^{k} \alpha_i &= \sum_{i=1}^{k} \alpha_i \int_{S^{n-1}} (u \cdot v_i)^2 d\nu(u) = \int_{S^{n-1}} \sum_{i=1}^{k} \alpha_i (u \cdot v_i)^2 d\nu(u) \\ &= \int_{S^{n-1}} (Tu \cdot u) d\nu(u).\end{aligned} \tag{9}$$

By (7) we have $|Tu| \leqslant 1 - (u \cdot y)$ for each $u \in \operatorname{supp}\nu$. Integrating both sides with respect to ν and using the fact that $\int_{S^{n-1}} u d\nu(u) = 0$ and (5) we get

$$\begin{aligned}&\int_{S^{n-1}} |Tu|^2 d\nu(u) \leqslant \int_{S^{n-1}} (1 - (u \cdot y))^2 d\nu(u) \\ &= \int_{S^{n-1}} d\nu(u) - 2\Big(\int_{S^{n-1}} u d\nu(u) \cdot y\Big) + \int_{S^{n-1}} (u \cdot y)^2 d\nu(u) \\ &= n + |y|^2,\end{aligned}$$

which, together with (8), gives

$$\frac{1}{k}\Big(\sum_{i=1}^{k} \alpha_i\Big)^2 \leqslant \sum_{i=1}^{k} \alpha_i^2 \leqslant n + |y|^2. \tag{10}$$

On the other hand, let $x_u = Tu$ for $u \in \operatorname{supp}\nu$. Then

$$\begin{aligned}\int_{S^{n-1}} x_u d\nu(u) &= \int_{S^{n-1}} Tu d\nu(u) = \int_{S^{n-1}} \sum_{i=1}^{k} \alpha_i (u \cdot v_i) v_i d\nu(u) \\ &= \sum_{i=1}^{k} \alpha_i \Big(\int_{S^{n-1}} u d\nu(u) \cdot v_i\Big) v_i \\ &= T\Big(\int_{S^{n-1}} u d\nu(u)\Big) = 0.\end{aligned}$$

Lemma 2 and (9) show that

$$\begin{aligned}\Big(\sum_{i=1}^{k}\alpha_i\Big)^2 &= \Big(\int_{S^{n-1}}(x_u\cdot u)d\nu(u)\Big)^2\\ &\leqslant \int_{S^{n-1}}\int_{S^{n-1}}|x_u||x_v|(1-(u\cdot v))d\nu(u)d\nu(v)\\ &= \int_{S^{n-1}}\int_{S^{n-1}}|Tu||Tv|(1-(u\cdot v))d\nu(u)d\nu(v).\end{aligned}$$

Since $1-(u\cdot v)\geqslant 0$ for all $u,v\in \operatorname{supp}\nu$, by (7), we have

$$\Big(\sum_{i=1}^{k}\alpha_i\Big)^2 \leqslant \int_{S^{n-1}}\int_{S^{n-1}}(1-(u\cdot v))(1-(u\cdot y))(1-(v\cdot y))d\nu(u)d\nu(v).$$

Expanding this product and using the fact that ν is centered, we obtain

$$\Big(\sum_{i=1}^{k}\alpha_i\Big)^2 \leqslant \Big(\int_{S^{n-1}}d\nu(u)\Big)^2-\int_{S^{n-1}}\int_{S^{n-1}}(u\cdot v)(u\cdot y)(v\cdot y)d\nu(u)d\nu(v),$$

which together with (4) and (5) yields

$$\Big(\sum_{i=1}^{k}\alpha_i\Big)^2 \leqslant n^2-|y|^2. \tag{11}$$

This inequality is added to (10) to give

$$\Big(1+\frac{1}{k}\Big)\Big(\sum_{i=1}^{k}\alpha_i\Big)^2 \leqslant n^2+n,$$

and hence

$$\Big(\sum_{i=1}^{k}\alpha_i\Big)^2 \leqslant \frac{kn(n+1)}{k+1}$$

as required. □

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