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OPTIMAL CRITICAL EXPONENT L^p INEQUALITIES OF HARDY TYPE ON THE SPHERE VIA XIAO'S METHOD

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Abstract. First, we correct the proof presented in [Sharp L^p Hardy type and uncertainty principle inequalities on the sphere, Journal of Mathematical Inequalities, 13, 4 (2019), 1011–1022] and obtain a sharp version of an L^p Hardy inequality on the sphere $\mathbb{S}^n$ for all $2 \leqslant p < n$. Secondly, we prove sharp critical exponent L^n inequalities on the sphere $\mathbb{S}^n$ in $\mathbb{R}^n$, $n \geqslant 2$. The singularity in this problem is the geodesic distance from an arbitrary point on the sphere. Moreover, we prove that neither the L^p Hardy inequality has a nontrivial maximizer in $W^{1,p}(\mathbb{S}^n)$ for any $2 \leqslant p < n$, nor does the limiting case L^n Hardy inequality have a nontrivial maximizer in $W^{1,n}(\mathbb{S}^n)$.

1. Introduction

To our best knowledge, the first successful attempt to adapt the ideas introduced in [4] to obtain inequalities of Hardy type on the n-dimensional sphere was that of Xiao's in [8]. He obtained sharp L^2 Hardy inequalities on the Euclidean sphere $\mathbb{S}^n$, $n \geqslant 3$. Later, sharp critical case L^2 results were proved in [2] on the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. Another extension was presented in [7] where subcritical optimal L^p inequalities were proved. The authors in [2, 7, 8] considered the geodesic distance from the pole. In this case, the geodesic distance is precisely the angular variable. Very recently, the author in [9] improved the L^2 results in [8] by taking the singularity to be the geodesic distance from an arbitrary point on the sphere. This was followed by an attempt to obtain the corresponding sharp L^p Hardy inequalities with the general geodesic distance in [1]. The author has proven in [3] various sharp L^p inequalities of Hardy type on the sphere in both the subcritical and critical exponent cases. The way we prove sharpness of our inequalities in [3] takes into account all the constants involved.

Let $u \in C^\infty(\mathbb{S}^n)$, $n \geqslant 3$. Assume that d denotes the geodesic distance on the sphere from an arbitrary point. It is claimed in ([1], Theorem 1) that the following inequality holds for all $2 \leqslant p < n$:

$$\left(\frac{n-p}{p}\right)^p \int_{\mathbb{S}^n} \frac{|u|^p d\sigma_n}{|\tan d|^p} \leqslant \int_{\mathbb{S}^n} |\nabla u|^p d\sigma_n + \left(\frac{n-p}{p}\right)^{p-1} \int_{\mathbb{S}^n} \frac{|u|^p d\sigma_n}{\sin^{p-2} d}. \tag{1}$$

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The proof suggested in [1] is inaccurate. We point out a missing factor in that proof. Once corrected, the proof no longer implies the inequality (1). Interestingly, we obtain an optimal L^p inequality, the inequality (11) below, proved in [3] using the divergence theorem, properties of the gradient and Laplacian of the geodesic distance, and Hölder and Young inequalities. Nevertheless, we believe that the inequality (1) probably holds true.

We also adapt Xiao's method to obtain a sharp critical exponent L^n Hardy type inequality on $\mathbb{S}^n$, $n \geqslant 2$, with the general geodesic distance from an arbitrary point on the sphere. See Theorem 1.

Finally, we prove the nonexistence of nontrivial maximizers in the Sobolev space $W^{1,p}(\mathbb{S}^n)$ for the L^p inequality (11) for any $2 \leqslant p < n$. We also show that the L^n inequality (12) of Theorem 1 has no nontrivial maximizer in $W^{1,n}(\mathbb{S}^n)$. See Theorem 2.

2. Preliminaries

Let $n \geqslant 2$ and let $\Theta_n := (\theta_j)_{j=1}^n \in [0,\pi]^{n-1} \times [0,2\pi]$. We can assign to each point on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ the spherical coordinates parametrization $(x_m(\Theta_n))_{m=1}^{n+1}$, where

$$x_m(\Theta_n) := \begin{cases} \cos\theta_1, & m = 1; \\ \prod_{j=1}^{m-1} \sin\theta_j \cos\theta_m, & 2 \leqslant m \leqslant n; \\ \prod_{j=1}^{n} \sin\theta_j, & m = n+1. \end{cases}$$

With this representation, the surface gradient $\nabla_{\mathbb{S}^n}$ on the sphere $\mathbb{S}^n$ is defined by

$$\nabla_{\mathbb{S}^n} = \frac{\partial}{\partial\theta_1}\widehat{\theta_1} + \frac{1}{\sin\theta_1}\frac{\partial}{\partial\theta_2}\widehat{\theta_2} + \cdots + \frac{1}{\sin\theta_1 \cdots \sin\theta_{n-1}}\frac{\partial}{\partial\theta_n}\widehat{\theta_n},$$

where $\left\{\widehat{\theta_j}\right\}$ is an orthonormal set of tangential vectors with the vector $\widehat{\theta_j}$ pointing in the direction of increase of θ_j. In addition, the Laplace-Beltrami operator $\Delta_{\mathbb{S}^n}$ takes the form

$$\begin{aligned}\Delta_{\mathbb{S}^{n-1}} = &\frac{1}{\sin^{n-2}\theta_1}\frac{\partial}{\partial\theta_1}\left(\sin^{n-2}\theta_1\frac{\partial}{\partial\theta_1}\right) + \frac{1}{\sin^2\theta_1\sin^{n-3}\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin^{n-3}\theta_2\frac{\partial}{\partial\theta_2}\right) \\ &+ \ldots + \frac{1}{\sin^2\theta_1\sin^2\theta_2\ldots\sin^2\theta_{n-2}\sin\theta_{n-1}}\frac{\partial}{\partial\theta_{n-1}}\left(\sin\theta_{n-1}\frac{\partial}{\partial\theta_{n-1}}\right) \\ &+ \frac{1}{\sin^2\theta_1\sin^2\theta_2\ldots\sin^2\theta_{n-1}}\frac{\partial^2}{\partial\theta_n^2}.\end{aligned}$$

Upon, identifying every point $(x_m(\Theta_{n-1}))_{m=1}^{n+1} \in \mathbb{S}^n$ with its parameters Θ_n, the geodesic distance $d(\Theta_n, \Phi_n)$ from a point $\Phi_n \in \mathbb{S}^n$ is defined by

$$d(\Theta_n, \Phi_n) = \arccos\left(\sum_{m=1}^{n+1} x_m(\Theta_n)x_m(\Phi_n)\right). \tag{2}$$

The following properties of the geodesic distance on the sphere are proved in [3]:

LEMMA 1. *Suppose Φ_n is a point on the sphere $\mathbb{S}^n$. Assume that $d(.,\Phi_n):\mathbb{S}^n \to [0,\pi]$ is the geodesic distance from Φ_n on $\mathbb{S}^n$ defined in (2). Then*

$$|\nabla_{\mathbb{S}^{n-1}} d| = 1, \tag{3}$$

$$\Delta_{\mathbb{S}^{n-1}} d = (n-1)\frac{\cos d}{\sin d}. \tag{4}$$

We will also need the following basic inequality that can be found in [5]:

$$|x+y|^p > |x|^p + p|x|^{p-2}\langle x,y\rangle, \quad x,y \in \mathbb{R}^n,\ p>1. \tag{5}$$

An equality holds in (5) if and only if $y=0$.

3. A correction of the proof in [1]

Let $\Phi_n \in \mathbb{S}^n$, $n \geqslant 3$, and let $2 \leqslant p < n$. Let $u \in C^\infty(\mathbb{S}^n)$ and write

$$u(\Theta_n) = \phi^\alpha(\Theta_n)\,\psi(\Theta_n), \tag{6}$$

where $\phi(\Theta_n) := \sin d(\Theta_n,\Phi_n)$, $\alpha = -(n-p)/p$. Clearly $\psi \in C^\infty(\mathbb{S}^n)$. Since the geodesic metric $d(\Theta_n,\Phi_n)=0$ only if $\Theta_n=\Phi_n$ and $d(\Theta_n,\Phi_n)=\pi$ only if Θ_n,Φ_n are antipodal, then $1/\phi \in C^\infty(\mathbb{S}^n\setminus\{\pm\Phi_n\})$. Taking the surface gradient of both sides of (6), then employing the inequality (5), it follows that on $\mathbb{S}^n\setminus\{\pm\Phi_n\}$ we have

$$\begin{aligned}
|\nabla_{\mathbb{S}^n} u|^p &= \left|\alpha\phi^{\alpha-1}\psi\nabla_{\mathbb{S}^n}\phi + \phi^\alpha\nabla_{\mathbb{S}^n}\psi\right|^p \\
&\geqslant |\alpha|^p|\phi|^{\alpha p-p}|\psi|^p|\nabla_{\mathbb{S}^n}\phi|^p \\
&\quad + p|\alpha|^{p-2}|\phi|^{(\alpha-1)(p-2)}|\psi|^{p-2}|\nabla_{\mathbb{S}^n}\phi|^{p-2}\langle\alpha\phi^{\alpha-1}\psi\nabla_{\mathbb{S}^n}\phi,\phi^\alpha\nabla_{\mathbb{S}^n}\psi\rangle \\
&= |\alpha|^p\phi^{\alpha p-p}|\psi|^p|\nabla_{\mathbb{S}^n}\phi|^p \\
&\quad + \alpha|\alpha|^{p-2}\phi^{\alpha p-p+1}\left(p|\psi|^{p-2}\psi\right)|\nabla_{\mathbb{S}^n}\phi|^{p-2}\langle\nabla_{\mathbb{S}^n}\phi,\nabla_{\mathbb{S}^n}\psi\rangle,
\end{aligned} \tag{7}$$

since $\phi>0$. So far, our proof is in accordance with that in [1]. Since $p>1$, then $|\psi|^p$ is differentiable and we have $\nabla_{\mathbb{S}^n}|\psi|^p = p|\psi|^{p-2}\psi\nabla_{\mathbb{S}^n}\psi$. Also, since $1/\phi$ is smooth on on $\mathbb{S}^n\setminus\{\pm\Phi_n\}$ and $\alpha p-p+2 = -(n-2)\neq 0$, then we can write $\phi^{\alpha p-p+1}\nabla_{\mathbb{S}^n}\phi = \frac{1}{\alpha p-p+2}\nabla_{\mathbb{S}^n}\phi^{\alpha p-p+2}$. Using this in (7) implies

$$\begin{aligned}
|\nabla_{\mathbb{S}^n} u|^p \geqslant& |\alpha|^p\phi^{\alpha p-p}|\psi|^p|\nabla_{\mathbb{S}^n}\phi|^p \\
&+\frac{\alpha|\alpha|^{p-2}}{\alpha p-p+2}|\nabla_{\mathbb{S}^n}\phi|^{p-2}\langle\nabla_{\mathbb{S}^n}\phi^{\alpha p-p+2},\nabla_{\mathbb{S}^n}|\psi|^p\rangle.
\end{aligned} \tag{8}$$

At this point, the factor $|\nabla_{\mathbb{S}^n}\phi|^{p-2}$ went unjustifiably missing in [1]. Moreover, the vector $\nabla_{\mathbb{S}^n}|\psi|^p$ is confused with $\nabla_{\mathbb{S}^n}\psi^p$. The next main step in [1] is to write

$$\langle\nabla_{\mathbb{S}^n}\phi^{\alpha p-p+2},\nabla_{\mathbb{S}^n}|\psi|^p\rangle = \operatorname{div}\left(|\psi|^p\nabla_{\mathbb{S}^n}\phi^{\alpha p-p+2}\right) - |\psi|^p\Delta_{\mathbb{S}^n}\phi^{\alpha p-p+2},$$

then use the divergence theorem that yields $\int_{\mathbb{S}^n}\operatorname{div}\left(|\psi|^p\nabla_{\mathbb{S}^n}\phi^{\alpha p-p+2}\right)d\sigma_n = 0$.

If the divergence theorem is to be used, it should rather be applied to $|\nabla_{\mathbb{S}^n}\phi|^{p-2}$ div $\left(|\psi|^p\nabla_{\mathbb{S}^n}\phi^{\alpha p-p+2}\right)$ whose integral does not simply vanish.

Let us proceed from (8). Substituting for α, ϕ and ψ, then using (3), and integrating both sides over $\mathbb{S}^n$, we find

$$\int_{\mathbb{S}^n}|\nabla_{\mathbb{S}^n}u|^p\,d\sigma_n \geqslant \int_{\mathbb{S}^n}\left(\frac{n-p}{p}\right)^p\frac{|u|^p}{|\tan d|^p}d\sigma_n+\frac{1}{n-2}\left(\frac{n-p}{p}\right)^{p-1}I_{n,p}, \tag{9}$$

where

$$I_{n,p}=\int_{\mathbb{S}^n}|\cos d|^{p-2}\left\langle\nabla_{\mathbb{S}^n}\frac{1}{\sin^{n-2}d},\nabla_{\mathbb{S}^n}\left(|u|^p\sin^{n-p}d\right)\right\rangle d\sigma_n.$$

Now, observe that, when $p\geqslant 2$, we can make sense of the gradient

$$\nabla_{\mathbb{S}^n}|\cos d|^{p-2}\cos d=-(p-1)|\cos d|^{p-2}\sin d\,\nabla_{\mathbb{S}^n}d.$$

Therefore, using (4), we may simplify $I_{n,p}$ by integration by parts on the compact manifold $\mathbb{S}^n$ to obtain

$$\begin{aligned}I_{n,p}&=(n-2)\int_{\mathbb{S}^n}|u|^p\sin^{n-p}d\,\mathrm{div}\left(|\cos d|^{p-2}\cos d\frac{\nabla_{\mathbb{S}^n}d}{\sin^{n-1}d}\right)d\sigma_n\\&=-(n-2)(p-1)\int_{\mathbb{S}^n}|u|^p\sin^{n-p}d\frac{|\cos d|^{p-2}}{\sin^{n-2}d}d\sigma_n\\&=-(n-2)(p-1)\int_{\mathbb{S}^n}\frac{|u|^p}{|\tan d|^{p-2}}d\sigma_n,\end{aligned}$$

because, using (3) and (4), it turns out that

$$\left\langle\nabla_{\mathbb{S}^n}\frac{1}{\sin^{n-1}d},\nabla_{\mathbb{S}^n}d\right\rangle=-\frac{\Delta_{\mathbb{S}^n}d}{\sin^{n-1}d},\ \Theta_n\neq\pm\Phi_n. \tag{10}$$

Plugging the integral $I_{n,p}$ into the inequality (9) we deduce the inequality

$$\begin{aligned}\int_{\mathbb{S}^n}|\nabla_{\mathbb{S}^n}u|^p\,d\sigma_n+(p-1)\left(\frac{n-p}{p}\right)^{p-1}&\int_{\mathbb{S}^n}\frac{|u|^p}{|\tan d|^{p-2}}d\sigma_n\\&\geqslant\left(\frac{n-p}{p}\right)^p\int_{\mathbb{S}^n}\frac{|u|^p}{|\tan d|^p}d\sigma_n.\end{aligned}\tag{11}$$

REMARK 1. The inequality (11) is obtained using a different method in [3]. It is also shown in [3] that all three coefficients in (11) are optimal for all $2\leqslant p<n$, using optimizing sequences in the Sobolev space $W^{1,p}(\mathbb{S}^n)$. By density, such functions can be approximated by smooth functions to give optimizing sequences in $C^\infty(\mathbb{S}^n)$.

REMARK 2. When $p=2$, the inequality (11) reduces to the main inequality proved in ([9], Theorem 1.1).

4. Critical $L^n(\mathbb{S}^n)$ Hardy inequality

We show how to apply Xiao's method [8] to prove an optimal critical exponent Hardy inequality on $\mathbb{S}^n$, $n \geqslant 2$, considering the geodesic distance $d(.,\Phi_n)$ defined in (2) from an arbitrary point $\Phi_n \in \mathbb{S}^n$. We will certainly make use of the properties (3) and (4).

THEOREM 1. *Fix $n \geqslant 2$ and assume $u \in C^\infty(\mathbb{S}^n)$. Let Φ_n be some point on the sphere $\mathbb{S}^n$ and consider the geodesic distance (2) from Φ_n. Then*

$$\begin{aligned}\int_{\mathbb{S}^n} |\nabla_{\mathbb{S}^n} u|^n \, d\sigma_n + (n-1)\left(\frac{n-1}{n}\right)^{n-1} \int_{\mathbb{S}^n} \frac{|u|^n}{|\tan d|^{n-2}\left(\log \frac{e}{\sin d}\right)^{n-1}} d\sigma_n \\ \geqslant \left(\frac{n-1}{n}\right)^n \int_{\mathbb{S}^n} \frac{|u|^n}{|\tan d|^n \left(\log \frac{e}{\sin d}\right)^n} d\sigma_n. \qquad (12)\end{aligned}$$

Proof. Suppose $u \in C^\infty(\mathbb{S}^n)$. Analogously to (6), we can write

$$u(\Theta_n) = \phi^\alpha(\Theta_n)\,\psi(\Theta_n),$$

where $\psi \in C^\infty(\mathbb{S}^n)$, but we modify the definition of α and that of ϕ as follows:

$$\phi(\Theta_n) := \log \frac{e}{\sin d(\Theta_n,\Phi_n)}, \qquad \alpha = \frac{n-1}{n}.$$

We note here that $\phi \in C^1\left(\mathbb{S}^n \setminus \{\pm\Phi_n\}\right)$ and we have

$$\nabla_{\mathbb{S}^n}\phi(\Theta_n) = -\frac{1}{\tan d(\Theta_n,\Phi_n)}\nabla_{\mathbb{S}^n} d(\Theta_n,\Phi_n), \quad \Theta_n \neq \pm\Phi_n. \qquad (13)$$

Consequently, in the light of (3), we see that

$$|\nabla_{\mathbb{S}^n}\phi(\Theta_n)| = \frac{1}{|\tan d(\Theta_n,\Phi_n)|}. \qquad (14)$$

We pick up the proof at (8) with $p = n$. Substituting for α, ϕ and ψ, while using (13) and (14), we get

$$|\alpha|^n \phi^{\alpha n - n}|\psi|^n |\nabla_{\mathbb{S}^n}\phi|^n = \left(\frac{n-1}{n}\right)^n \frac{|u|^n}{|\tan d|^n \left(\log \frac{e}{\sin d}\right)^n}, \quad \Theta_n \neq \pm\Phi_n, \qquad (15)$$

$$\begin{aligned}&\frac{\alpha|\alpha|^{n-2}}{\alpha n - n + 2}|\nabla_{\mathbb{S}^n}\phi|^{n-2}\langle\nabla_{\mathbb{S}^n}\phi^{\alpha n - n + 2}, \nabla_{\mathbb{S}^n}|\psi|^n\rangle \\ &= \left(\frac{n-1}{n}\right)^{n-1} \frac{1}{|\tan d|^{n-2}}\left\langle \nabla_{\mathbb{S}^n}\left(\log\frac{e}{\sin d}\right), \nabla_{\mathbb{S}^n}|\psi|^n \right\rangle \qquad (16) \\ &= -\left(\frac{n-1}{n}\right)^{n-1} \frac{1}{|\tan d|^{n-2}\tan d}\langle \nabla_{\mathbb{S}^n} d, \nabla_{\mathbb{S}^n}|\psi|^n\rangle, \quad \Theta_n \neq \pm\Phi_n.\end{aligned}$$

Taking into account the calculations (15) and (16), we integrate both sides of (8), with $p = n$, over $\mathbb{S}^n$. It follows that

$$\int_{\mathbb{S}^n} |\nabla_{\mathbb{S}^n} u|^n \, d\sigma_n \geqslant \left(\frac{n-1}{n}\right)^n \int_{\mathbb{S}^n} \frac{|u|^n}{|\tan d|^n \left(\log \frac{e}{\sin d}\right)^n} d\sigma_n - \left(\frac{n-1}{n}\right)^{n-1} J_{n,p}, \tag{17}$$

where

$$J_{n,p} = \int_{\mathbb{S}^n} \frac{|\cos d|^{n-2} \cos d}{\sin^{n-1} d} \langle \nabla_{\mathbb{S}^n} d, \nabla_{\mathbb{S}^n} |\psi|^n \rangle \, d\sigma_n.$$

Invoking the divergence theorem, we see that

$$J_{n,p} = -\int_{\mathbb{S}^n} |\psi|^n \operatorname{div} \left(\frac{|\cos d|^{n-2} \cos d}{\sin^{n-1} d} \nabla_{\mathbb{S}^n} d \right) d\sigma_n$$

Using the identity (10), we immediately realize

$$\begin{aligned} J_{n,p} &= -\int_{\mathbb{S}^n} \frac{|u|^n}{\left(\log \frac{e}{\sin d(\Theta_n, \Phi_n)}\right)^{n-1}} \left\langle \frac{\nabla_{\mathbb{S}^n} d}{\sin^{n-1} d}, \nabla_{\mathbb{S}^n} |\cos d|^{n-2} \cos d \right\rangle d\sigma_n \\ &= (n-1) \int_{\mathbb{S}^n} \frac{|u|^n}{\left(\log \frac{e}{\sin d(\Theta_n, \Phi_n)}\right)^{n-1}} \frac{1}{|\tan d|^{n-2}} d\sigma_n. \end{aligned} \tag{18}$$

The inequality (12) finally follows from (17) and (18). □

REMARK 3. The inequality (12) is derived in [3] using a different method. It is noteworthy that all constants of (12) are optimal. This is also proved in [3] utilizing optimizing sequences in the Sobolev space $W^{1,n}(\mathbb{S}^n)$. The constants are therefore optimal for smooth functions. For, arguing by contradiction, if a constant in (12) could be improved for smooth functions, then the improved inequality would also be valid for $W^{1,n}(\mathbb{S}^n)$ functions by density.

5. Nonexistence of maximizers

THEOREM 2. *The inequality* (11) *does not have a nonzero maximizer in* $W^{1,p}(\mathbb{S}^n)$ *for any* $2 \leqslant p < n$. *Analogously, the inequality* (12) *does attain a nonzero maximizer in* $W^{1,n}(\mathbb{S}^n)$.

Proof. With the exception of (7), equality persists throughout the proof of (11). The inequality (7) is an application of (5) to the vectors $x := \alpha\phi^{\alpha-1}\psi\nabla_{\mathbb{S}^n}\phi$ and $y := \phi^\alpha \nabla_{\mathbb{S}^n}\psi$. Observe that the inequality (5) is strict if and only if $y \neq 0$. Indeed, when $y \neq 0$, (5) becomes a restating of the strict convexity of the mapping $x \mapsto |x|^p$ on $\mathbb{R}^n$. In other words, equality occurs in (5) if and only if $y = 0$. Hence equality holds in (7), equivalently in (11), if and only if

$$\phi^\alpha \nabla_{\mathbb{S}^n} \psi = 0. \tag{19}$$

Recall that $\phi(\Theta_n) := \sin d(\Theta_n,\Phi_n) \geqslant 0$ and notice that $\phi(\Theta_n)=0$ if and only if Θ_n lies in the negligible set $\{\Theta_n \in \mathbb{S}^n : \Theta_n = \pm\Phi_n\}$. By (6), the equality (19) is equivalent to

$$\nabla_{\mathbb{S}^n}\psi = \nabla_{\mathbb{S}^n}\frac{u}{\phi^\alpha} = 0 \tag{20}$$

for almost every $\Theta_n \in \mathbb{S}^n$. Solving (20) for $u \neq 0$ we find

$$u(\Theta_n) = \frac{c}{\sin^{\frac{n-p}{p}} d(\Theta_n,\Phi_n)}. \tag{21}$$

Taking the surface gradient of (21) and using the identity (3) implies

$$|\nabla_{\mathbb{S}^n}u(\Theta_n)| = |c|\frac{n-p}{p}\frac{|\cos d(\Theta_n,\Phi_n)|}{\sin^{\frac{n}{p}} d(\Theta_n,\Phi_n)}.$$

Obviously $|\nabla_{\mathbb{S}^n}u(\Theta_n)|$ is not essentially bounded. Furthermore, it follows from the definition (2) that $\cos d(\Theta_n,\Phi_n) = \Theta_n\cdot\Phi_n$ and $\sin d(\Theta_n,\Phi_n) = \sqrt{1-(\Theta_n\cdot\Phi_n)^2}$. So, to investigate the integrability of $|\nabla_{\mathbb{S}^n}u|$ on $\mathbb{S}^n$, we exploit the following change of variables (see for example [6]):

$$\int_{\mathbb{S}^n} f(\nu\cdot\Theta_n)\,d\sigma_n = C_n\int_{-1}^{1} f(|\nu|s)\left(1-s^2\right)^{\frac{n-2}{2}} ds, \tag{22}$$

where C_n is a constant that depends only on the dimension n. The identity (22) simplifies the integration of functions of the form $\Theta_n \mapsto f(\nu\cdot\Theta_n)$ on the sphere $\mathbb{S}^n$. Observe that $|\nabla_{\mathbb{S}^n}u(\Theta_n)|$ is a function of $\Phi_n\cdot\Theta_n$. Let $c_{n,p} := |c|^p\left((n-p)/p\right)^p$. Since $|\Phi_n|=1$ then using (22) yields

$$\int_{\mathbb{S}^n}|\nabla_{\mathbb{S}^n}u(\Theta_n)|^p\,d\sigma_n = c_{n,p}\int_{\mathbb{S}^n}\frac{|\Theta_n\cdot\Phi_n|^p}{\left(1-(\Theta_n\cdot\Phi_n)^2\right)^{\frac{n}{2}}}\,d\sigma_n = C_n c_{n,p}\int_{-1}^{1}\frac{|s|^p}{1-s^2}ds,$$

which diverges for every p. Therefore the only nonzero maximizer of (11) is not in $W^{1,p}(\mathbb{S}^n)$ for any $1\leqslant p\leqslant\infty$.

Next, an equality holds in (12) if and only if $u=0$ or

$$u(\Theta_n) = \bar{c}\left(\log\frac{e}{\sin d(\Theta_n,\Phi_n)}\right)^{\frac{n-1}{n}}$$

Consequently, by the identity (3), we have

$$\begin{aligned}|\nabla_{\mathbb{S}^n}u(\Theta_n)|^p &= \tilde{c}_{n,p}\frac{|\cos d(\Theta_n,\Phi_n)|^p}{\sin^p d(\Theta_n,\Phi_n)}\left(\log\frac{e}{\sin d(\Theta_n,\Phi_n)}\right)^{-\frac{p}{n}}\\ &= \tilde{c}_{n,p}\frac{|\Theta_n\cdot\Phi_n|^p}{\left(1-(\Theta_n\cdot\Phi_n)^2\right)^{\frac{p}{2}}}\left(\log\frac{e}{\sqrt{1-(\Theta_n\cdot\Phi_n)^2}}\right)^{-\frac{p}{n}},\end{aligned}$$

with $\tilde{c}_{n,p} = |\bar{c}|^p\,((n-1)/n)^p$. Using (22) we see that

$$\int_{\mathbb{S}^n} |\nabla_{\mathbb{S}^n} u(\Theta_n)|^p\, d\sigma_n = C_n \tilde{c}_{n,p} \int_{-1}^{1} \frac{|s|^p}{(1-s^2)^{\frac{p-n+2}{2}}} ds,$$

which converges only if $p < n$. □

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REFERENCES

[1] ABIMBOLA ABOLARINWA, KAMILU RAUF, SONGTING YIN, *Sharp L^p Hardy type and uncertainty principle inequalities on the sphere*, Journal of Mathematical Inequalities, **13**, 4 (2019), 1011–1022.

[2] AHMED A. ABDELHAKIM, *Limiting case hardy inequalities on the sphere*, Mathematical Inequalities and Applications, **21**, 4 (2018), 1079–1090.

[3] AHMED A. ABDELHAKIM, *Sharp subcritical and critical L^p Hardy inequalities on the sphere*, arXiv:2006.05473.

[4] ISMAIL KOMBE AND MURAD ÖZAYDIN, *Improved Hardy and Rellich inequalities on Riemannian manifolds*, Transactions of the American Mathematical Society, **361**, 12 (2009), 6191–6203.

[5] P. LINDQVIST, *On the equation* $div(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2} u = 0$, Proc. Amer. Math. Soc. **109**, (1990), 157–164.

[6] LOUKAS GRAFAKOS, *Classical Fourier Analysis*, 2nd ed., Springer, 2008.

[7] X. SUN, F. PAN, *Hardy type inequalities on the sphere*, J. Inequalities and Applications, (2017) 2017:148.

[8] Y. XIAO, *Some Hardy inequalities on the sphere*, J. Math. Inequal. 10 (2016), 793–805.

[9] SONGTING YIN, *A sharp Hardy type inequality on the sphere*, New York Journal of Mathematics, 24 (2018) 1101–1110.

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