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SOME UNIQUENESS PROBLEMS CONCERNING MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we investigate some uniqueness problems for forward differences $\Delta_\eta^n f(z)$ and $\Delta_\eta f(z)$ of a meromorphic function $f(z)$. When $f(z)$ is an entire function with a Borel exceptional small function $a(z)$, we obtain the concrete expression of $f(z)$ under the condition that $\Delta_\eta^n f(z)$ and $\Delta_\eta f(z)$ share a small function $b(z)$ CM. When $f(z)$ is a meromorphic function with a small deficient function $a(z)$, we obtain the relationship between $\Delta_\eta^n f(z)$ and $\Delta_\eta f(z)$ who share a small function $b(z)$ and ∞ CM.

1. Introduction and results

We assume that the reader is familiar with the basic notations of Nevanlinna's value distribution theory (see [16, 17, 24]). In particular, we denote by $S(r,f)$ any function satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside a set of r of finite logarithmic measure. A meromorphic function $\alpha(z)$ is said to be a small function of $f(z)$, if $T(r,\alpha(z)) = S(r,f)$, and denote by $S(f)$ the set of functions which are small compare to $f(z)$. In addition, the hyper-order of $f(z)$ is defined by

$$\rho_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r}.$$

The deficiency of $a(z) \in S(f)$ is defined by

$$\delta(a,f) = 1 - \overline{\lim_{r\to\infty}} \frac{N(r,\frac{1}{f-a})}{T(r,f)}.$$

If $\delta(a,f) > 0$, then $a(z)$ is called a small deficient function of $f(z)$.

Furthermore, for a nonzero constant η, the difference operators $\Delta_\eta^n f(z)$ are defined by (see [1, 22]) $\Delta_\eta f(z) = f(z+\eta) - f(z)$ and $\Delta_\eta^{n+1} f(z) = \Delta_\eta^n f(z+\eta) - \Delta_\eta^n f(z)$, where $n = 1,2,\cdots$.

Let f and g be two nonconstant meromorphic functions and $a \in \mathbb{C}$. We say f and g share a CM provided that $f-a$ and $g-a$ have the same zeros counting

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multiplicities, and f and g share ∞ CM provided that f and g have the same poles counting multiplicities. Using the same method, we can define f and g sharing a function $a(z)$ CM, where $a(z) \in S(f) \bigcap S(g)$.

Nevanlinna's four values theorem [20] says that if two nonconstant meromorphic functions f and g share four values CM, then $f \equiv g$ or f is a Möbius transformation of g. To reduce the number of shared values quickly, many authors began to consider the case that $f(z)$ and $g(z)$ have some special relationship. One successful attempt in this direction was created by Rubel and Yang [21] in 1977, and they proved that for a non-constant entire function $f(z)$, if $f(z)$ and $f'(z)$ share two distinct finite values a, b CM, then $f(z) \equiv f'(z)$. Later on, many scholars began to investigate the uniqueness of meromorphic functions sharing values with their derivatives. Now we first recall the following result proved by Li and Wang [19] in 2007.

THEOREM 1. ([19]) *Let $f(z)$ be an entire function, a be a finite nonzero value, and let $n(\geqslant 2)$ be a positive integer. If $f(z)$, $f'(z)$ and $f^{(n)}(z)$ share the value a CM, then $f(z)$ assumes the form*

$$f(z) = be^{cz} + a - \frac{a}{c},$$

where b, c are nonzero constants and $c^{n-1} = 1$.

It is well known that $\Delta_\eta^n f(z)$ is regarded as the difference counterpart of $f^{(n)}(z)$, where $n \geqslant 1$. Recently, value distribution in difference analogues of meromorphic functions has become an interesting subject, and many results of complex difference equations are rapidly obtained (see [6, 7, 9, 12]). In particular, some authors started to consider the uniqueness of meromorphic functions sharing small function with their shifts or difference operators (see [2, 3, 10, 14, 15]). In 2014, Chen and Li [2] considered the difference analogue of Theorem 1, and obtained the following result.

THEOREM 2. ([2]) *Let $f(z)$ be a nonconstant entire function of finite order, and let $a(z)(\not\equiv 0) \in S(f)$ be a periodic entire function with period η. If $f(z)$, $\Delta_\eta f(z)$ and $\Delta_\eta^n f(z)$ $(n \geqslant 2)$ share $a(z)$ CM, then $\Delta_\eta^n f(z) \equiv \Delta_\eta f(z)$.*

In [2], the following example was given to shows that the conclusion of Theorem 2 can occur.

EXAMPLE 1. Let $f(z) = g(z)e^{(\frac{\pi}{4}+\ln\sqrt{2})z} + 1 + i$ and $\eta = 1$, where $g(z)$ is a periodic entire function with period η. Then, $\Delta_\eta f(z) \equiv \Delta_\eta^5 f(z) = ig(z)e^{(\frac{\pi}{4}+\ln\sqrt{2})z}$, and hence $f(z), \Delta_\eta f(z), \Delta_\eta^5 f(z)$ share 1 CM, but $f(z) \not\equiv \Delta_\eta f(z)$.

REMARK 1. This example shows that the order of the function $f(z)$ in Theorem 2 is not always one if let $\rho(g) > 1$ in Example 1, and also shows that the conclusion $\Delta_\eta^n f(z) \equiv \Delta_\eta f(z)$ in Theorem 2 cannot be extended to $f(z) \equiv \Delta_\eta f(z)$ in general.

EXAMPLE 2. Let $f(z) = e^{zln2}$ and $\eta = 1$. By simple calculation, we get $\Delta_\eta f(z) = \Delta_\eta^n f(z) = e^{zln2} = f(z)$. Consequently, for any $a \in \mathbb{C}$, $f(z), \Delta_\eta f(z)$ and $\Delta_\eta^n f(z)$ share a CM and we can easily see that $\Delta_\eta^n f(z) \equiv \Delta_\eta f(z) \equiv f(z)$.

EXAMPLE 3. Let $f(z) = e^{zln2} + 1$ and $\eta = 1$. By calculation, we see that $\Delta_\eta f(z) = e^{zln2}$ and $\Delta_\eta^n f(z) = e^{zln2}$ share every finite value b CM, and so $\Delta_\eta^n f(z) \equiv \Delta_\eta f(z)$, but $\Delta_\eta f(z) \not\equiv f(z)$.

Example 3 shows that $f(z)$ doesn't satisfy the condition "$f(z)$, $\Delta_\eta f(z)$ and $\Delta_\eta^n f(z)$ share a CM", but the conclusion "$\Delta_\eta^n f(z) \equiv \Delta_\eta f(z)$" still holds and doesn't involve the function $f(z)$. What's more, the condition "$f(z)$, $\Delta_\eta f(z)$ and $\Delta_\eta^n f(z)$ share $a(z)$ CM" in Theorem 2 is relatively strong. So some problems arise naturally.

QUESTION 1. It's natural to ask what will happen if we replace the condition "$f(z)$, $\Delta_\eta f(z)$ and $\Delta_\eta^n f(z)$ share $a(z)$ CM" by "$\Delta_\eta f(z)$ and $\Delta_\eta^n f(z)$ share $a(z)$ CM" in Theorem 2?

QUESTION 2. Can we omit the condition "$a(z)(\not\equiv 0)$ be a periodic entire function with period η" and only retain "$a(z)$ is an entire function" in Theorem 2?

QUESTION 3. Can we obtain the specific expression of the function $f(z)$ as in Theorem 1?

In reality, the entire functions $f(z)$ appearing in Examples 2 and 3 both have a finite Borel exceptional value. Therefor, we answer the above questions partly from the point of view of Borel exceptional value by proving the following theorem and give the precise expression of $f(z)$, which is more profound than the conclusion in Theorem 2, and implies that the order of the entire function we considered is always one, which is different from that we point out in Remark 1. The method we used in this article is completely different from that used in [2], and basically comes from [8].

THEOREM 3. *Let $n(\geqslant 2)$ be a positive integer, and let $f(z)$ be a finite order transcendental entire function such that $\lambda(f - a(z)) < \rho(f)$, where $a(z) \in S(f)$ is an entire function and satisfies $\rho(a(z)) < 1$. If $\Delta_\eta^n f(z)$ and $\Delta_\eta f(z)$ share entire function $b(z) \in S(f)$ CM, where $b(z) \not\equiv \Delta_\eta a(z)$, $\rho(b(z)) < 1$, and let $\eta\,(\in \mathbb{C})$ satisfies $\Delta_\eta^n f(z) \not\equiv 0$, then*

$$f(z) = a(z) + De^{cz},$$

where c, D are two nonzero constants.

REMARK 2. It is obvious that Example 3 satisfies Theorem 3. From the conclusion of Theorem 3, we know $f(z)$ has the specific expression $f(z) = a(z) + De^{cz}$, which implies that $f(z)$ is an entire function of regular growth. Combining $a(z), b(z) \in S(f)$, we can see that the condition "$\rho(a) < 1$, $\rho(b) < 1$" is reasonable.

Noting that if $a(z) \equiv a$ is a constant in Theorem 3, then $\Delta_\eta a(z) = 0$. Hence, we can deduce the following Corollary.

COROLLARY 1. *Let $n(\geqslant 2)$ be a positive integer, and let $f(z)$ be a finite order transcendental entire function with a finite Borel exceptional value a. If $\Delta_\eta^n f(z)$ and $\Delta_\eta f(z)$ share a finite value $b(\neq 0)$ CM, where $\eta\,(\in \mathbb{C})$ satisfies $\Delta_\eta^n f(z) \not\equiv 0$, then*

$$f(z) = a + De^{cz},$$

where c, D are two nonzero constants.

During the proof of Theorem 3, we obtain (33). If $a(z) \equiv a$ in the hypotheses of the Theorem 3, then $\Delta_\eta^n a(z) = \Delta_\eta^n a = 0 \ (n \geqslant 1)$. Hence, we have $A = \frac{\Delta_\eta^n a(z) - b(z)}{\Delta a(z) - b(z)} = 1$ by (11), which appears in the proof of Lemma 8. Thus, we can get the following corollary, whose conclusion is the same as that of Theorem 2.

COROLLARY 2. *Let $n (\geqslant 2)$ be a positive integer, and let $f(z)$ be a finite order transcendental entire function with a finite Borel exceptional value a, and let $\eta \, (\in \mathbb{C})$ be a constant satisfies $\Delta_\eta^n f(z) \not\equiv 0$. If $\Delta_\eta^n f(z)$ and $\Delta_\eta f(z)$ share a finite value $b (\neq 0)$ CM, then*

$$\Delta_\eta^n f(z) \equiv \Delta_\eta f(z).$$

The condition "$f(z)$ is a finite order transcendental entire function with $\lambda(f - a) < \rho(f)$" in Theorem 3 implies $\delta(a,f) = 1$. A natural question is what can be obtained if we relax the above restriction? For example, replace $\delta(a,f) = 1$ with $\delta(a,f) > 0$, or let $f(z)$ be a transcendental meromorphic function, or let the order of $f(z)$ be infinite. Next, we consider the above question and obtain the following theorem.

THEOREM 4. *Let $n (\geqslant 2)$ be a positive integer, $f(z)$ be a transcendental meromorphic function with $\rho_2(f) < 1$, and $a(z), b(z) \in S(f)$ such that $b(z) \not\equiv \Delta_\eta^i a(z)$ $(i = 1, n)$ and $\max\{\rho(a), \rho(b)\} < 1$. If $\Delta_\eta^n f(z)$ and $\Delta_\eta f(z)$ share $b(z), \infty$ CM and $\delta(a,f) > 0$, then*

$$\frac{\Delta_\eta^n f(z) - b(z)}{\Delta_\eta f(z) - b(z)} = D$$

for some nonzero constant D. In particular, if the deficient function $a(z) \equiv 0$ and $b(z) \not\equiv 0$, then $\Delta_\eta^n f(z) \equiv \Delta_\eta f(z)$.

2. Lemmas for the Proofs of Theorems

LEMMA 1. ([11, 23]) *Suppose that $n \geqslant 2$ and let $f_1(z), \cdots, f_n(z)$ be meromorphic functions and $g_1(z), \cdots, g_n(z)$ be entire functions such that*

(i) $\Sigma_{j=1}^n f_j(z) \exp\{g_j(z)\} = 0$;

(ii) *when* $1 \leqslant j < k \leqslant n$, $g_j(z) - g_k(z)$ *is not constant;*

(iii) *when* $1 \leqslant j \leqslant n$, $1 \leqslant h < k \leqslant n$,

$$T(r, f_j) = o\{T(r, \exp\{g_h - g_k\})\} \, (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ has finite linear measure or logarithmic measure.

Then $f_j(z) \equiv 0$, $j = 1, \cdots, n$.

ε-set. Following Hayman [16], we define an ε-set to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an ε-set then the set of $r \geqslant 1$ for which the circle $S(0,r)$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

LEMMA 2. ([1]) *Let f be a function transcendental and meromorphic in the plane of order <1. Let $h>0$. Then there exists an ε-set E such that*

$$f(z+c)-f(z)=cf'(z)(1+o(1)), \quad as\ z\to\infty\ in\ \mathbb{C}\backslash E,$$

uniformly in c for $|c|\leqslant h$.

LEMMA 3. ([1]) *Let f be a transcendental function of order less than 1 and h be a positive constant. Then there exists an $\varepsilon-$set E such that*

$$\frac{f'(z+\eta)}{f(z+\eta)}\to 0,\ \frac{f(z+\eta)}{f(z)}\to 1, \quad as\ z\to\infty\ in\ \mathbb{C}\backslash E.$$

uniformly in η for $|\eta|\leqslant h$. Further, the set E may be chosen so that for large $|z|\notin E$, the function f has no zeros or poles in $|\zeta-z|\leqslant h$.

LEMMA 4. ([18]) *Let f be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$W(z,f)P(z,f)=Q(z,f),$$

where $W(z,f), P(z,f), Q(z,f)$ are difference polynomials such that the total degree $\deg W(z,f)\ =n$ *in $f(z)$ and its shifts, and* $\deg Q(z,f)\leqslant n$. *Moreover, we assume that $W(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then, for each $\varepsilon>0$,*

$$m(r,P(z,f))=O(r^{\rho-1+\varepsilon})+S(r,f),$$

possibly outside of an exceptional set of finite logarithmic measure.

LEMMA 5. ([5]) *Let $P_n(z),\cdots,P_0(z)$ be polynomials such that $P_nP_0\not\equiv 0$ and satisfy*

$$P_n(z)+\cdots+P_0(z)\not\equiv 0. \tag{1}$$

Then every finite order transcendental meromorphic solution $f(z)$ $(\not\equiv 0)$ of the equation

$$P_n(z)f(z+n)+P_{n-1}(z)f(z+n-1)+\cdots+P_0(z)f(z)=0 \tag{2}$$

satisfy $\sigma(f)\geqslant 1$, and $f(z)$ assumes every nonzero value $a\in\mathbb{C}$ infinitely often and $\lambda(f-a)=\sigma(f)$.

REMARK 3. If the equation (2) satisfies the condition (1) and all $P_j(z)$ are constants, we can see that the equation (2) does not possess any nonzero polynomial solution. In fact, suppose that $P(z)=a_kz^k+a_{k-1}z^{k-1}+\cdots+a_0$ $(k\geqslant 0, a_k\neq 0)$ is a solution of equation (2). Then we have

$$a_k\cdot(P_n+\cdots+P_0)\cdot z^k+O(z^{k-1})\equiv 0. \tag{3}$$

From (1) and $a_k\neq 0$, we can see that (3) is a contradiction.

LEMMA 6. ([5]) *Let* $F(z), P_n(z), \cdots, P_0(z)$ *be polynomials such that* $FP_nP_0 \not\equiv 0$. *Then every finite order transcendental meromorphic solution* $f(z)$ $(\not\equiv 0)$ *of the equation*

$$P_n(z)f(z+n)+P_{n-1}(z)f(z+n-1)+\cdots+P_0(z)f(z)=F(z) \tag{4}$$

satisfies $\lambda(f)=\sigma(f)\geqslant 1$.

REMARK 4. From the proof of the Lemmas 5, 6 in [5], we can see that if we replace equation (2) by

$$P_n(z)f(z+n\eta)+P_{n-1}(z)f(z+(n-1)\eta)+\cdots+P_0(z)f(z)=0,$$

or equation (4) by

$$P_n(z)f(z+n\eta)+P_{n-1}(z)f(z+(n-1)\eta)+\cdots+P_0(z)f(z)=F(z),$$

then the corresponding conclusion still holds.

LEMMA 7. ([19]) *Suppose that* h *is a nonconstant meromorphic function satisfying*

$$\overline{N}(r,h)+\overline{N}(r,1/h)=S(r,h).$$

Let $f=a_0h^p+a_1h^{p-1}+\cdots+a_p$ *and* $g=b_0h^q+b_1h^{q-1}+\cdots+b_q$ *be polynomials in* h *with coefficients* $a_0, a_1, \cdots, a_p, b_0, b_1, \cdots, b_q$ *being small functions of* h *and* $a_0b_0a_p \not\equiv 0$. *If* $q\leqslant p$, *then* $m(r,g/f)=S(r,h)$.

LEMMA 8. *Let* $n(\geqslant 2)$ *be a positive integer, and let* $f(z)$ *be a finite order transcendental entire function with* $\lambda(f-a(z))<\rho(f)$, *where* $a(z)$ *is an entire function and satisfies* $\rho(a)<1$. *If* $\Delta_\eta^n f(z)\not\equiv 0$ *for some constant* $\eta\,(\in\mathbb{C})$, *and*

$$\frac{\Delta_\eta^n f(z)-b(z)}{\Delta_\eta f(z)-b(z)}=A, \tag{5}$$

where A *is a nonzero constant and* $b(z)$ $(\not\equiv \Delta_\eta a(z))$ *is an entire function satisfying* $\rho(b)<1$, *then*

$$f(z)=a(z)+De^{cz} \quad \text{and} \quad A=\frac{\Delta_\eta^n a(z)-b(z)}{\Delta a(z)-b(z)},$$

where c, D *are two nonzero constants.*

Proof. By the assumptions and Hadamard's factorization theorem, $f(z)$ can be written as

$$f(z)=a(z)+D(z)e^{h(z)}, \tag{6}$$

where $D(z)(\not\equiv 0)$ is an entire function, $h(z)$ is a polynomial with $\deg h = k$ $(k \geqslant 1)$, $H(z)$ and $h(z)$ satisfy

$$\lambda(D) = \rho(D) = \lambda(f - a(z)) = \rho_1 < \rho(f) = \deg h. \tag{7}$$

Substituting (6) into (5), we can get that

$$\frac{\Delta_\eta^n f(z) - b(z)}{\Delta_\eta f(z) - b(z)} = \frac{\sum_{j=0}^{n} (-1)^j C_n^j D(z+(n-j)\eta) e^{h(z+(n-j)\eta)} + v_2(z)}{D(z+\eta)e^{h(z+\eta)} - D(z)e^{h(z)} + v_1(z)} = A, \tag{8}$$

where $v_2(z) = \Delta_\eta^n a(z) - b(z)$, $v_1(z) = \Delta_\eta a(z) - b(z)$.

Obviously, since $\rho(\Delta_\eta^l a(z)) \leqslant \rho(a(z)) < 1$ $(l = 1, n)$, we have

$$\rho(v_j(z)) \leqslant \max\{\rho(a(z)), \rho(b(z))\} < 1 \ (j = 1,2). \tag{9}$$

Rewrite (8) in the form

$$\begin{aligned}
&\sum_{j=0}^{n-2} (-1)^j C_n^j D(z+(n-j)\eta) e^{h(z+(n-j)\eta)-h(z)} \\
&+ [(-1)^{n-1} C_n^{n-1} - A] D(z+\eta) e^{h(z+\eta)-h(z)} + [(-1)^n + A] D(z) \\
&= [Av_1(z) - v_2(z)] e^{-h(z)}.
\end{aligned} \tag{10}$$

Firstly, we assert that $Av_1(z) - v_2(z) \equiv 0$. On the contrary, we suppose that $Av_1(z) - v_2(z) \not\equiv 0$. Then, by (9), we have $\max\{\rho(v_1(z)), \rho(v_2(z))\} < 1 \leqslant k$. Hence, $\rho(Av_1(z) - v_2(z)) < 1 \leqslant k$. From $\rho(D(z)) < \deg h(z) = k$ and $\deg(h(z+j\eta) - h(z)) = k-1$ $(j = 1, 2, \cdots, n)$, we can obtain a contradiction by comparing the growth order of both sides of (10). Hence, $Av_1(z) - v_2(z) \equiv 0$, that is

$$A = \frac{v_2(z)}{v_1(z)} = \frac{\Delta_\eta^n a(z) - b(z)}{\Delta_\eta a(z) - b(z)}. \tag{11}$$

Thus, (10) can be written as

$$\begin{aligned}
&\sum_{j=0}^{n-2} (-1)^j C_n^j D(z+(n-j)\eta) e^{h(z+(n-j)\eta)-h(z)} \\
&+ [(-1)^{n-1} C_n^{n-1} - A] D(z+\eta) e^{h(z+\eta)-h(z)} + [(-1)^n + A] D(z) = 0.
\end{aligned} \tag{12}$$

Secondly, we prove that $\rho(f) = k = 1$. On the contrary, we suppose that $\rho(f) = k \geqslant 2$. Then we will deduce a contradiction for both $A = -(-1)^n$ and $A \neq -(-1)^n$ respectively.

Case 1. Suppose that $A = -(-1)^n \neq 0$, that is, $A + (-1)^n = 0$. Therefore, (12) can be rewritten as

$$\begin{aligned}
&\sum_{j=0}^{n-2} (-1)^j C_n^j D(z+(n-j)\eta) e^{h(z+(n-j)\eta)} \\
&+ [(-1)^{n-1} C_n^{n-1} - A] D(z+\eta) e^{h(z+\eta)} = 0.
\end{aligned} \tag{13}$$

Thus, for the positive integer $n(\geqslant 2)$, there are two subcases: (1) $n=2$; (2) $n\geqslant 3$.

Subcase 1.1. $n=2$. Then $A=-1$ and (13) implies that

$$e^{h(z+2\eta)-h(z+\eta)}=\frac{D(z+\eta)}{D(z+2\eta)}. \tag{14}$$

Set $S_1(z)=\frac{D(z+\eta)}{D(z+2\eta)}$. Then (14) indicates that $S_1(z)$ is a nonconstant entire function. Set $\rho(D)=\rho_1$. Then by a version of the difference analogue of the logarithmic derivative lemma ([9]), for each $\varepsilon_1(0<5\varepsilon_1<k-\rho_1)$, there exists a set $E_1\subset(1,\infty)$ of finite logarithmic measure, such that for all z satisfying $|z|=r\notin[0,1]\cup E_1$, we have

$$\exp\{-r^{\rho_1-1+\varepsilon_1}\}\leqslant\left|\frac{D(z+\eta)}{D(z+2\eta)}\right|\leqslant\exp\{r^{\rho_1-1+\varepsilon_1}\}. \tag{15}$$

Since $S_1(z)$ is an entire function, (15) implies that

$$T(r,S_1(z))=m(r,S_1(z))=m\left(r,\frac{D(z+\eta)}{D(z+2\eta)}\right)\leqslant r^{\rho_1-1+\varepsilon_1},$$

and we have $\rho(S_1(z))\leqslant\rho_1-1+\varepsilon_1<k-1$. Together with $\deg(h(z+\eta)-h(z))=k-1$, we can get a contradiction by (14).

Subcase 1.2. $n\geqslant 3$. Then (13) implies that

$$\sum_{j=0}^{n-2}(-1)^jC_n^j\frac{D(z+(n-j)\eta)}{D(z+\eta)}e^{h(z+(n-j)\eta)-h(z+\eta)}+[(-1)^{n-1}C_n^{n-1}-A]=0. \tag{16}$$

So $S_2(z):=e^{h(z+2\eta)-h(z+\eta)}$ is a transcendental entire function since $\rho(S_2(z))=k-1\geqslant 1$. For $j=3,4,\cdots,n$, we have

$$\begin{aligned}&e^{h(z+j\eta)-h(z+\eta)}\\&=e^{h(z+j\eta)-h(z+(j-1)\eta)}e^{h(z+(j-1)\eta)-h(z+(j-2)\eta)}\cdots e^{h(z+2\eta)-h(z+\eta)}\\&=\underbrace{S_2(z+(j-2)\eta)S_2(z+(j-3)\eta)\cdots S_2(z)}_{j-1\ \text{terms}}.\end{aligned}$$

Thus, (16) can be rewritten as

$$W_2(z,S_2(z))\cdot S_2(z)=A-(-1)^{n-1}C_n^{n-1}, \tag{17}$$

where

$$\begin{aligned}W_2(z,S_2)=&\frac{D(z+n\eta)}{D(z+\eta)}S_2(z+(n-2)\eta)S_2(z+(n-3)\eta)\cdots S_2(z+\eta)\\&-C_n^1\frac{D(z+(n-1)\eta)}{D(z+\eta)}S_2(z+(n-3)\eta)S_2(z+(n-4)\eta)\cdots S_2(z+\eta)\\&+\cdots+(-1)^{n-2}C_n^{n-2}\frac{D(z+2\eta)}{D(z+\eta)}.\end{aligned}$$

According to $A=-(-1)^n$, we get $A-(-1)^{n-1}C_n^{n-1}=(-1)^{n-1}(1-C_n^{n-1})=(-1)^n(n-2)\not\equiv 0$. Obviously, $W_2(z,S_2(z))\not\equiv 0$ by (17). Set $\rho(D)=\rho_2$, then $\rho_2<k$. Since $S_2(z)$ is of regular growth and $\rho(S_2(z))=k-1$, for any given ε_2 $(0<5\varepsilon_2<k-\rho_2)$ and all $r>r_0$ (>0), we have

$$T(r,S_2(z))>r^{k-1-\varepsilon_2}. \tag{18}$$

On the other hand, the difference analogue of the logarithmic derivative lemma ([9]) gives that for ε_2, there exists a set $E_2\subset(1,\infty)$ of finite logarithmic measure, such that for all z satisfying $|z|=r\notin[0,1]\cup E_2$, we have

$$\exp\{-r^{\rho_2-1+\varepsilon_2}\}\leqslant\left|\frac{D(z+j\eta)}{D(z+\eta)}\right|\leqslant\exp\{r^{\rho_2-1+\varepsilon_2}\}\quad (j=2,3,\cdots,n). \tag{19}$$

Thus, (18) and (19) give

$$m\left(r,\frac{D(z+j\eta)}{D(z+\eta)}\right)=S(r,S_2)\quad (j=2,3,\cdots,n). \tag{20}$$

Noting that $\deg_{S_2}W_2(z,S_2)=n-2\geqslant 1$ and applying Lemma 4 to (17), we have

$$T(r,S_2)=m(r,S_2)=S(r,S_2).$$

But $S_2(z)$ is an transcendental entire function, it is absurd.

Case 2. Suppose $A\neq-(-1)^n$, that is, $A+(-1)^n\neq 0$. Then (12) gives that

$$\begin{aligned}&\sum_{j=0}^{n-2}(-1)^jC_n^j\frac{D(z+(n-j)\eta)}{D(z)}e^{h(z+(n-j)\eta)-h(z)}\\&+[(-1)^{n-1}C_n^{n-1}-A]\frac{D(z+\eta)}{D(z)}e^{h(z+\eta)-h(z)}=(-1)^{n+1}-A.\end{aligned} \tag{21}$$

Denote $S_3(z):=e^{h(z+\eta)-h(z)}$, then $S_3(z)$ is a nonconstant entire function. Thus, (21) can be written as

$$W_3(z,S_3(z))\cdot S_3(z)=-[(-1)^n+A], \tag{22}$$

where

$$\begin{aligned}W_3(z,S_3)=&\frac{D(z+n\eta)}{D(z)}S_3(z+(n-1)\eta)S_3(z+(n-2)\eta)\cdots S_3(z+\eta)\\&-C_n^1\frac{D(z+(n-1)\eta)}{D(z)}S_3(z+(n-2)\eta)S_3(z+(n-3)\eta)\cdots S_3(z+\eta)\\&+\cdots\\&+(-1)^{n-2}C_n^{n-2}\frac{D(z+2\eta)}{D(z)}S_3(z+\eta)+[(-1)^{n-1}C_n^{n-1}+A]\frac{D(z+\eta)}{D(z)}.\end{aligned}$$

Obviously, $W_3(z,S_3(z))\not\equiv 0$. Thus, for $\deg_{S_3}W_3(z,S_3)=n-1\geqslant 1$, we can get $T(r,S_3)=m(r,S_3)=S(r,S_3)$ by apply the method of proof in Subcase 1.2 and deduce a contradiction.

So $\rho(f) = \deg h(z) = 1$. Let $h(z) = cz + c_0$, where $c\ (\neq 0), c_0$ are two constants. Then $f(z)$ has the form

$$f(z) = a(z) + D_*(z)e^{cz+c_0} = a(z) + D(z)e^{cz}, \tag{23}$$

where $D(z) = D_*(z)e^{c_0}\ (\not\equiv 0)$ is an entire function satisfying

$$\rho(D(z)) = \lambda(D(z)) = \lambda(f(z) - a(z)) < \rho(f) = 1.$$

Thirdly, we assert that $D(z)\ (\not\equiv 0)$ is a constant. To our purpose, we only need to show $D'(z) \equiv 0$. According to (12) and $h(z) = cz + c_0$, we deduce that

$$\begin{aligned}&\sum_{j=0}^{n-2}(-1)^j C_n^j e^{(n-j)c\eta} D(z+(n-j)\eta)\\&+[(-1)^{n-1}C_n^{n-1} - A]e^{c\eta}D(z+\eta) + [(-1)^n + A]D(z) = 0.\end{aligned} \tag{24}$$

We show that the sum of all coefficients of (24) is equal to zero, that is

$$\begin{aligned}&\sum_{j=0}^{n-2}(-1)^j C_n^j e^{(n-j)c\eta} + [(-1)^{n-1}C_n^{n-1} - A]e^{c\eta} + ((-1)^n + A)\\&= (e^{c\eta} - 1)[(e^{c\eta} - 1)^{n-1} - A] = 0.\end{aligned} \tag{25}$$

On the contrary, suppose that

$$e^{nc\eta} - C_n^1 e^{(n-1)c\eta} + \cdots + [(-1)^{n-1}C_n^{n-1} - A]e^{c\eta} + [(-1)^n + A] \neq 0.$$

Then we have $\rho(D) \geqslant 1$ by applying Lemma 5 to (24) and noting Remarks 3–4, a contradiction. Hence, (25) holds and gives

$$e^{c\eta} = 1 \quad \text{or} \quad A = (e^{c\eta} - 1)^{n-1}. \tag{26}$$

On the other hand, (25) implies that

$$(-1)^n + A = -\left[e^{nc\eta} - C_n^1 e^{(n-1)c\eta} + \cdots + \left((-1)^{n-1}C_n^{n-1} - A\right)e^{c\eta}\right].$$

Substituting the above expression into equation (24), we obtain

$$\begin{aligned}&\sum_{j=0}^{n-2}(-1)^j C_n^j e^{(n-j)c\eta}\,(D(z+(n-j)\eta) - D(z))\\&+\left[(-1)^{n-1}C_n^{n-1} - A\right]e^{c\eta}(D(z+\eta) - D(z)) = 0.\end{aligned} \tag{27}$$

Lemma 2 implies that there exists ε-sets E_j^* such that for $j = 1, 2, \cdots, n$, we have

$$D(z + j\eta) - D(z) = j\eta D'(z) + o_j(1)D'(z) \quad \text{as } z \to \infty \text{ in } \mathbb{C}\backslash E_j^*. \tag{28}$$

Together with (27), we obtain

$$\eta D'(z) \cdot K_1 + D'(z) \cdot o(1) = 0 \quad \text{as } z \to \infty \text{ in } \mathbb{C}\backslash E, \tag{29}$$

where $E = \bigcup_{j=1}^{n} E_j^*$ and K_1 is a constant satisfying

$$K_1 = ne^{nc\eta} - (n-1)C_n^1 e^{(n-1)c\eta} + \cdots + (-1)^{n-2} 2C_n^{n-2} e^{2c\eta} + [(-1)^{n-1}C_n^{n-1} - A]e^{c\eta}.$$

We assert that $K_1 \neq 0$. On the contrary, suppose $K_1 = 0$. Noting that $C_n^j \cdot (n-j) = n \cdot C_{n-1}^j$ $(j = 0, 1, \cdots, n-1)$, we have

$$\begin{aligned} & ne^{nc\eta} - C_n^1(n-1)e^{(n-1)c\eta} + \cdots + (-1)^{n-2}C_n^{n-2}2e^{2c\eta} + [(-1)^{n-1}C_n^{n-1} - A]e^{c\eta} \\ = {} & ne^{c\eta}\left[e^{(n-1)c\eta} + (-1)C_{n-1}^1 e^{(n-2)c\eta} + \cdots + (-1)^{n-2}C_{n-1}^{n-2}e^{c\eta} + (-1)^{n-1}\right] - Ae^{c\eta} \\ = {} & e^{c\eta}[n(e^{c\eta} - 1)^{n-1} - A] = 0, \end{aligned}$$

which implies

$$A = n(e^{c\eta} - 1)^{n-1}. \tag{30}$$

Together with (26), if $e^{c\eta} = 1$, then we have $A = 0$ by (30), a contradiction. If $A = (e^{c\eta} - 1)^{n-1}$, then we have

$$n(e^{c\eta} - 1)^{n-1} = (e^{c\eta} - 1)^{n-1},$$

which implies $e^{c\eta} = 1$ since $n \geqslant 2$. Hence, (30) gives $A = n(e^{c\eta} - 1)^{n-1} = 0$, it is absurd.

So $K_1 \neq 0$. (29) implies $D'(z) \equiv 0$. Thus, we can see that $D(z)$ is a nonzero constant. Hence, $f(z)$ has the form

$$f(z) = a(z) + De^{cz}, \tag{31}$$

where c, D are two nonzero constants. $\square$

3. Proof of Theorem 3

According to the conditions of the Theorem 3, we can see that (6) and (7) still hold. Since $\Delta_\eta^n f(z)$ and $\Delta_\eta f(z)$ share $b(z) (\not\equiv \Delta_\eta a(z))$ CM, then

$$\frac{\Delta_\eta^n f(z) - b(z)}{\Delta_\eta f(z) - b(z)} = \frac{\sum_{j=0}^{n} (-1)^{n-j} C_n^j D(z+j\eta) e^{h(z+j\eta)} + v_2(z)}{D(z+\eta)e^{h(z+\eta)} - D(z)e^{h(z)} + v_1(z)} = e^{P(z)}, \tag{32}$$

where $P(z)$ is a polynomial, $v_2(z) = \Delta_\eta^n a(z) - b(z)$, $v_1(z) = \Delta_\eta a(z) - b(z)$, and $\rho(v_j(z)) < 1$ $(j = 1, 2)$.

In order to prove Theorem 3, we only need to prove

$$\frac{\Delta_\eta^n f(z) - b(z)}{\Delta_\eta f(z) - b(z)} = D, \tag{33}$$

where D is a nonzero constant. Then by Lemma 8, we can obtain the conclusion of Theorem 1.1.

If $P(z) \equiv 0$, then (33) obviously holds by (32). So we only need to suppose that $P(z) \not\equiv 0$ and assert that $\deg P(z) = 0$.

Set $\deg P(z) = s$ and

$$h(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0, \quad P(z) = b_s z^s + b_{s-1} z^{s-1} + \cdots + b_0, \tag{34}$$

where $k = \rho(f) \geqslant 1$, $a_k (\neq 0)$, $a_{k-1}, \cdots, a_0$, $b_s (\neq 0)$, $b_{s-1}, \cdots, b_0$ are constants. Then (32) implies that

$$0 \leqslant \deg P = s \leqslant \deg h = k.$$

In what follows, we prove that neither $1 \leqslant s = k$ nor $1 \leqslant s < k$ holds.

Case 1. Suppose that $1 \leqslant s = k$. Then, (32) implies that

$$G_{11}(z) e^{P(z)} + G_{12} e^{P(z)-h(z)} + G_{13} e^{-h(z)} + G_{14} = 0, \tag{35}$$

where

$$\begin{aligned} &T(r, G_{11}) = D(z+\eta) e^{h(z+\eta)-h(z)} - D(z); \\ &T(r, G_{12}) = v_1(z); \\ &T(r, G_{13}) = -v_2(z); \\ &T(r, G_{14}) = -\sum_{j=0}^{n} (-1)^{n-j} C_n^j D(z+j\eta) e^{h(z+j\eta)-h(z)}. \end{aligned}$$

Thus, there are three subcases: (1) $b_k \neq \pm a_k$; (2) $b_k = a_k$; (3) $b_k = -a_k$.

Subcase 1.1. Suppose that $b_k \neq \pm a_k$. Then, we can deduce from $\rho(D) < k$ and $\deg(h(z+j\eta) - h(z)) = k-1$ $(j = 1, 2, \cdots, n)$ that $\rho(G_{1m}(z)) < k$ $(m = 1, 2, 3, 4)$. On the other hand, $b_k \neq \pm a_k$ implies that $\deg(P \pm h) = \deg(P) = \deg(-h) = k$. Considering $e^{P \pm h}$, e^P and e^{-h} are of regular growth, and $\rho(G_{1m}) < k$ $(m = 1, 2, 3, 4)$, we can obtain that

$$\begin{aligned} &T(r, G_{1m}) = o\left(T\left(r, e^{P \pm h}\right)\right); \\ &T(r, G_{1m}) = o\left(T\left(r, e^{P}\right)\right); \\ &T(r, G_{1m}) = o\left(T\left(r, e^{-h}\right)\right), \end{aligned} \tag{36}$$

for $m = 1, 2, 3, 4$.

Thus, applying Lemma 1 to (35) and combining (36), we have $G_{1m}(z) \equiv 0$ $(m = 1, 2, 3, 4)$. Thus, $\Delta_\eta a(z) - b(z) = v_1(z) = G_{12}(z) \equiv 0$, which contracts the assumption $b(z) \not\equiv \Delta_\eta a(z)$.

Subcase 1.2. Suppose that $b_k = a_k$. Rewrite (35) as

$$G_{21}(z) e^{P(z)} + G_{22} e^{-h(z)} + G_{23} = 0, \tag{37}$$

where $h_0(z) \equiv 0$ and

$$\begin{aligned} &T(r, G_{21}) = D(z+\eta) e^{h(z+\eta)-h(z)} - D(z); \\ &T(r, G_{22}) = -v_2(z); \\ &T(r, G_{23}) = v_1(z) e^{P(z)-h(z)} - \sum_{j=0}^{n} (-1)^{n-j} C_n^j D(z+j\eta) e^{h(z+j\eta)-h(z)}. \end{aligned}$$

Since $\rho(D) < k$ and $\deg(h(z+j\eta)-h(z)) = k-1$ $(j = 1,2,\cdots,n)$, we can see that $\rho(G_{2m}(z)) < k$ $(m = 1, 2, 3)$. On the other hand, by $b_k = a_k$, we can see that $\deg(P+h) = \deg(P) = \deg(-h) = k$. Since e^{P+h}, e^P and e^{-h} are of regular growth, and $\rho(G_{2m})$ $< k$ $(m = 1, 2, 3)$, we can see that

$$\begin{aligned} T(r, G_{2m}) &= o\left(T\left(r, e^{P+h}\right)\right); \\ T(r, G_{2m}) &= o\left(T\left(r, e^{P}\right)\right); \\ T(r, G_{2m}) &= o\left(T\left(r, e^{-h}\right)\right), \end{aligned} \tag{38}$$

for $m = 1, 2, 3$.

Thus, applying Lemma 1 to (37), by (38), we can obtain

$$G_{2m}(z) \equiv 0 \quad (m = 1, 2, 3).$$

From $G_{21}(z) = 0$, we get $D(z+\eta)e^{h(z+\eta)} \equiv D(z)e^{h(z)}$. Together with $f(z) = a(z) + D(z)e^{h(z)}$, we have $\Delta_\eta f(z) = \Delta_\eta a(z)$, which implies $\Delta_\eta^n f(z) = \Delta_\eta^n a(z)$. Thus, by $G_{22}(z) = v_2(z) \equiv 0$, we have $\Delta_\eta^n f(z) - b(z) = \Delta_\eta^n a(z) - b(z) = v_2(z) = -G_{22}(z) \equiv 0$. It is impossible by (32).

Subcase 1.3. Suppose that $b_k = -a_k$. Then (35) is rewritten as

$$G_{31}(z)e^{P(z)} + G_{32}e^{P(z)-h(z)} + G_{33} = 0, \tag{39}$$

where

$$\begin{aligned} T(r, G_{31}) &= D(z+\eta)e^{h(z+\eta)-h(z)} - D(z) - v_2(z)e^{-P(z)-h(z)}; \\ T(r, G_{32}) &= v_1(z); \\ T(r, G_{33}) &= -\sum_{j=0}^{n}(-1)^{n-j}C_n^j D(z+j\eta)e^{h(z+j\eta)-h(z)}. \end{aligned}$$

Using a proof similar to that of subcase 1.1, we have $G_{3m}(z) \equiv 0$ $(m = 1, 2, 3)$. So $\Delta_\eta a(z) - b(z) = v_1(z) = G_{32}(z) \equiv 0$, which contracts $b(z) \not\equiv \Delta_\eta a(z)$.

Case 2. Suppose that $1 \leqslant s < k$. Then, (32) implies that

$$\begin{aligned} &\sum_{j=0}^{n}(-1)^{n-j}C_n^j D(z+j\eta)e^{h(z+j\eta)-h(z)} \\ &- \left[D(z+\eta)e^{h(z+\eta)-h(z)} - D(z)\right]e^{P(z)} = \left[v_1(z)e^{P(z)} - v_2(z)\right]e^{-h(z)}. \end{aligned} \tag{40}$$

By $\rho(v_j(z)) < 1$ $(j = 1,2)$ and $1 \leqslant \deg P(z) = s < k$, we know that $v_1(z)e^{P(z)} - v_2(z) \not\equiv 0$ and $\rho(v_1(z)e^{P(z)} - v_2(z)) = \deg P(z) = s < k$. Comparing the growth order of both sides of equation (40), we get a contradiction.

4. Proof of Theorem 4

Since $\Delta_\eta^n f(z)$ and $\Delta_\eta f(z)$ share $b(z)$ and ∞ CM, we obtain

$$\frac{\Delta_\eta^n f(z) - b(z)}{\Delta_\eta f(z) - b(z)} = e^{P(z)}, \tag{41}$$

where $P(z)$ is an entire function. By (41), we have

$$T(r,e^{P(z)}) = O(T(r,f)),$$

which implies

$$S(r,e^{P(z)}) = S(r,f).$$

By $T(r,\Delta_\eta^j f(z)) = O(T(r,f))$, we get $S(r,\Delta_\eta^j f(z)) = S(r,f)$, $(j=1,n)$.

Now we prove that $P(z)$ is a constant. Suppose that, on the contrary, $P(z)$ is not a constant. Since for $j=1,n$

$$\max\{\rho(a),\rho(b)\}<1, \max\{\rho(\Delta_\eta^j a(z)),\rho(\Delta_\eta^j b(z))\} \leqslant \max\{\rho(a),\rho(b)\}<1$$

and $e^{P(z)}$ is of regular growth with $\rho(e^P)\geqslant 1$, we have

$$\begin{aligned}\max\{T(r,a(z)),T(r,b(z))\} &= o\left(T\left(r,e^P\right)\right);\\ \max\{T(r,\Delta_\eta^j a(z)),T(r,\Delta_\eta^j b(z))\} &= o\left(T\left(r,e^P\right)\right),(j=1,n).\end{aligned} \tag{42}$$

From (41), we have

$$\Delta_\eta^n(f(z)-a(z)) - e^{P(z)}\Delta_\eta(f(z)-a(z)) = b(z)-\Delta_\eta^n a(z)+[\Delta_\eta a(z)-b(z)]e^{P(z)}.$$

We assert that $b(z)-\Delta_\eta^n a(z)+(\Delta_\eta a(z)-b(z))e^{P(z)}\not\equiv 0$. Otherwise, suppose $b(z)-\Delta_\eta^n a(z)+(\Delta_\eta a(z)-b(z))e^{P(z)}\equiv 0$, then

$$e^{P(z)} = \frac{\Delta_\eta^n a(z)-b(z)}{\Delta_\eta a(z)-b(z)},$$

which implies that

$$\rho(e^{P(z)}) \leqslant \max\{\rho(\Delta_\eta^n a), \rho(\Delta_\eta a), \rho(b)\} \leqslant \max\{\rho(b)\rho(a)\}<1,$$

contradicting $\rho(e^P)\geqslant 1$. So $b(z)-\Delta_\eta^n a(z)+(\Delta_\eta a(z)-b(z))e^{P(z)}\not\equiv 0$.

Dividing the above equality by $[b(z)-\Delta_\eta^n a(z)+(\Delta_\eta a(z)-b(z))e^{P(z)}](f(z)-a(z))$, we have

$$\frac{\frac{\Delta_\eta^n(f(z)-a(z))}{f(z)-a(z)} - \frac{\Delta_\eta(f(z)-a(z))}{f(z)-a(z)}e^{P(z)}}{b(z)-\Delta_\eta^n a(z)+(\Delta_\eta a(z)-b(z))e^{P(z)}} = \frac{1}{f(z)-a(z)}. \tag{43}$$

Since $b(z)-\Delta_\eta^j a(z)\not\equiv 0$ $(j=1,n)$, from Lemma 7 and (42), we deduce that

$$m\left(r,\frac{1}{b(z)-\Delta_\eta^n a(z)+(\Delta_\eta a(z)-b(z))e^{P(z)}}\right) = S(r,e^P),$$

$$m\left(r,\frac{e^{P(z)}}{b(z)-\Delta_\eta^n a(z)+(\Delta_\eta a(z)-b(z))e^{P(z)}}\right) = S(r,e^P).$$

Furthermore, by a version of the difference analogue of the logarithmic derivative in [13], we can obtain

$$m\left(r, \frac{\Delta_\eta^j (f(z)-a(z))}{f(z)-a(z)}\right) = S(r,f), \quad j=1,n.$$

So by (43), we have

$$m\left(r, \frac{1}{f(z)-a(z)}\right) = S(r,e^P)+S(r,f) = S(r,f),$$

which gives $\delta(a,f)=0$, contradicting the assumption $\delta(a,f)>0$. Hence we have proved that $P(z)$ is a constant. Setting $e^{P(z)}=D$, we have

$$\frac{\Delta_\eta^n f(z)-b(z)}{\Delta_\eta f(z)-b(z)} = D. \tag{44}$$

Next we consider the case $a(z)\equiv 0$ and $b(z)\not\equiv 0$, then (44) implies that

$$\Delta_\eta^n f(z) - D\Delta_\eta f(z) = (1-D)b(z).$$

If $D\neq 1$, then dividing the above equality by $(1-D)b(z)f(z)$, we obtain

$$\frac{1}{(1-D)b(z)}\frac{\Delta_\eta^n f(z)}{f(z)} - \frac{D}{(1-D)b(z)}\frac{\Delta_\eta f(z)}{f(z)} = \frac{1}{f(z)}.$$

So by (42) and the difference analogue of the logarithmic derivative in [14], we get

$$m\left(r, \frac{1}{f(z)}\right) = S(r,f),$$

which gives $\delta(0,f)=0$, contradicts $\delta(0,f)>0$. Hence $D=1$ and $\Delta_\eta^n f(z)\equiv\Delta_\eta f(z)$.

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