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WEAKLY SINGULAR HENRY–GRONWALL–BIHARI TYPE INEQUALITIES AND THEIR APPLICATIONS

NGUYEN MINH DIEN

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Abstract. In this work, we establish some new weakly singular Henry-Gronwall-Bihari type inequalities that are generalized from some recent works. Unlike most previous papers, in this work, we consider integral inequalities that include two integrals with doubly singular kernels, and obtain the bounds by an exponential function. We apply the obtained results to investigate the existence and uniqueness of solution of a fractional differential equation and a class of integral equations with weakly singular sources.

1. Introduction

It is well known that Henry-Gronwall-Bihari type integral inequalities play a significant role in the study of quantitative properties of solutions of differential and integral equations. There are numerous versions of Henry-Gronwall-Bihari inequalities and their applications, we refer to [2, 3, 5, 8, 9, 11, 12, 13, 14, 15, 16, 21], and the references therein.

In the last few decades, the integral inequalities with weakly singular kernels have attracted the attention of many researchers due to its applications to various problems related to fractional derivatives. Usually, the integrals involving this type of inequalities have a singularity, but some problems of theory and practicality require us to solve integral inequalities with doubly singularities (see e.g. [4, 5, 19, 20, 21]).

In the literature, integral inequalities with doubly singular kernels have been studied by many researchers. In fact, Henry [8] considered the following integral inequality

$$u(t) \leqslant p+q\int_0^t (t-\tau)^{\beta-1}\tau^{\gamma-1}u(\tau)\,\mathrm{d}\tau \ \text{ for } \ \beta>0, \gamma>0, \ \text{ and } \ \beta+\gamma>1. \quad (1)$$

Under an appropriate assumption, the author obtained the bounds by Mittag-Leffler function. In 2019, Webb [21] extended the result of Henry by investigating the integral inequality (1), found the bounds by an exponential function and applied it to study fractional differential equations with weakly singular sources. Ma and Pecaric [11] considered the following integral inequality

$$u^p(t) \leqslant p(t)+q(t)\int_0^t (t^\alpha-\tau^\alpha)^{\beta-1}\tau^{\gamma-1}f(\tau)u^q(\tau)\,\mathrm{d}\tau,$$

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obtained some bounds and applied them to study fractional differential and integral equations. Medved [12] considered the integral inequality

$$u(t) \leqslant p(t) + \int_0^t (t-\tau)^{\alpha-1}\tau^{\gamma-1} f(\tau)\omega(u(\tau))\, \mathrm{d}\tau.$$

To obtain the bounds, the author used the following restrictions $\alpha > 1/2,\ \gamma > 1/2$, or $\alpha = 1/(m+1)$ and $\gamma = 1 - 1/\kappa(m+2)$ for some $m \geqslant 1,\ \kappa > 1$. The author also applied the obtained results to study semilinear evolution equations. With the above restrictions, we emphasize that the problem can not discuss when $\gamma \leqslant 1/2$. Therefore, in this work, we would like to relax the above restrictions.

For the case, the function u may be singular, Henry [8] considered a function u belonging to $L^1[0,T]$ and satisfies the following integral inequality

$$u(t) \leqslant at^{-\alpha} + b\int_0^t (t-\tau)^{-\beta} u(\tau)\, \mathrm{d}\tau \ \text{ for } \ \alpha,\beta \in [0,1), a \geqslant 0, b > 0,$$

and proved that $u(t) \leqslant Ct^{-\alpha}$. The extension of the result of Henry was made by Webb [21], beginning with the following integral inequality

$$u(t) \leqslant at^{-\alpha} + b + c\int_0^t (t-\tau)^{-\beta}\tau^{-\gamma} u(\tau)\, \mathrm{d}\tau \ \text{ for } \ a,b \geqslant 0, c > 0, \alpha+\beta+\gamma < 1,$$

the author obtained the bounds by exponential function, and applied the obtained inequality to study a fractional differential equation involving the Riemann-Liouville fractional derivative with a weakly singular source. Zhu [23] considered the integral inequality

$$u(t) \leqslant at^{-\alpha} + bt^{-\delta}\int_0^t (t-\tau)^{\beta-1} l(\tau) u(\tau)\, \mathrm{d}\tau \ \text{ for } \ a,b \geqslant 0, \alpha > \delta \geqslant 0.$$

Here the author used the assumptions $t^{-\alpha} l(t) \in L^q_{Loc}[0,+\infty)$ with $q > 1/\beta$ and $t^\alpha u(t)$ a continuous and non-negative function on $[0,+\infty)$.

Recently, integral inequalities for an appropriate function was proposed by Sousa et al [17], in which the following integral inequality

$$u(t) \leqslant p(t) + \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} u(\tau)\, \mathrm{d}\tau \qquad (2)$$

was considered. Under some appropriate assumptions, the authors found the bounds by Mittag-Leffler function and applied it to study the Cauchy-type problem with ψ-Hilfer operator. Very recently, Boulares et al [2] also considered the integral inequality (2) and obtain the bounds by an exponential function. However, we can not find any paper deal with Henry-Gronwall-Bihari type inequalities involving an appropriate function ψ and doubly singular kernels.

Integral inequalities with two singular kernels were studied by Ding et al [6], starting by considering the following integral inequality

$$u(t) \leqslant a(t) + b(t)\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) u(s)\, \mathrm{d}s + c(t)\int_0^t (t-s)^{\beta-1} u(s)\, \mathrm{d}s,$$

obtained the bounds as follows

$$u(t) \leqslant b(t) + \sum_{k=1}^{\infty}\sum_{i=1}^{n} C(k,n) \int_0^t (t-s)^{i\alpha+(k-i)\beta-1} E^i_{\beta,i\alpha+(k-i)\beta}\left(\lambda(t-s)^\beta\right) a(s)\, \mathrm{d}s,$$

where $C(k,n) = \frac{k!(a(t))^i(c(t)\Gamma(\beta))^{n-i}}{i!(k-i)!}\frac{(\Gamma(\beta))^i}{\Gamma(i\beta+n-i)}$. In particular, for $a(t) = A$, $b(t) = b$, $c(t) = c$, the authors showed that $u(t) \leqslant AE_\alpha(ct^\alpha)\left(1+t^\beta E_{\beta,\beta+1}(b+\lambda)t^\beta\right)$. In [3, 4], we consider some integral inequalities with two singular kernel for an appropriate function and obtained the bounded by Mittag-Leffler function.

Motivated by the above analysis, we investigate some integral inequalities including the sum of two integrals with doubly singular kernels. The main contributions of our work are that:

- Establish some new generalized Henry-Gronwall-Bihari type inequalities involving an appropriate function ψ with doubly singular kernels. The results are generalized from the results of some previous works such as [3, 6, 8, 12, 21].
- Relax some restrictions of previous works.
- Apply our results to investigate the existence and uniqueness of solution of a fractional differential equation and an integral equation with weakly singular sources.

The current paper is structured as follows. In section 2, we present the main results of this paper. Section 3 is devoted to introducing some applications of our results. Conclusions are given in section 4.

2. Weakly singular Henry-Gronwall-Bihari type inequalities

This section presents some new generalized Henry-Gronwall-Bihari type inequalities with doubly singular kernels. Let us begin by giving some notations. For $a,b \in \mathbb{R}$, $a < b$, we denote $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\}$. For $\varphi \in L^\infty[a,b]$ and $p \geqslant 1$, we define

$$\varphi^*(t) = \operatorname*{ess\ sup}_{a\leqslant\tau\leqslant t} |\varphi(\tau)|, \tag{3}$$

and

$$\begin{aligned} C^1_+[a,b] &= \left\{\varphi : \varphi \in C^1[a,b] \text{ and } \varphi'(t) > 0 \text{ for all } t \in [a,b]\right\}, \\ L^p_+[a,b] &= \{\varphi : \varphi \in L^p[a,b] \text{ and } \varphi(t) \geqslant 0 \text{ a.e. } t \in [a,b]\}, \\ L^\infty_+[a,b] &= \{\varphi : \varphi \in L^\infty[a,b] \text{ and } \varphi(t) \geqslant 0 \text{ a.e. } t \in [a,b]\}. \end{aligned}$$

For brevity, for $x,y,z \in \mathbb{R}$, $a \leqslant s < t \leqslant b$, and $\psi \in C^1_+[a,b]$, let us define

$$\Lambda_{x,y,z}(s,t,\psi) = (\psi'(s))^x(\psi(t)-\psi(s))^{y-1}(\psi(s)-\psi(a))^{-z}.$$

From the above notations, we can state an essential lemma. Readers can find the proof of this lemma in [4, Lemma 2.4].

LEMMA 1. *Let* $a,b\in\mathbb{R},\ b>a,\ \beta>0,\ \gamma<1$, *and let* $\psi\in C^1_+[a,b]$. *For* $a\leqslant\tau\leqslant s\leqslant t\leqslant b$, *we put*

$$\Upsilon_{\beta,\gamma}(\tau,s,t,\psi)=\int_\tau^s \Lambda_{1,\beta,\gamma}(\xi,t,\psi)\ \mathrm{d}\xi.$$

Then

$$\Upsilon_{\beta,\gamma}(\tau,s,t,\psi)=(\psi(t)-\psi(a))^{\beta-\gamma}\int_{\nu(\tau)}^{\nu(s)}(1-y)^{\beta-1}y^{-\gamma}\ \mathrm{d}y,$$

where $\nu(s)=(\psi(s)-\psi(a))/(\psi(t)-\psi(a))$. *Consequently,* $\lim_{|s-\tau|\to 0}\Upsilon_{\beta,\gamma}(\tau,s,t,\psi)=0$. *Moreover*

$$\Upsilon_{\beta,\gamma}(a,t,t,\psi)=(\psi(t)-\psi(a))^{\beta-\gamma}B(\beta,1-\gamma)\ \textit{for any}\ t\in[a,b],$$

where $B(\cdot,\cdot)$ *is the Beta function.*

We now state and prove the main results of this section. We begin by presenting some Henry-Gronwall type inequalities with weakly singular kernels.

THEOREM 1. *Let* $a,b\in\mathbb{R}$ *with* $a<b$, *and let* $\beta>0,\ \gamma\in\mathbb{R}$, *and* $\gamma<\min\{1,\beta\}$. *Let* $p\in L^\infty_+[a,b]$, *and* $\psi\in C^1_+[a,b]$. *Suppose that* $u\in L^\infty_+[a,b]$ *satisfies the inequality*

$$u(t)\leqslant p(t)+\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}(\psi(s)-\psi(a))^{-\gamma}k(s)u(s)\ \mathrm{d}s. \tag{4}$$

For $\rho\in\mathbb{R}$ *such that* $0<\rho<\min\{1,1-\gamma\}$ *and* $\beta-\gamma-\rho\geqslant 0$, *if* $k\in L^{1/\rho}_+[a,b]$, *then*

$$u(t)\leqslant\left(P(t)+\int_a^t Q(s)P(s)\exp\left(\int_s^t Q(\tau)\ \mathrm{d}\tau\right)\ \mathrm{d}s\right)^\rho.$$

Moreover, we have

$$u(t)\leqslant 2^{1-\rho}p^*(t)\exp\left(\rho\int_a^t Q(\tau)\ \mathrm{d}\tau\right),$$

where p^* *defined as in (3). In particular, if* $p=0$ *a.e., then* $u=0$ *a.e. on* $[a,b]$. *Herein* $P(t)=2^{1/\rho-1}p^{1/\rho}(t)$, *and* $Q(t)=2^{1/\rho-1}D^{1/\rho}\psi'(t)k^{1/\rho}(t)$ *with* $D=B^{1-\rho}\Big((\beta-\rho)/(1-\rho),(1-\gamma-\rho)/(1-\rho)\Big)(\psi(b)-\psi(a))^{\beta-\gamma-\rho}$.

REMARK 1. It is worth noting that Henry [8] and Webb [21] considered the case $a=0,\ b=T,\ \psi(t)=t,\ k(t)=C$ and required $\alpha,\beta\in(0,1)$ and $\gamma<\beta<1$. Herein, our results are general and hold for all $\beta>0$. Besides, our proofs are new and different from the proofs of previous works but easier.

Proof. Using the fact that

$$(\psi'(s))^\rho\Lambda_{1-\rho,\beta,\gamma}(s,t,\psi)=\psi'(s)(\psi(t)-\psi(s))^{\beta-1}(\psi(s)-\psi(a))^{-\gamma},$$

and the Holder inequality, we have

$$
\begin{aligned}
u(t) &\leqslant p(t)+\int_a^t (\psi'(s))^{\rho}\Lambda_{1-\rho,\beta,\gamma}(s,t,\psi)k(s)u(s)\ \mathrm{d}s\\
&\leqslant p(t)+\left(\int_a^t \left(\Lambda_{1-\rho,\beta,\gamma}(s,t,\psi)\right)^{\frac{1}{1-\rho}}\ \mathrm{d}s\right)^{1-\rho}\left(\int_a^t \left((\psi'(s))^{\rho}k(s)u(s)\right)^{1/\rho}\ \mathrm{d}s\right)^{\rho}\\
&= p(t)+\left(\int_a^t \Lambda_{1,\frac{\beta-\rho}{1-\rho},\frac{\gamma}{1-\rho}}(s,t,\psi)\ \mathrm{d}s\right)^{1-\rho}\left(\int_a^t \psi'(s)k^{1/\rho}(s)u^{1/\rho}(s)\ \mathrm{d}s\right)^{\rho}.
\end{aligned}
$$

In view of Lemma 1 and direct computations, we obtain

$$
u(t)\leqslant p(t)+D\left(\int_a^t \psi'(s)k^{1/\rho}(s)u^{1/\rho}(s)\ \mathrm{d}s\right)^{\rho},
$$

where $D=B^{1-\rho}\Big((\beta-\rho)/(1-\rho),(1-\gamma-\rho)/(1-\rho)\Big)(\psi(b)-\psi(a))^{\beta-\gamma-\rho}$. Applying the inequality $(c+d)^r\leqslant 2^{r-1}(c^r+d^r)$ for any $c,d>0$ and $r\geqslant 1$, one has

$$
u^{1/\rho}(t)\leqslant P(t)+\int_a^t Q(s)u^{1/\rho}(s)\ \mathrm{d}s,
$$

where $P(t)=2^{1/\rho-1}p^{1/\rho}(t)$, and $Q(t)=2^{1/\rho-1}D^{1/\rho}\psi'(t)k^{1/\rho}(t)$. By virtue of the Gronwall inequality, we get

$$
u(t)\leqslant\left(P(t)+\int_a^t Q(s)P(s)\exp\left(\int_s^t Q(\tau)\ \mathrm{d}\tau\right)\ \mathrm{d}s\right)^{\rho} \tag{5}
$$

for a.e. $t\in[a,b]$. Moreover, if $p\in L_+^{\infty}[a,b]$ then $P^*(t)$ the non-decreasing function, and $P(t)\leqslant P^*(t)=2^{(1-\rho)/\rho}(p^*(t))^{1/\rho}$ for all $t\in[a,b]$. Using the Gronwall inequality, we have

$$
u^{1/\rho}(t)\leqslant P^*(t)\exp\left(\int_a^t Q(\tau)\ \mathrm{d}\tau\right)\quad\text{for a.e. } t\in[a,b]. \tag{6}
$$

The latter inequality leads to

$$
u(t)\leqslant 2^{1-\rho}p^*(t)\exp\left(\rho\int_a^t Q(\tau)\ \mathrm{d}\tau\right).
$$

Particularly, if $p=0$ a.e. then $P^*(t)=0$ a.e., and we have $v=0$ a.e. on $[a,b]$. Combining (5) and (6), we obtain the desired results of Theorem. □

THEOREM 2. *Let $a,b\in\mathbb{R}$ with $a<b$, and let $\beta_i>0$, $\gamma_i\in\mathbb{R}$, and $\gamma_i<\min\{1,\beta_i\}$ for $i=1,2$. Let $\psi\in C_+^1[a,b]$, and $p\in L_+^{\infty}[a,b]$. Suppose that $u\in L_+^{\infty}[a,b]$ satisfies the inequality*

$$
\begin{aligned}
u(t)\leqslant p(t)&+\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta_1-1}(\psi(s)-\psi(a))^{-\gamma_1}k_1(s)u(s)\ \mathrm{d}s\\
&+\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta_2-1}(\psi(s)-\psi(a))^{-\gamma_2}k_2(s)u(s)\ \mathrm{d}s.
\end{aligned} \tag{7}
$$

For $\rho_i \in \mathbb{R}$ such that $0 < \rho_i < \min\{1, 1-\gamma_i\}$ and $\beta_i - \gamma_i - \rho_i \geqslant 0$ with $i = 1,2$, if $k_i \in L_+^{1/\rho_i}[a,b]$ for $i = 1,2$.

(i). Without lost of generality, we suppose further, that $\beta_1 \geqslant 1$ and p the non-decreasing function on $[a,b]$, then we have

$$u(t) \leqslant 2^{2-\rho_1-\rho_2} p(t) \exp\left(\int_a^t (\rho_2 Q_2(\tau) + \rho_1 Q_1(\tau))\, d\tau \right),$$

where $Q_2(t) = 2^{1/\rho_2 - 1} D_2^{1/\rho_2} \psi'(t) k_2^{1/\rho_2}(t)$ and $Q_1(t) = 2^{1/\rho_1 - 1} D_1^{1/\rho_1} \psi'(t) h^{1/\rho_1}(t)$ with $D_i = B^{1-\rho_i}\Big((\beta_i - \rho_i)/(1-\rho_i), (1-\gamma_i-\rho_i)/(1-\rho_i)\Big)(\psi(b) - \psi(a))^{\beta_i - \gamma_i - \rho_i}$, $(i = 1,2)$ and $h(t) = 2^{1-\rho_2} \exp\left(\rho_2 \int_a^b Q_2(\tau)\, d\tau\right) k_1(t)$.

(ii). If $\max\{\beta_1,\beta_2\} < 1$ and $\max\{\gamma_1,\gamma_2\} < \min\{\beta_1,\beta_2\}$, we put $\eta = \min\{\beta_1,\beta_2\}$, $\sigma = \max\{\gamma_1,\gamma_2\}$, $\vartheta(t) = k_1(t)(\psi(b)-\psi(a))^{\beta_1+\sigma-\eta-\gamma_1} + k_2(t)(\psi(b)-\psi(a))^{\beta_2+\sigma-\eta-\gamma_2}$. We suppose further that $\vartheta \in L_+^{1/\rho}[a,b]$ for some $0 < \rho < 1-\sigma$ such that $\eta - \sigma - \rho \geqslant 0$, then

$$u(t) \leqslant 2^{1-\rho} p^*(t) \exp\left(\rho \int_a^t Q(\tau)\, d\tau \right),$$

where $Q(t) = 2^{1/\rho - 1} D^{1/\rho} \psi'(t) \vartheta^{1/\rho}(t)$ with $D = B^{1-\rho}\Big((\eta-\sigma)/(1-\sigma), (1-\eta-\sigma)/(1-\sigma)\Big)(\psi(b)-\psi(a))^{\eta-\sigma-\rho}$, and p^* defined as in (3).

REMARK 2. Note that the results of Theorem 2 seem to be new and still not study in recently papers. In fact, Ding et al [6] only consider the case $\psi(t) = t$, $\gamma_1 = \gamma_2 = 0$ and $k_1(t) = C_1, k_2(t) = C_2$ with some constants $C_1, C_2 > 0$. In [3], we considered the case $k_1(t) = C_1, k_2(t) = C_2$ and obtain the bounds by the Mittag-Leffler function.

Proof. (i). Let us define the following function

$$Su(t) = p(t) + \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\beta_1 - 1}(\psi(\tau)-\psi(a))^{-\gamma_1} k_1(\tau) u(\tau)\, d\tau.$$

Then, we have Su the non-decreasing (with respect to t) for $\beta_1 \geqslant 1$. Indeed, for $a \leqslant s < t \leqslant b$, we have $(\psi(t)-\psi(\tau))^{\beta_1-1} - (\psi(s)-\psi(\tau))^{\beta_1-1} \geqslant 0$. This gives

$$\begin{aligned} Su(t) - Su(s) = (p(t)-p(s)) &+ \int_a^s \psi'(\tau)\left((\psi(t)-\psi(\tau))^{\beta_1-1} - (\psi(s)-\psi(\tau))^{\beta_1-1}\right) \\ &\times (\psi(\tau)-\psi(a))^{-\gamma_1} k_1(\tau) u(\tau)\, d\tau \\ &+ \int_s^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\beta_1-1}(\psi(\tau)-\psi(a))^{-\gamma_1} k_1(\tau) u(\tau)\, d\tau \geqslant 0. \end{aligned}$$

This gives $(Su)^*(t) = Su(t)$. Applying Theorem 1, we get

$$\begin{aligned} u(t) &\leqslant 2^{1-\rho_2} Su(t) \exp\left(\rho_2 \int_a^t Q_2(\tau)\, \mathrm{d}\tau\right) \\ &\leqslant 2^{1-\rho_2} p(t) \exp\left(\rho_2 \int_a^t Q_2(\tau)\, \mathrm{d}\tau\right) \\ &\quad + \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\beta-1}(\psi(\tau)-\psi(a))^{-\gamma} h(\tau) u(\tau)\, \mathrm{d}\tau, \end{aligned} \tag{8}$$

where $Q_2(t) = 2^{1/\rho_2 - 1} D_2^{1/\rho_2} \psi'(t) k_2^{1/\rho_2}(t)$ with $D_2 = B^{1-\rho_2}\Big((\beta_2-\rho_2)/(1-\rho_2), (1-\gamma_2-\rho_2)/(1-\rho_2)\Big)(\psi(b)-\psi(a))^{\beta_2-\gamma_2-\rho_2}$, and $h(t) = 2^{1-\rho_2}\exp\left(\rho_2 \int_a^b Q_2(\tau)\, \mathrm{d}\tau\right) k_1(t)$. It is clear to see that $Q(t) = 2^{1-\rho_2} p(t) \exp\left(\rho_2 \int_a^t Q_2(\tau)\, \mathrm{d}\tau\right)$ is a non-decreasing function. Thus, we can apply Theorem 1 for the inequality (8) to obtain the result of part (i).

(ii). Let us put $\eta = \min\{\beta_1, \beta_2\}$ and $\sigma = \max\{\gamma_1, \gamma_2\}$. By elementary computations, we can verify that

$$\begin{aligned} (\psi(t)-\psi(\tau))^{\beta_1-1} &\leqslant (\psi(b)-\psi(a))^{\beta_1-\eta}(\psi(t)-\psi(\tau))^{\eta-1}, \\ (\psi(t)-\psi(\tau))^{\beta_2-1} &\leqslant (\psi(b)-\psi(a))^{\beta_2-\eta}(\psi(t)-\psi(\tau))^{\eta-1}, \\ (\psi(\tau)-\psi(a))^{-\gamma_1} &\leqslant (\psi(b)-\psi(a))^{\sigma-\gamma_1}(\psi(\tau)-\psi(a))^{-\sigma}, \\ (\psi(\tau)-\psi(a))^{-\gamma_2} &\leqslant (\psi(b)-\psi(a))^{\sigma-\gamma_2}(\psi(\tau)-\psi(a))^{-\sigma} \end{aligned}$$

for any $a \leqslant \tau < t$. Pushing the above inequalities into (7), we get

$$u(t) \leqslant p(t) + \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\eta-1}(\psi(\tau)-\psi(a))^{-\sigma} \vartheta(\tau) u(\tau)\, \mathrm{d}\tau,$$

where $\vartheta(t) = k_1(t)(\psi(b)-\psi(a))^{\beta_1+\sigma-\eta-\gamma_1} + k_2(t)(\psi(b)-\psi(a))^{\beta_2+\sigma-\eta-\gamma_2}$. Applying the result of Theorem 1, we obtain the result of part (ii). This completes the proof of Theorem. □

Now, we present some Henry-Gronwall type inequalities in which the function u may be singular.

THEOREM 3. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $\alpha \geqslant 0$, $\beta > 0$, $\gamma \in \mathbb{R}$, and $\alpha + \gamma < \min\{1, \beta\}$. Let $p \in L_+^\infty[a,b]$, and $\psi \in C_+^1[a,b]$. Suppose that $(\psi(.) - \psi(a))^\alpha u \in L_+^\infty[a,b]$ satisfies the inequality*

$$\begin{aligned} u(t) &\leqslant (\psi(t)-\psi(a))^{-\alpha} p(t) \\ &\quad + \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}(\psi(s)-\psi(a))^{-\gamma}\kappa(s) u(s)\, \mathrm{d}s. \end{aligned} \tag{9}$$

For $\rho \in \mathbb{R}$ such that $0 < \rho < \min\{1, 1-\alpha-\gamma\}$ and $\beta - \gamma - (\alpha+1)\rho \geqslant 0$, if $\kappa \in L_+^{1/\rho}[a,b]$, then

$$u(t) \leqslant (\psi(t)-\psi(a))^{-\alpha}\left(P(t) + \int_a^t Q(s) P(s) \exp\left(\int_s^t Q(\tau)\, \mathrm{d}\tau\right) \mathrm{d}s\right)^{\rho}.$$

Moreover, we have

$$u(t) \leqslant 2^{1-\rho} p^*(t)(\psi(t)-\psi(a))^{-\alpha} \exp\left(\rho \int_a^t Q(\tau)\,\mathrm{d}\tau\right).$$

In particular, if $p = 0$ *a.e., then* $u = 0$ *a.e. on* $[a,b]$. *Herein* $P(t) = 2^{1/\rho-1} p^{1/\rho}(t)$, *and* $Q(t) = 2^{1/\rho-1} D^{1/\rho} \psi'(t) \kappa^{1/\rho}(t)$ *with* $D = B^{1-\rho}\Big((\beta-\rho)/(1-\rho), (1-\alpha-\gamma-\rho)/(1-\rho)\Big)(\psi(b)-\psi(a))^{\beta-\gamma-(\alpha+1)\rho}$.

REMARK 3. We emphasize that Henry [8] and Webb [21] considered the case $a = 0, b = T$, $\psi(t) = t$, $\kappa(t) = C$ and required $\alpha, \beta \in (0,1)$ and $\alpha+\gamma < \beta < 1$. Zhu [23] also used the condition $\beta \in (0,1)$. Our results require $\alpha+\gamma < \min\{1,\beta\}$ and $\beta > 0$.

Proof. Let us put

$$v(t) = (\psi(t)-\psi(a))^{\alpha} u(t). \tag{10}$$

It is obvious to see that

$$\Lambda_{1,\beta,\gamma}(s,t,\psi)u(s) = \Lambda_{1,\beta,\alpha+\gamma}(s,t,\psi)(\psi(s)-\psi(a))^{\alpha}u(s) = \Lambda_{1,\beta,\alpha+\gamma}(s,t,\psi)v(s).$$

Thus, the inequality (9) gives

$$\begin{aligned} v(t) &\leqslant p(t) + (\psi(t)-\psi(a))^{\alpha} \int_a^t \Lambda_{1,\beta,\alpha+\gamma}(s,t,\psi)\kappa(s)v(s)\,\mathrm{d}s \\ &\leqslant p(t) + (\psi(b)-\psi(a))^{\alpha} \int_a^t \Lambda_{1,\beta,\alpha+\gamma}(s,t,\psi)\kappa(s)v(s)\,\mathrm{d}s. \end{aligned}$$

Applying Theorem 1 with $\gamma := \alpha+\gamma$ and $k(t) = (\psi(b)-\psi(a))^{\alpha}\kappa(t)$, we have

$$v(t) \leqslant \left(P(t) + \int_a^t Q(s)P(s)\exp\left(\int_s^t Q(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right)^{\rho}. \tag{11}$$

We also have

$$v(t) \leqslant \left(P^*(t)\exp\left(\int_a^t Q(\tau)\,\mathrm{d}\tau\right)\right)^{\rho}. \tag{12}$$

where $P(t) = 2^{1/\rho-1}p^{1/\rho}(t)$, and $Q(t) = 2^{1/\rho-1}D^{1/\rho}\psi'(t)\kappa^{1/\rho}(t)$ with $D = B^{1-\rho}\Big((\beta-\rho)/(1-\rho), (1-\alpha-\gamma-\rho)/(1-\rho)\Big)(\psi(b)-\psi(a))^{\beta-\gamma-(\alpha+1)\rho}$. Pushing (10) into (11) and (12), we obtain the results of Theorem. □

THEOREM 4. *Let* $a, b \in \mathbb{R}$ *with* $a < b$, *and let* $\beta_i > 0$, $\gamma_i \in \mathbb{R}$, *and* $\gamma_i + \alpha < \min\{1, \beta_i\}$ *for* $i = 1, 2$. *Let* $\psi \in C_+^1[a,b]$, *and* $p \in L_+^{\infty}[a,b]$ *the non-decreasing function* *on* $[a,b]$. *Suppose that* $u \in L_+^\infty[a,b]$ *satisfies the inequality*

$$\begin{aligned} u(t) \leqslant\ & p(t)(\psi(t)-\psi(a))^{-\alpha} \\ & + \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta_1-1}(\psi(s)-\psi(a))^{-\gamma_1}k_1(s)u(s)\,\mathrm{d}s \\ & + \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta_2-1}(\psi(s)-\psi(a))^{-\gamma_2}k_2(s)u(s)\,\mathrm{d}s. \end{aligned}$$

For $\rho_i \in \mathbb{R}$ *such that* $0<\rho_i<\min\{1,1-\alpha-\gamma_i\}$ *and* $\beta_i-\alpha-\gamma_i-\rho_i \geqslant 0$ *with* $i=1,2$, *if* $k_i \in L_+^{1/\rho_i}[a,b]$ *for* $i=1,2$.

(i). Without lost of generality, we suppose further, that $\beta_1 \geqslant 1$, *then we have*

$$u(t) \leqslant 2^{2-\rho_1-\rho_2}p(t)(\psi(t)-\psi(a))^{-\alpha}\exp\left(\int_a^t (\rho_2 Q_2(\tau)+\rho_1 Q_1(\tau))\,\mathrm{d}\tau\right),$$

where $Q_2(t)=2^{1/\rho_2-1}D_2^{1/\rho_2}\psi'(t)g^{1/\rho_2}(t)$ *and* $Q_1(t)=2^{1/\rho_1-1}D_1^{1/\rho_1}\psi'(t)h^{1/\rho_1}(t)$ *with* $D_i=B^{1-\rho_i}\Big((\beta_i-\rho_i)/(1-\rho_i),(1-\gamma_i-\rho_i)/(1-\rho_i)\Big)(\psi(b)-\psi(a))^{\beta_i-\gamma_i-\rho_i}\ (i=1,2)$, $g(t)=(\psi(b)-\psi(a))^{\alpha}k_2(t)$ *and* $h(t)=2^{1-\rho_2}\exp\left(\rho_2\int_a^b Q_2(\tau)\,\mathrm{d}\tau\right)(\psi(b)-\psi(a))^{\alpha}k_1(t)$.

(ii). If $\max\{\beta_1,\beta_2\}<1$ *and* $\alpha+\max\{\gamma_1,\gamma_2\}<\min\{\beta_1,\beta_2\}$, *we put* $\eta=\min\{\beta_1,\beta_2\}$, $\sigma=\alpha+\max\{\gamma_1,\gamma_2\}$ *and*

$$\vartheta(t)=k_1(t)(\psi(b)-\psi(a))^{\beta_1+\sigma-\eta-\gamma_1}+k_2(t)(\psi(b)-\psi(a))^{\beta_2+\sigma-\eta-\gamma_2}.$$

We suppose further that $\vartheta \in L_+^{1/\rho}[a,b]$ *for some* $0<\rho<1-\sigma$ *such that* $\eta-\sigma-\rho \geqslant 0$, *then*

$$u(t) \leqslant 2^{1-\rho}p(t)\exp\left(\rho\int_a^t Q(\tau)\,\mathrm{d}\tau\right),$$

where $Q(t)=2^{1/\rho-1}D^{1/\rho}\psi'(t)\vartheta^{1/\rho}(t)$ *with* $D=B^{1-\rho}\Big((\eta-\sigma)/(1-\sigma),(1-\eta-\sigma)/(1-\sigma)\Big)(\psi(b)-\psi(a))^{\eta-\sigma-\rho}$.

Proof. We put

$$v(t)=(\psi(t)-\psi(a))^{\alpha}u(t).$$

Similar the proof of Theorem 2 and by direct computation, we obtain the desired results of Theorem. We omit to perform it in detail here. □

We now establish some Bihari-type inequalities with doubly singular kernels. Firstly, we have the following result.

THEOREM 5. *Let* $a,b \in \mathbb{R}$ *with* $a<b$, *and let* $\beta>0$, $\gamma<\min\{1,\beta\}$. *Let* $p \in L_+^\infty[a,b]$, *and* $\psi \in C_+^1[a,b]$. *Suppose that* $u \in L_+^\infty[a,b]$ *satisfies the inequality*

$$u(t) \leqslant p(t)+\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}(\psi(s)-\psi(a))^{-\gamma}k(s)H(u(s))\,\mathrm{d}s,$$

where $H:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous and non-decreasing function. For $\rho\in\mathbb{R}$ such that $0<\rho<\min\{1,1-\gamma\}$ and $\beta-\gamma-\rho\geqslant 0$, we define

$$G(t)=\int_{t_0}^{t}\frac{\mathrm{d}s}{H^{1/\rho}(s^{\rho})},\ \ t_0\geqslant 0.$$

If $k\in L_+^{1/\rho}[a,b]$, then

$$u(t)\leqslant\left(G^{-1}\left(G(P(t))+\int_a^t Q(s)\ \mathrm{d}s\right)\right)^{\rho}\ \text{for all}\ \ t\in[a,T],$$

where $P(t)=2^{1/\rho-1}p^{1/\rho}(t)$, and $Q(t)=2^{1/\rho-1}C^{1/\rho}\psi'(t)k^{1/\rho}(t)$ with $C=B^{1-\rho}\Big((\beta-\rho)/(1-\rho),(1-\gamma-\rho)/(1-\rho)\Big)(\psi(b)-\psi(a))^{\beta-\gamma-\rho}$, and G^{-1} is the inverse function of G and $T>a$ is chosen so that $G(P(t))+\int_a^t Q(s)\ \mathrm{d}s\in\mathrm{Dom}\left(G^{-1}\right)$ for any $t\in[a,T]$.

REMARK 4. Medved [12] considered the case $a=0,b=T$, $\psi(t)=t$, and required $\beta>1/2$ and $\gamma<1/2$, or $\beta=1/(m+1)$ and $\gamma<1/\kappa(m+2)$ for some $\kappa>1$. Our result is generalized and relaxed the above restrictions.

Proof. By the same method of the proof of Theorem 1, we can verify that

$$u^{1/\rho}(t)\leqslant P(t)+\int_a^t Q(s)H^{1/\rho}(u(s))\ \mathrm{d}s:=L(t), \tag{13}$$

where $P(t)=2^{1/\rho-1}p^{1/\rho}(t)$, and $Q(t)=2^{1/\rho-1}C^{1/\rho}\psi'(t)k^{1/\rho}(t)$ with

$$C=B^{1-\rho}\Big((\beta-\rho)/(1-\rho),(1-\gamma-\rho)/(1-\rho)\Big)(\psi(b)-\psi(a))^{\beta-\gamma-\rho}.$$

From (13), we find that $G^{1/\rho}(u(t))\leqslant G^{1/\rho}\left(L^{\rho}(t)\right)$. Therefore, we get

$$\begin{aligned}(G(L(t)))'=\frac{L'(t)}{H^{1/\rho}\left(L^{\rho}(t)\right)}&=\frac{P'(t)+Q(t)H^{1/\rho}\left(u(t)\right)}{H^{1/\rho}\left(L^{\rho}(t)\right)}\\ &\leqslant\frac{P'(t)}{H^{1/\rho}\left(P^{\rho}(t)\right)}+Q(t)=(G(P(t))'+Q(t)\end{aligned}$$

due to $P(t)\leqslant L(t)$ for any $t\in[a,b]$. By integrating both side the last inequality on $[a,t]$, we get

$$G\left(u^{1/\rho}(t)\right)\leqslant G(L(t))\leqslant G(P(t))+\int_a^t Q(s)\ \mathrm{d}s.$$

The latter inequality leads to the desired result of Theorem. □

To close this section, we propose a Bihari-type inequality, in which the function u may be singular. The result can be stated as follows.

THEOREM 6. *Let $a,b \in \mathbb{R}$ with $a<b$, and let $\alpha \geqslant 0$, $\beta>0$, $\gamma<\min\{1,\beta\}$. Let $p \in L_+^\infty[a,b]$, and $\psi \in C_+^1[a,b]$. Suppose that $(\psi(.)-\psi(a))^\alpha u \in L_+^\infty[a,b]$ satisfies the inequality*

$$u(t) \leqslant (\psi(t)-\psi(a))^{-\alpha} p(t) + \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}(\psi(s)-\psi(a))^{-\gamma}\kappa(s) W(s,\psi,u(s))\,\mathrm{d}s, \qquad (14)$$

where $W:[a,b]\times \in C_+^1[a,b]\times \mathbb{R}_+ \to \mathbb{R}_+$.

If $\kappa \in L_+^{1/\rho}[a,b]$, and there exists a continuous and non-decreasing function $\omega: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$W(t,\psi,u(t)) \leqslant \omega\left[(\psi(t)-\psi(a))^\alpha u(t)\right], \text{ for any } t \in [a,b],$$

then, for $\rho \in \mathbb{R}$ such that $0<\rho<\min\{1,1-\gamma\}$ and $\alpha+\beta-\gamma-(\alpha+1)\rho \geqslant 0$, we have

$$u(t) \leqslant (\psi(t)-\psi(a))^{-\alpha}\left(F^{-1}\left(F(P(t))+\int_a^t Q(s)\,\mathrm{d}s\right)\right)^\rho \text{ for all } t \in [a,T],$$

where $F(t)=\int_{t_0}^t \frac{\mathrm{d}s}{\omega^{1/\rho}(s^\rho)}$, $t_0 \geqslant 0$, $P(t)=2^{1/\rho-1}p^{1/\rho}(t)$, $Q(t)=2^{1/\rho-1}C^{1/\rho}\psi'(t)\kappa^{1/\rho}(t)$ with $C=B^{1-\rho}\Big((\beta-\rho)/(1-\rho),(1-\gamma-\rho)/(1-\rho)\Big)(\psi(b)-\psi(a))^{\alpha+\beta-\gamma-(\alpha+1)\rho}$, and F^{-1} is the inverse function of F and $T>a$ is chosen so that $F(P(t))+\int_a^t Q(s)\,\mathrm{d}s \in \mathrm{Dom}\left(F^{-1}\right)$ for any $t \in [a,T]$.

REMARK 5. The results in Theorem 6 seem to be new and are still under consideration in previous works.

Proof. Putting $v(t)=(\psi(t)-\psi(a))^\alpha u(t)$. Then the inequality (14) gives

$$v(t) \leqslant p(t)+(\psi(b)-\psi(a))^\alpha \int_a^t \Lambda_{1,\beta,\gamma}(s,t,\psi)\kappa(s)\omega(v(s))\,\mathrm{d}s,$$

Applying Theorem 5 with $k(t):=(\psi(b)-\psi(a))^\alpha\kappa(t)$, we get

$$v(t) \leqslant \left(F^{-1}\left(F(P(t))+\int_a^t Q(s)\,\mathrm{d}s\right)\right)^\rho,$$

where $P(t)=2^{1/\rho-1}p^{1/\rho}(t)$, and $Q(t)=2^{1/\rho-1}C^{1/\rho}\psi'(t)\kappa^{1/\rho}(t)$ with $C=B^{1-\rho}\Big((\beta-\rho)/(1-\rho),(1-\gamma-\rho)/(1-\rho)\Big)(\psi(b)-\psi(a))^{\alpha+\beta-\gamma-(\alpha+1)\rho}$, and F^{-1} is the inverse function of F, and $T>a$ is chosen so that $F(P(t))+\int_a^t Q(s)\,\mathrm{d}s \in \mathrm{Dom}\left(F^{-1}\right)$ for any $t \in [a,T]$. The last inequality leads to the desired result of Theorem. □

3. Applications

We apply our results to study the existence and uniqueness of solution of a differential equation involving ψ-Hiffer fractional derivatives with weakly singular sources and a class of integral with two singular kernels. We start by introducing the definitions of fractional integrals and ψ-Hilfer fractional derivatives.

DEFINITION 1. (see [1, 10]) Let $\alpha > 0$, $\psi \in C^1_+[a,b]$, and f an integrable function. Fractional integral of a function f with respect to another function ψ is defined by

$$I^{\alpha;\psi}_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} f(\tau) \, \mathrm{d}\tau.$$

DEFINITION 2. Let $n \in \mathbb{N}$, $n-1 < \alpha < n$, $\psi \in C^1_+[a,b]$, and $f \in C^n[a,b]$. The left ψ-Hilfer fractional derivative ${}^H D^{\alpha,\beta;\psi}_{a+}(.)$ of function of order α and type $\beta \in [0,1]$ is defined as [18]

$${}^H D^{\alpha,\beta;\psi}_{a+} f(t) = I^{\beta(n-\alpha);\psi}_{a+} \left(\frac{1}{\psi'(t)} \frac{\mathrm{d}}{\mathrm{d}t} \right)^n I^{(1-\beta)(n-\alpha);\psi}_{a+} f(t).$$

For complete surveys of basic properties of the fractional integrals and ψ-Hilfer fractional derivatives, we refer to [10, 18].

In this section, let us define the Banach space

$$C_{1-\gamma}[a,b] = \left\{ u \in C((a,b],\mathbb{R}) : (\psi(.) - \psi(a))^{1-\gamma} u \in C[a,b] \right\}.$$

We denote the norm associated the space $C_{1-\gamma}[a,b]$ by

$$\|u\|_\gamma = \sup_{t\in[a,b]} (\psi(t) - \psi(a))^{1-\gamma} |u(t)|.$$

For $k \in L^\infty[a,b]$, we also denote $|||k|||_b = \operatorname{ess\,sup}_{a \leqslant t \leqslant b} |k(t)|$.

Using the above notations, we state and prove an useful lemma as follows.

LEMMA 2. *Let* $\alpha > 0$, $\gamma \in (0,1]$, *and* $a < b$, $a \leqslant \tau < t \leqslant b$. *Let* $\psi \in C^1_+[a,b]$, $u \in C_{1-\gamma}([a,b];\mathbb{R})$. *We denote*

$$Su(t) = \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} f(\tau,\psi,u(\tau)) \, \mathrm{d}\tau,$$
$$B_R = \{u \in C_{1-\gamma}([a,b];\mathbb{R}) : \|u\|_\gamma \leqslant R\}.$$

Suppose that there exist $\rho \in \mathbb{R}$ *with* $\rho + 1 - \gamma < \min\{1,\alpha\}$ *and* $p > 0$ *such that*

$$|f(t,\psi,u)| \leqslant (\psi(t) - \psi(a))^{-\rho} (A_1|u|^p + A_2) \ \text{ for any } \ t \in (a,b],\ u \in \mathbb{R}.$$

Then $S(B_R)$ *is equicontinuous on* $[a,b]$ *and uniformly bounded.*

Proof. The proof is similar to the one [3, Lemma 4.5], hence we omit it. □

3.1. The initial problem with ψ-Hilfer fractional derivatives

In this part, we consider the existence and uniqueness of solution of a nonlinear differential equation with respect to ψ-Hilfer fractional derivatives. Precisely, let $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, $\psi \in C_+^1[a,b]$, and $f:(a,b)\times C_+^1[a,b]\times\mathbb{R}\to\mathbb{R}$. We consider the following problem

$$\begin{cases} {}^H D_{a+}^{\alpha;\beta;\psi} u(t) = f(t,\psi,u), \quad a < t \leqslant b, \\ I^{1-\gamma}u(a) = u_a, \quad u_a \in \mathbb{R}, \quad \gamma = \alpha+\beta-\alpha\beta. \end{cases} \tag{15}$$

We known from [17] that $u(t)$ satisfies initial value problem (15) if $u(t)$ satisfies the second kind Volterra integral equation

$$u(t) = \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}u_a + \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,\psi,u(s))\,\mathrm{d}s. \tag{16}$$

DEFINITION 3. The solution of the equation (16) is called mild solution of the problem (15).

To investigate the existence and uniqueness, the following assumptions are posed.

- *Assumption* $(\mathscr{C}1)$. $f \in C\left((a,b]\times C_+^1[a,b]\times\mathbb{R};\mathbb{R}\right)$, and there exist $M>0$, $p \geqslant 0$, $\rho\in\mathbb{R}$, and $\kappa_1:[a,b]\to\mathbb{R}_+$ such that

$$|f(t,\psi,u)| \leqslant (\psi(t)-\psi(a))^{-\rho_1}(\kappa_1(t)|u|^p + M) \text{ for any } u\in\mathbb{R},\ t\in(a,b].$$

- *Assumption* $(\mathscr{C}2)$. There exist $\rho_1\in\mathbb{R}$, and $h\in C([a,b]\times\mathbb{R},\mathbb{R})$ such that

$$|f(t,\psi,u)-f(t,\psi,v)| \leqslant (\psi(t)-\psi(a))^{-\rho_1}|h(t,u)-h(t,v)| \text{ for all } u,v\in\mathbb{R},\ t\in(a,b].$$

- *Assumption* $(\mathscr{C}3)$. There exist $\rho\in\mathbb{R}$, and $\kappa:[a,b]\to\mathbb{R}_+$ such that

$$|f(t,\psi,u)-f(t,\psi,v)| \leqslant \kappa(t)(\psi(t)-\psi(a))^{-\rho}|u-v| \text{ for any } u,v\in\mathbb{R},\ t\in(a,b],$$
$$|f(t,\psi,0)| \leqslant \kappa(t)(\psi(t)-\psi(a))^{-\rho}.$$

THEOREM 7. *Let* $0<\alpha\leqslant 1$, $0\leqslant\beta\leqslant 1$ *such that* $\alpha+\gamma>1$ *with* $\gamma=\alpha+\beta-\alpha\beta$, *and let* $\psi\in C_+^1[a,b]$, *and* $\rho,\rho_1\in\mathbb{R}_+$ *such that* $r=\alpha+\gamma-\rho-1, r_1=\alpha+\gamma-\rho_1-1>0$. *Suppose that the assumptions* $(\mathscr{C}1)$ *and* $(\mathscr{C}2)$ *hold. Then,*

(i). The problem (15) *has at least one mild solution belonging to* $C_{1-\gamma}[a,b]$ *whenever one of the following assertions is valid*

(A1). $0\leqslant p\leqslant 1$ *and* $k_1\in L_+^{1/r_1}[a,b]\cap L_+^\infty[a,b]$.

(A2). $p>1$, $k_1\in L_+^\infty[a,b]$, *and there exists a* $\mathscr{M}>0$ *such that*

$$\mathscr{M} > E + \frac{\Gamma(1-\rho_1-p(1-\gamma))}{\Gamma(1+\alpha-\rho_1-p(1-\gamma))}\|\kappa_1\|_b\mathscr{M}^p,$$

where $E=|u_a|/\Gamma(\gamma)+M\Gamma(1-\rho)/\Gamma(\alpha-\rho+1)(\psi(b)-\psi(a))^{1+\alpha-\rho-\gamma}$.

(ii). If the assumption $(\mathscr{C}3)$ *holds, and* $\kappa\in L_+^{1/r}[a,b]\cap L_+^\infty[a,b]$. *Then the problem* (15) *has a unique belonging to* $C_{1-\gamma}[a,b]$.

Proof. We consider the operator $Q : C_{1-\gamma}[a,b] \to C_{1-\gamma}[a,b]$ defined by

$$Qu(t) = \frac{u_a}{\Gamma(\gamma)}(\psi(t)-\psi(a))^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,\psi,u(s))\,\mathrm{d}s.$$

(i). We put

$$B_R = \{u \in C_{1-\gamma}([a,b];\mathbb{R}) : \|u\|_\gamma \leqslant R\}.$$

Firstly, we consider the case the assertion (A1) holds. For $0 \leqslant p \leqslant 1$, from the assumption $(\mathscr{C}1)$, we can find that

$$\begin{aligned}|f(t,\psi,u)| &\leqslant (\psi(t)-\psi(a))^{-\rho_1}(\|\kappa_1\|_b |u|^p + M)\\ &\leqslant (\psi(t)-\psi(a))^{-\rho_1}(\|\kappa_1\|_b |u| + M_1) \ \text{ for any } \ u\in\mathbb{R},\ t\in(a,b],\end{aligned}$$

where $M_1 = M + \|\kappa_1\|_b$. Therefore, using Lemma 2, we can verify that $Q(B_R)$ is equicontinuous and uniformly bounded. Base on the assumption $(\mathscr{C}2)$, by straightforward, we find that Q is a continuous operator. So we conclude that Q is a compact operator in $C_{1-\gamma}[a,b]$.

Continuously, let us denote

$$\Omega = \left\{u \in C_{1-\gamma}[a,b] : u = \lambda Qu \text{ for some } \lambda \in [0,1]\right\}.$$

If $u \in \Omega$ then $|u(t)| \leqslant |Qu(t)|$, by straightforward, one has

$$\begin{aligned}|u(t)| \leqslant &\left(\frac{|u_a|}{\Gamma(\gamma)} + M_1\frac{\Gamma(1-\rho_1)}{\Gamma(\alpha-\rho_1+1)}(\psi(t)-\psi(a))^{1+\alpha-\rho_1-\gamma}\right)(\psi(t)-\psi(a))^{\gamma-1}\\ &+\frac{1}{\Gamma(\alpha)}\int_a^t \Lambda_{1,\alpha,\rho_1}(s,t,\psi)\kappa_1(s)|u(s)|\,\mathrm{d}s \end{aligned} \tag{17}$$

due to $B(\alpha,1-\rho_1)/\Gamma(\alpha) = \Gamma(1-\rho_1)/\Gamma(\alpha-\rho_1+1)$. We can apply Theorem 3 with $r_1 = \alpha+\gamma-\rho_1-1$ to (17), and obtain that

$$(\psi(t)-\psi(a))^{1-\gamma}|u(t)| \leqslant \left(\Phi(t) + \int_a^t \Psi(s)\Phi(s)\exp\left(\int_s^t \Psi(s)\,\mathrm{d}s\right)\mathrm{d}s\right)^{r_1},$$

where $\Psi(t) = D^{1/r_1}\psi'(t)\kappa_1^{1/r_1}(t)$ with D being a positive constant and

$$\Phi(t) = 2^{(1-r_1)/r_1}\left(|u_a|/\Gamma(\gamma) + M_1\Gamma(1-\rho_1)/\Gamma(\alpha-\rho_1+1)(\psi(t)-\psi(a))^{1+\alpha-\rho_1-\gamma}\right)^{1/r_1}.$$

From the latter inequality, we conclude that Ω is bounded. By virtue of the Leray-Schauder fixed point theorem, we find that Q has at least one fixed point in $C_{1-\gamma}([a,b],\mathbb{R})$, which is a mild solution of the problem (15) in $C_{1-\gamma}([a,b],\mathbb{R})$.

Secondly, we consider the case the assertion (A2) holds. For $p > 1$, we put

$$\Omega = \left\{w \in C_{1-\gamma}([a,b],\mathbb{R}) : \|w\|_\gamma < \mathscr{M}\right\}.$$

Suppose that there exists $u \in \partial\Omega$ such that $u = \lambda Qu$ for some $\lambda \in (0,1)$, by direct computation, we get

$$
\begin{aligned}
&(\psi(t)-\psi(a))^{1-\gamma}|u(t)| \\
\leqslant\ & \frac{|u_a|}{\Gamma(\gamma)+M\frac{\Gamma(1-\rho_1)}{\Gamma(\alpha-\rho_1+1)}(\psi(t)-\psi(a))^{1+\alpha-\rho_1-\gamma}} \\
&+\frac{1}{\Gamma(\alpha)}\int_a^t \Lambda_{1,\alpha,\rho_1+p(1-\gamma)}(s,t,\psi)\kappa_1(s)\left((\psi(s)-\psi(a))^{1-\gamma}|u(s)|\right)^p\ \mathrm{d}s \\
\leqslant\ & E+\frac{|||\kappa_1|||_b\|u\|_\gamma^p}{\Gamma(\alpha)}\int_a^t \Lambda_{1,\alpha,\rho_1+p(1-\gamma)}(s,t,\psi)\ \mathrm{d}s \\
\leqslant\ & E+\frac{\Gamma(1-\rho_1-p(1-\gamma))}{\Gamma(1+\alpha-\rho_1-p(1-\gamma))}|||\kappa_1|||_b\|u\|_\gamma^p,
\end{aligned}
$$

where $E = |u_a|/\Gamma(\gamma)+M\Gamma(1-\rho_1)/\Gamma(\alpha-\rho_1+1)(\psi(b)-\psi(a))^{1+\alpha-\rho_1-\gamma}$. It implies that

$$
\mathscr{M} \leqslant E+\frac{\Gamma(1-\rho_1-p(1-\gamma))}{\Gamma(1+\alpha-\rho_1-p(1-\gamma))}|||\kappa_1|||_b\mathscr{M}^p.
$$

This contradicts the hypothesis. So, we conclude that there dose not exist $u \in \partial\Omega$ such that $u = \lambda Qu$ for some $\lambda \in [0,1]$. Therefore, the nonlinear Leray-Schauder alternatives fixed point theorem (see [7, p.4]) implies that Q has a fixed point $u \in \overline{\Omega}$, which is a mild solution of the problem (15) in $C_{1-\gamma}([a,b],\mathbb{R})$.

(ii). From the assumption $(\mathscr{C}3)$, we find that

$$
|f(t,\psi,u)| \leqslant (\psi(t)-\psi(a))^{-\rho}(\kappa(t)|u|+|||\kappa|||_b)\ \text{ for any }\ u\in\mathbb{R},\ t\in(a,b].
$$

Using result in part (i), we conclude that the problem (15) has at least one mild solution in $C_{1-\gamma}([a,b],\mathbb{R})$.

Finally, we prove that the uniqueness of solution of our problem. To this aim, let us consider two mild solution u_1, u_2 of our problem. Base on the assumption $(\mathscr{C}3)$, by direct computations, we have

$$
|u_1(t)-u_2(t)| \leqslant \frac{1}{\Gamma(\alpha)}\int_a^t \Lambda_{1,\alpha,\rho}(s,t,\psi)\kappa(s)|u_1(s)-u_2(s)|\ \mathrm{d}s.
$$

Using Theorem 1, we obtain the desired result of Theorem. $\square$

3.2. A class of integral equation with two singular kernels

In this part, let $\alpha,\beta > 0$, $\phi \in C[a,b]$, and $f,g : (a,b]\times C_+^1[a,b]\times\mathbb{R}\to\mathbb{R}$, we consider the following integral equation

$$
\begin{aligned}
u(t) = \phi(t) &+ \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s,\psi,u(s))\ \mathrm{d}s \\
&+ \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}g(s,\psi,u(s))\ \mathrm{d}s. \qquad (18)
\end{aligned}
$$

REMARK 6. The integral equation (18) is generalized from some of the integral equations that appear in the study fractional Langevin equations (see [4, 22]).

In order to investigate the existence results, we use the following assumptions.

• *Assumption* $(\mathscr{C}4)$. $g \in C\left((a,b] \times C_+^1[a,b] \times \mathbb{R};\mathbb{R}\right)$, and there exist $M > 0$, $\rho_2 \in \mathbb{R}$, and $\kappa_2 : [a,b] \to \mathbb{R}_+$ such that

$$|g(t,\psi,u)| \leqslant (\psi(t)-\psi(a))^{-\rho_2}(\kappa_2(t)|u|+M) \text{ for any } u \in \mathbb{R},\ t \in (a,b].$$

• *Assumption* $(\mathscr{C}5)$. There exist $\rho_2 \in \mathbb{R}$, and $\ell \in C([a,b] \times \mathbb{R},\mathbb{R})$ such that

$$|g(t,\psi,u)-g(t,\psi,v)| \leqslant (\psi(t)-\psi(a))^{-\rho_2}|\ell(t,u)-\ell(t,v)| \text{ for all } u,v \in \mathbb{R},\ t \in (a,b].$$

THEOREM 8. *Let $a,b \in \mathbb{R}$ with $a<b$, $\beta_1,\beta_2>0$, and let $\rho_1,\rho_2 \in \mathbb{R}$ such that $\rho_1 < \{1,\beta_1\}$ and $\rho_2 < \{1,\beta_2\}$. Let $\psi \in C_+^1[a,b]$, and let the assumptions $(\mathscr{C}1)-(\mathscr{C}2)$ and $(\mathscr{C}4)-(\mathscr{C}5)$ hold with $\kappa_1,\kappa_2 \in L_+^\infty[a,b]$. Then the problem (18) has at least one solution in $C[a,b]$ if one of the following conditions is satisfied.*

(i). $\max\{\beta_1,\beta_2\} \geqslant 1$, *$\phi$ the non-decreasing function on $[a,b]$, and $\kappa_i \in L^{1/r_i}$ for some $0<r_i<\min\{1,1-\rho_i\}$ and $\beta_i-\rho_i-r_i \geqslant 0$ $(i=1,2)$.*

(ii). $\max\{\beta_1,\beta_2\}<1$, *and $\sigma := \max\{\rho_1,\rho_2\} < \min\{\beta_1,\beta_2\} := \eta$, and $\kappa_i \in L_+^{1/r}$ for some $0<r<1-\sigma$ and $\eta-\sigma-r \geqslant 0$.*

Proof. We consider the operator $P : C[a,b] \to C[a,b]$ defined by

$$\begin{aligned} Pu(t) = \phi(t) &+ \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta_1-1} f(s,\psi,u(s))\, \mathrm{d}s \\ &+ \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta_2-1} g(s,\psi,u(s))\, \mathrm{d}s. \end{aligned}$$

From the assumptions $(\mathscr{C}1)$ and $(\mathscr{C}4)$, we have

$$|f(t,\psi,u)| \leqslant (\psi(t)-\psi(a))^{-\rho_1}(|||\kappa_1|||_b|u|+M) \text{ for any } u \in \mathbb{R},\ t \in (a,b]$$

and

$$|g(t,\psi,u)| \leqslant (\psi(t)-\psi(a))^{-\rho_2}(|||\kappa_2|||_b|u|+M) \text{ for any } u \in \mathbb{R},\ t \in (a,b].$$

Therefore, using Lemma 2, we can verify that Pu is equicontinuous (with respect to t) and bounded. Base on the assumptions $(\mathscr{C}2)$ and $(\mathscr{C}5)$, by straightforward, we find that P is a continuous operator. So we conclude that P is a compact operator in $C[a,b]$.

We denote

$$\Omega = \{u \in C[a,b] : u = \lambda Pu \text{ for some } \lambda \in [0,1]\}.$$

For $u \in \Omega$, using Lemma 1, the assumptions $(\mathscr{C}1)$ and $(\mathscr{C}4)$, we have

$$\begin{aligned} |u(t)| \leqslant |Pu(t)| \leqslant \phi(t) &+ \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta_1-1}|f(s,\psi,u(s))|\, \mathrm{d}s \\ &+ \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta_2-1}|g(s,\psi,u(s))|\, \mathrm{d}s \end{aligned}$$

$$\leqslant H(t)+\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta_1-1}(\psi(s)-\psi(a))^{-\rho_1}\kappa_1(\tau)|u(\tau)|\;\mathrm{d}s$$
$$+\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\beta_2-1}(\psi(s)-\psi(a))^{-\rho_2}\kappa_2(\tau)|u(\tau)|\;\mathrm{d}s, \tag{19}$$

where

$$H(t)=\phi(t)+MB(\beta_1,1-\rho_1)(\psi(t)-\psi(a))^{\beta_1-\rho_1}+MB(\beta_2,1-\rho_2)(\psi(t)-\psi(a))^{\beta_2-\rho_2}.$$

We note that if ϕ is the non-decreasing function then H is also the non-decreasing function. Therefore, applying Theorem 2 to integral inequality (19), we conclude that Ω is bounded. So, using the Leray-Schauder fixed point theorem we obtain the desired results of Theorem. □

4. Conclusions

We proposed and proved some new weakly Henry-Gronwall-Bihari type inequalities for an appropriate function ψ, which are generalized from some recent works. We applied the obtained results to study the existence and uniqueness of solution of a fractional differential equation and a integral equation with weakly singular sources.

In future work, we hope that we can extend our results for the sum of n-integral terms with doubly singular kernels and apply them to study nonlinear fractional differential or fractional partial differential equations involving generalized fractional derivatives with weakly singular sources.

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Nguyen Minh Dien
Faculty of Education
Thu Dau Mot University
Thu Dau Mot City, Binh Duong Province, Vietnam
e-mail: diennm@tdmu.edu.vn,
nmdien81@gmail.com