Journal of Mathematical Inequalities
Volume 16, Number 1 (2022), 347–354 doi:10.7153/jmi-2022-16-25

A NOTE ON "AN EIGENVALUE INEQUALITY FOR POSITIVE SEMIDEFINITE $k \times k$ BLOCK MATRICES"

FENG ZHANG

(*Communicated by M. Krnić*)

Abstract. Zhang and Xu recently obtained some new matrix norm inequalities in [1]. In this note, we provide alternative proofs and some applications for these results.

1. Introduction

Given a real vector $x = (x_1, x_2, \cdots, x_n) \in R^n$, we rearrange its components as $x_{[1]} \geqslant x_{[2]} \geqslant \cdots \geqslant x_{[n]}$. For $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in R^n$, if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]} \ k = 1, 2, \cdots, n$$

then we say that x is weakly majorized by y denoted $x \prec_w y$. If $x \prec_w y$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ hold, we say that x is majorized by y denoted $x \prec y$.

Let M_n be the set of $n \times n$ complex matrices. The identity matrix of M_n is denoted by I. In [1], Zhang and Xu obtained the following result

THEOREM 1. *Let $A_1, A_2, \ldots, A_k \in M_n$ with $A_i^* A_j = -A_j^* A_i$, $(1 \leqslant i < j \leqslant k)$. Then*

$$\lambda \left(\sum_{i=1}^{k} A_i A_i^* \right) \prec \lambda \left(\sum_{i=1}^{k} A_i^* A_i \right).$$

The way that Zhang and Xu proved Theorem 1 is original. Considering that alternative proofs may provide new perspectives to these elegant results, we take the chance to do it here.

Recall that a norm $\|\bullet\|$ on M_n is unitarily invariant if $\|UAV\| = \|A\|$ for any unitary matrices $U, V \in M_n$ and any $A \in M_n$. In the sequel, $\|\bullet\|$ stands for any unitarily invariant norm. Due to Ky Fan's result (see [2]), it's known that $\|X\| \leqslant \|Y\|$ if and only if $s(X) \prec_w s(Y)$ for $X, Y \in M_n$. Let f be a convex increasing function on $[0, +\infty)$. If $x \prec_w y$, then [2]

$$f(x) \prec_w f(y). \tag{1}$$

We use the following notations throughout this paper:

Mathematics subject classification (2020): 15A42, 06A06.
Keywords and phrases: Norm inequality, block matrix.

1. For $A \in M_n$, $s_i(A)$ is the i-th largest singular value of A and $s(A) = (s_1(A), \cdots, s_n(A))$.

2. $\lambda_i(A)$ is the i-th largest eigenvalue of A and $\lambda(A) = (\lambda_1(A), \cdots, \lambda_n(A))$.

3. $s_i(A) = \lambda_i(|A|)$ for $|A| = (A^*A)^{\frac{1}{2}}$, where A^* is the conjugate transpose of A.

4. When A, B are Hermitian, $A \geqslant B$ means that $A - B$ is positive semidefinite.

5. $A \oplus B$ is the direct sum of A and B.

6. We use $[X_{ij}]$ to present a block matrix in which the i, j -th block is X_{ij}.

For $a \in [0,1]$, Audenaert proved in [3] that

$$\|AB^*\|^2 \leqslant \|aA^*A + (1-a)B^*B\| \times \|(1-a)A^*A + aB^*B\|. \tag{2}$$

As explained in [3], inequality (2) interpolates between the Arithmetic–Geometric mean ($a = \frac{1}{2}$) and Cauchy–Schwarz ($a = 0$) matrix norm inequalities.

In [5], Zou and Jiang proved that

$$\|AB\| \leqslant \frac{1}{4a(1-a)} \left\|(aA + (1-a)B)^2\right\| \times \left\|((1-a)A + aB)^2\right\|. \tag{3}$$

In [6], Wu obtained a generalization of inequality (3) as follows

$$\left\||AB|^{2r}\right\| \leqslant \left[\frac{1}{4a(1-a)}\right]^r \left\|(aA + (1-a)B)^{2rp}\right\|^{\frac{1}{p}} \times \left\|((1-a)A + aB)^{2rq}\right\|^{\frac{1}{q}}, \tag{4}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p,\ q > 1$, $a \in (0,1)$ and $r \geqslant \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Further, we notice that inequality (4) is equivalent to the following inequality:

$$\left\||XY^*|^{2r}\right\| \leqslant \left[\frac{1}{4a(1-a)}\right]^r \left\|(a(X^*X)^{\frac{1}{2}} + (1-a)(Y^*Y)^{\frac{1}{2}})^{2rp}\right\|^{\frac{1}{p}} \times \left\|((1-a)(X^*X)^{\frac{1}{2}} + a(Y^*Y)^{\frac{1}{2}})^{2rq}\right\|^{\frac{1}{q}}. \tag{5}$$

As an application of our proof, we use block matrix technique to present some generalizations of inequality (2) and inequality(5).

The following Lemma will be needed in our proof.

LEMMA 2. [4] *Let $A, B \in M_n$ be Hermitian matrices. Then* $2\lambda(A) \prec \lambda(A-B) + \lambda(A+B)$.

2. Main result

In this senction, we present the new proof of Theorem 1.

Proof of Theorem 1. We use mathematical induction to prove this result. The base case $m = 2$, i.e. Theorem 1.1 was treated in [4].

Suppose the asserted inequality is true for $m = k$ for some $k \geqslant 2$.

Then we consider the $m = k+1$ case.

In this case, we consider the following matrices:

$$A_1, A_2, \cdots, A_{k-1}, A_k + A_{k+1}$$

and

$$A_1, A_2, \cdots, A_{k-1}, A_k - A_{k+1},$$

where $A_i^* A_j = -A_j^* A_i \ (1 \leqslant i < j \leqslant k+1)$.

By the inductive hypothesis,

$$\begin{aligned}
&\lambda\left(\sum_{i=1}^{k-1} A_i A_i^* + (A_k + A_{k+1})(A_k + A_{k+1})^*\right) \\
&\prec \lambda\left(\sum_{i=1}^{k-1} A_i^* A_i + (A_k + A_{k+1})^*(A_k + A_{k+1})\right) \\
&= \lambda\left(\sum_{i=1}^{k-1} A_i^* A_i + \left(A_k^* A_k + A_{k+1}^* A_{k+1} + A_k^* A_{k+1} + A_{k+1}^* A_{k+1}\right)\right) \\
&= \lambda\left(\sum_{i=1}^{k+1} A_i^* A_i\right)
\end{aligned}$$

and

$$\begin{aligned}
&\lambda\left(\sum_{i=1}^{k-1} A_i A_i^* + (A_k - A_{k+1})(A_k - A_{k+1})^*\right) \\
&\prec \lambda\left(\sum_{i=1}^{k-1} A_i^* A_i + (A_k - A_{k+1})^*(A_k - A_{k+1})\right) \\
&= \lambda\left(\sum_{i=1}^{k-1} A_i^* A_i + \left(A_k^* A_k + A_{k+1}^* A_{k+1} - A_k^* A_{k+1} - A_{k+1}^* A_k\right)\right) \\
&= \lambda\left(\sum_{i=1}^{k+1} A_i^* A_i\right).
\end{aligned}$$

Now we proceed to estimating the eigenvalue of $\sum_{i=1}^{k+1} A_iA_i^*$, by Lemma 2, we obtain

$$\begin{aligned}
&\lambda\left(\sum_{i=1}^{k+1} A_iA_i^*\right)\\
&\prec \frac{1}{2}\lambda\left(\sum_{i=1}^{k-1} A_iA_i^* + A_kA_k^* + A_{k+1}A_{k+1}^* + A_kA_{k+1}^* + A_{k+1}A_k^*\right)\\
&\quad + \frac{1}{2}\lambda\left(\sum_{i=1}^{k-1} A_iA_i^* + A_kA_k^* + A_{k+1}A_{k+1}^* - A_kA_{k+1}^* - A_{k+1}A_k^*\right)\\
&\prec \frac{1}{2}\lambda\left(\sum_{i=1}^{k-1} A_iA_i^* + (A_k + A_{k+1})(A_k + A_{k+1})^*\right)\\
&\quad + \frac{1}{2}\lambda\left(\sum_{i=1}^{k-1} A_iA_i^* + (A_k - A_{k+1})(A_k - A_{k+1})^*\right)\\
&\prec \frac{1}{2}\left[\lambda\left(\sum_{i=1}^{k+1} A_i^*A_i\right) + \lambda\left(\sum_{i=1}^{k+1} A_i^*A_i\right)\right] = \lambda\left(\sum_{i=1}^{k+1} A_i^*A_i\right).
\end{aligned}$$

Thus the asserted inequality is true for $m = k+1$, so the proof of induction step is complete. □

REMARK 3. For $s \times t \ (s > t)$ matrices $A_1, A_2, \cdots, A_k$ with $A_i^*A_j = -A_j^*A_i \ (1 \leqslant i < j \leqslant k)$, by Theorem 1, we get

$$\begin{aligned}
\lambda\left(\sum_{i=1}^{k} A_iA_i^*\right) &= \lambda\left(\sum_{i=1}^{k} [A_i \ 0]\begin{bmatrix} A_i^* \\ 0 \end{bmatrix}\right)\\
&\prec \lambda\left(\sum_{i=1}^{k} \begin{bmatrix} A_i^* \\ 0 \end{bmatrix}[A_i \ 0]\right)\\
&\prec \lambda\left(\left(\sum_{i=1}^{k} A_i^*A_i\right) \oplus 0\right).
\end{aligned}$$

Hence, we show the following Proposition:

PROPOSITION 4. [1] *Let $H = [A_{ij}] \in M_{sn} \ (s \geqslant 2)$ be positive semidefinite matrix with $A_{ij} = -A_{ij}^* \ (i \neq j,\ i,j = 1,2,\ldots,s)$. Then*

$$\lambda(H) \prec \lambda\left(\sum_{i=1}^{s} A_{ii} \oplus 0\right). \tag{6}$$

THEOREM 5. *Let $A_i, B_i \in M_n,\ a \in [0,1]$. Then*

$$\left\|\sum_{i=1}^{n} A_iB_i^*\right\|^2 \leqslant \left\|\sum_{i=1}^{n} aA_i^*A_i + \sum_{i=1}^{n}(1-a)B_i^*B_i\right\| \left\|\sum_{i=1}^{n}(1-a)A_i^*A_i + \sum_{i=1}^{n} aB_i^*B_i\right\|,$$

where $A_i^* A_j = -A_j^* A_i$, $B_i^* B_j = -B_j^* B_i$ *(* $i \neq j$ *).*

Proof. By inequality (2), we have

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} A_i B_i^* \oplus 0 \right\|^2 \\
&= \left\| \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} B_1 & B_2 & \cdots & B_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}^* \right\|^2 \\
&\leqslant \left\| a\left[A_i^* A_j\right] + (1-a)\left[B_i^* B_j\right] \right\| \left\| (1-a)\left[A_i^* A_j\right] + a\left[B_i^* B_j\right] \right\|.
\end{aligned}$$

Let $M = \left[A_i^* A_j\right]$ and $N = \left[B_i^* B_j\right]$.

By inequality (6), we obtain

$$\|aM + (1-a)N\| \leqslant \left\| a \sum_{i=1}^{n} A_i^* A_i + (1-a) B_i^* B_i \oplus 0 \right\|$$

and

$$\|(1-a)M + aN\| \leqslant \left\| (1-a) \sum_{i=1}^{n} A_i^* A_i + a B_i^* B_i \oplus 0 \right\|.$$

Hence,

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} A_i B_i^* \right\|^2 \\
&\leqslant \left\| \sum_{i=1}^{n} a A_i^* A_i + \sum_{i=1}^{n} (1-a) B_i^* B_i \right\| \left\| \sum_{i=1}^{n} (1-a) A_i^* A_i + \sum_{i=1}^{n} a B_i^* B_i \right\|. \quad \square
\end{aligned}$$

REMARK 6. Let $n = 2$, $A_1 = B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = B_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = A_1^* A_2 \neq A_2^* A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \ B_1^* B_2 \neq B_2^* B_1$$

and

$$A_1 A_1^* + A_2 A_2^* = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \ A_1^* A_1 + A_2^* A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Therefore,

$$\lambda_1\left(A_1 A_1^* + A_2 A_2^*\right) = \|A_1 A_1^* + A_2 A_2^*\|_1 > \|A_1^* A_1 + A_2^* A_2\|_1 = \lambda_1\left(A_1^* A_1 + A_2^* A_2\right)$$

without $A_i^* A_j = -A_j^* A_i$, $B_i^* B_j = -B_j^* B_i$ $(i \neq j)$ and $\|A\|_1 = s_1(A)$.

COROLLARY 7. [3]*Let $A,B\in M_n$. Then*

$$\|AB^*\|^2 \leqslant \|aA^*A+(1-a)B^*B\|\times\|(1-a)A^*A+aB^*B\|.$$

Next we give a generalization of inequality (5), we require the following result for our purpose.

LEMMA 8. [2] *Let $0\leqslant A\leqslant B$ and $0<t<1$, then $A^t\leqslant B^t$.*

THEOREM 9. *Let $\frac{1}{p}+\frac{1}{q}=1$, $p,\ q>1$, $a\in(0,1)$ and $r\geqslant\max\left\{\frac{1}{p},\frac{1}{q}\right\}$. Then*

$$\left\|\left|\sum_{i=1}^n X_iY_i^*\right|^{2r}\right\| \leqslant \left[\frac{1}{4a(1-a)}\right]^r \left\|\left(a\sum_{i=1}^n (X_i^*X_i)^{\frac12}+(1-a)\sum_{i=1}^n (Y_i^*Y_i)^{\frac12}\right)^{2rp}\right\|^{\frac1p} \times\left\|\left((1-a)\sum_{i=1}^n (X_i^*X_i)^{\frac12}+a\sum_{i=1}^n (Y_i^*Y_i)^{\frac12}\right)^{2rq}\right\|^{\frac1q}$$

under the conditions that the off-diagonal elements of $[X_i^*X_j]^{\frac12}$ and $[Y_i^*Y_j]^{\frac12}$ are skew-Hermitian.

Proof. Let $M=\begin{bmatrix}X_1 & X_2 & \cdots & X_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}$, $N=\begin{bmatrix}Y_1 & Y_2 & \cdots & Y_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}$. By inequality (5), we have

$$\begin{aligned}\left\|\left|\sum_{i=1}^n X_iY_i^*\right|^{2r}\oplus 0\right\| &= \left\||MN^*|^{2r}\right\|\\ &\leqslant \left[\frac{1}{4a(1-a)}\right]^r\left\|\left(a(M^*M)^{\frac12}+(1-a)(N^*N)^{\frac12}\right)^{2rp}\right\|^{\frac1p}\\ &\quad\times\left\|\left((1-a)(M^*M)^{\frac12}+a(N^*N)^{\frac12}\right)^{2rq}\right\|^{\frac1q}.\end{aligned}$$

Under the conditions that the off-diagonal elements of $M_1=[X_i^*X_j]^{\frac12}$ and $N_1=[Y_i^*Y_j]^{\frac12}$ are skew-Hermitian.

We write $M_1=[A_{ij}]=P^*P$ with $P=[P_1,\cdots,P_n]$, then $A_{ii}=P_i^*P_i$. By calculation,

$$M_1^2=(P^*P)^2=P^*(PP^*)P$$

and

$$X_i^*X_i=P_i^*\left(\sum_{i=1}^n P_iP_i^*\right)P_i\geqslant P_i^*(P_iP_i^*)P_i=(P_i^*P_i)^2=A_{ii}^2.$$

By Lemma 8,

$$(X_i^* X_i)^{\frac{1}{2}} \geqslant A_{ii}. \tag{7}$$

Let $N_1 = [B_{ij}]$, we also have

$$(Y_i^* Y_i)^{\frac{1}{2}} \geqslant B_{ii}. \tag{8}$$

By inequality (7) and (8), we obtain

$$\left\| a (M^* M)^{\frac{1}{2}} + (1-a)(N^* N)^{\frac{1}{2}} \right\| \leqslant \left\| \left(a \sum_{i=1}^{n} (X_i^* X_i)^{\frac{1}{2}} + (1-a) \sum_{i=1}^{n} (Y_i^* Y_i)^{\frac{1}{2}} \right) \oplus 0 \right\|.$$

Since $r \geqslant \frac{1}{p}$, by inequality (1)

$$\left\| \left(a (M^* M)^{\frac{1}{2}} + (1-a)(N^* N)^{\frac{1}{2}} \right)^{2rp} \right\|^{\frac{1}{p}}$$

$$\leqslant \left\| \left(a \sum_{i=1}^{n} (X_i^* X_i)^{\frac{1}{2}} + (1-a) \sum_{i=1}^{n} (Y_i^* Y_i)^{\frac{1}{2}} \right)^{2rp} \oplus 0 \right\|^{\frac{1}{p}}.$$

By the same way, we have

$$\left\| \left((1-a)(M^* M)^{\frac{1}{2}} + a (N^* N)^{\frac{1}{2}} \right)^{2rq} \right\|^{\frac{1}{q}}$$

$$\leqslant \left\| \left((1-a) \sum_{i=1}^{n} (X_i^* X_i)^{\frac{1}{2}} + a \sum_{i=1}^{n} (Y_i^* Y_i)^{\frac{1}{2}} \right)^{2rq} \oplus 0 \right\|^{\frac{1}{q}}.$$

Therefore,

$$\left\| \left| \sum_{i=1}^{n} X_i Y_i^* \right|^{2r} \right\| \leqslant \left[\frac{1}{4a(1-a)} \right]^{r} \left\| \left(a \sum_{i=1}^{n} (X_i^* X_i)^{\frac{1}{2}} + (1-a) \sum_{i=1}^{n} (Y_i^* Y_i)^{\frac{1}{2}} \right)^{2rp} \right\|^{\frac{1}{p}}$$

$$\times \left\| \left((1-a) \sum_{i=1}^{n} (X_i^* X_i)^{\frac{1}{2}} + a \sum_{i=1}^{n} (Y_i^* Y_i)^{\frac{1}{2}} \right)^{2rq} \right\|^{\frac{1}{q}}. \quad \square$$

REMARK 10. Let $X_1 = Y_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $X_2 = Y_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $a = \frac{1}{2}$, $r = \frac{1}{2}$, $p = q = 2$, we find that

$$s_1 (X_1 X_1^* + X_2 X_2^*) = 2, s_1 \left((X_1^* X_1)^{\frac{1}{2}} + (X_2^* X_2)^{\frac{1}{2}} \right)^2 = 1.$$

Thus,

$$s_1 (X_1 X_1^* + X_2 X_2^*) > s_1 \left((X_1^* X_1)^{\frac{1}{2}} + (X_2^* X_2)^{\frac{1}{2}} \right)^2$$

without the conditions that the off-diagonal elements of $[X_i^* X_j]^{\frac{1}{2}}$ and $[Y_i^* Y_j]^{\frac{1}{2}}$ are skew-Hermitian.

REFERENCES

[1] F. ZHANG, J. XU, *An eigenvalue inequality for positive semidefinite $k \times k$ block matrices*, Journal of Mathematical Inequalities, 2020 (14), 1383–1388.

[2] R. BHATIA, *Matrix Analysis*, Springer, New York (1997).

[3] K. M. R. AUDENAERT, *Interpolating between the arithmetic-geometric mean and Cauchy-Schwarz matrix norm inequalities*, Oper. Matrices, 2015 (9), 475–479.

[4] M. LIN, H. WOLKOWICZ, *An eigenvalue majorization inequality for positive semidefinite block matrices*, Linear Multilinear A., 2012 (60), 1365–1368.

[5] L. ZOU, Y. JIANG, *A note on interpolation between the arithmetic-geometric mean and Cauchy-Schwarz matrix norm inequalities*, Journal of Mathematical Inequalities, 2016 (10), 1119–1122.

[6] X. WU, *Two inequalities of unitarily invariant norms for matrices*, Science Asia, 2019 (45), 395–397.

(Received February 27, 2021)

Feng Zhang
Department of Mathematics
Northeast Forestry University
Harbin 150040, P. R. China