Journal of
Mathematical
Inequalities
Volume 16, Number 1 (2022), 355–362 doi:10.7153/jmi-2022-16-26

SOME NOTES ON THE INCLUSION BETWEEN MORREY SPACES

PHILOTHEUS E. A. TUERAH AND NICKY K. TUMALUN*

(*Communicated by L. Liu*)

Abstract. In this paper, we show that the Morrey spaces $\mathscr{M}_{q_1}^p(\mathbb{R}^n)$ cannot be contained in the weak Morrey spaces $w\mathscr{M}_{q_2}^p(\mathbb{R}^n)$ for $q_1 \neq q_2$. We also show that the vanishing Morrey spaces $\mathscr{V}\mathscr{M}_q^p(\mathbb{R}^n)$ are not empty and properly contained in the Morrey spaces $\mathscr{M}_q^p(\mathbb{R}^n)$.

1. Introduction

Let $1 \leqslant p \leqslant q < \infty$ and $n \geqslant 2$. The *Morrey space* $\mathscr{M}_q^p(\mathbb{R}^n)$ is the set of all functions $f \in L_{\text{loc}}^p(\mathbb{R}^n)$ for which

$$\|f\|_{\mathscr{M}_q^p} = \sup_{x\in\mathbb{R}^n, r>0} |B(x,r)|^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p(B(x,r))} < \infty,$$

where

$$\|f\|_{L^p(B(x,r))} = \left(\int_{B(x,r)} |f(y)|^p dy\right)^{\frac{1}{p}}.$$

Here $B(x,r)$ is the open ball in Euclidean space $\mathbb{R}^n$ with center x and radius r, and $|B(x,r)|$ denotes its Lebesgue measure. Meanwhile, the *weak Morrey space* $w\mathscr{M}_q^p(\mathbb{R}^n)$ is defined to be the set of all functions $f \in wL_{\text{loc}}^p(\mathbb{R}^n)$ for which

$$\|f\|_{w\mathscr{M}_q^p} = \sup_{x\in\mathbb{R}^n, r>0} |B(x,r)|^{\frac{1}{q}-\frac{1}{p}} \|f\|_{wL^p(B(x,r))} < \infty,$$

where

$$\|f\|_{wL^p(B(x,r))} = \sup_{t>0} t\,|\{y \in B(x,r) : |f(y)| > t\}|^{\frac{1}{p}},$$

and $|\{y \in B(x,r) : |f(y)| > t\}|$ also denotes the Lebesgue measure of the set $\{y \in B(x,r) : |f(y)| > t\}$. Now, we define

$$\mathscr{V}\mathscr{M}_q^p(\mathbb{R}^n) = \left\{f \in \mathscr{M}_q^p(\mathbb{R}^n) : \lim_{r\to 0} \mathscr{M}_f(r) = 0\right\},$$

Mathematics subject classification (2020): 46E30, 42B35.
Keywords and phrases: Morrey spaces, vanishing Morrey spaces.
* Corresponding author.

Paper JMI-16-26

where

$$\mathscr{M}_f(r) = \sup_{x\in\mathbb{R}^n} |B(x,r)|^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p(B(x,r))}.$$

The set $\mathscr{V}\mathscr{M}_q^p(\mathbb{R}^n)$ is called the *vanishing Morrey space*. It is clear that $\mathscr{V}\mathscr{M}_q^p(\mathbb{R}^n)$ is a subset of $\mathscr{M}_q^p(\mathbb{R}^n)$.

The Morrey spaces were introduced by C. B. Morrey [1] and the vanishing Morrey spaces were introduced in [2]. Recently, many authors are attracted in studying the inclusion properties between Morrey spaces [3, 4, 5, 6, 7, 8]. One interesting result stated in [5, Remark 4.5], that is, the weak Morrey spaces $w\mathscr{M}_{q_1}^p(\mathbb{R}^n)$ cannot be contained in the weak Morrey spaces $w\mathscr{M}_{q_2}^p(\mathbb{R}^n)$ and vice versa, for distinct values q_1 and q_2. This statement was deduced by a characterization of inclusion between weak Morrey spaces and its parameters, which is proved by using Closed Graph Theorem and Morrey norm estimate for the characteristic functions of balls [5, Theorem 4.4]. Regarding to the inclusion between vanishing Morrey spaces and Morrey spaces over a bounded domain, it was stated in [9] that the vanishing Morrey spaces are properly contained in the Morrey spaces without giving an explicit counter example.

In this paper, we will prove that the Morrey spaces $\mathscr{M}_{q_1}^p(\mathbb{R}^n)$ cannot be contained in the weak Morrey spaces $w\mathscr{M}_{q_2}^p(\mathbb{R}^n)$ and the Morrey spaces $\mathscr{M}_{q_2}^p(\mathbb{R}^n)$ cannot be contained in the weak Morrey spaces $w\mathscr{M}_{q_1}^p(\mathbb{R}^n)$, for different values q_1 and q_2. This result is more general and sharp than the previous result in [5] since we can recover that previous result and the fact that $\mathscr{M}_q^p(\mathbb{R}^n)$ is a proper subset of $w\mathscr{M}_q^p(\mathbb{R}^n)$ [6, Theorem 1.2]. We also note that our method here is different than in [5] because we give a function which belongs to $\mathscr{M}_{q_1}^p(\mathbb{R}^n)\backslash w\mathscr{M}_{q_2}^p(\mathbb{R}^n)$ and a function which belongs to $\mathscr{M}_{q_2}^p(\mathbb{R}^n)\backslash w\mathscr{M}_{q_1}^p(\mathbb{R}^n)$. Furthermore, by using the idea in [8], we also show that the vanishing Morrey spaces $\mathscr{V}\mathscr{M}_q^p(\mathbb{R}^n)$ are non empty and properly contained in $\mathscr{M}_q^p(\mathbb{R}^n)$ by providing some examples.

The positive constant C that appears in the proofs of all theorems may vary from line to line and the notation $C = C(n,p,q)$ indicates that C depends only on n,p and q.

2. A note on the inclusion between weak Morrey spaces

Let $1 \leqslant p < q < \infty$ and $\gamma = \frac{n}{q} < n$. Define a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ by the formula

$$f(y) = \begin{cases} |y|^{-\gamma}, & y \neq 0, \\ 0, & y = 0. \end{cases} \tag{1}$$

It is clear that $1 - \frac{n}{p\gamma} < 0$ and $n - p\gamma > 0$ by observing to the given assumptions.

The function f, that appears in Lemma 1 and 3, is defined by (1).

LEMMA 1. *If* $x \in \mathbb{R}^n$ *and* $r > 0$*, then* $\|f\|_{L^p(B(x,r))} \leqslant Cr^{\frac{n}{p}-\gamma} = Cr^{\frac{n}{p}-\frac{n}{q}}$*, where* $C = C(n,p,q)$.

Proof. Note that

$$\int_{B(x,r)} |f(y)|^p dy = \int_{\{|y|\leqslant|x-y|<r\}} |y|^{-p\gamma} dy + \int_{\{|x-y|<|y|\}\cap\{|x-y|<r\}} |y|^{-p\gamma} dy$$
$$= I + II.$$

Since $n - p\gamma > 0$, we have

$$I \leqslant \int_{\{|y|<r\}} |y|^{-p\gamma} dy = C\int_0^r t^{n-p\gamma-1} dt = Cr^{n-p\gamma}$$

and

$$II \leqslant \int_{\{|x-y|<r\}} |x-y|^{-p\gamma} dy = C\int_0^r t^{n-p\gamma-1} dt = Cr^{n-p\gamma},$$

by using polar coordinate for radial function. Therefore

$$\|f\|_{L^p(B(x,r))} = \left(\int_{B(x,r)} |f(y)|^p dy\right)^{\frac{1}{p}} \leqslant C\left(Cr^{\frac{n}{p}-\gamma}\right)^{\frac{1}{p}} = Cr^{\frac{n}{p}-\frac{n}{q}},$$

which proves the lemma. □

The following lemma is not hard to prove. We leave its proof to the reader.

LEMMA 2. *Let $s > 0$, $M \geqslant 0$, and $\varphi : (0,\infty) \longrightarrow [0,\infty)$. If*

$$\sup_{0<t\leqslant s} \varphi(t) = M = \sup_{s<t<\infty} \varphi(t),$$

then

$$\sup_{t>0} \varphi(t) = M.$$

Using the above lemma, we can compute the weak Lebesgue norm of f on the ball $B(0,r)$ with arbitrary radius r.

LEMMA 3. *If $r > 0$, then $\|f\|_{wL^p(B(0,r))} = Cr^{-\gamma+\frac{n}{p}} = Cr^{-\frac{n}{q}+\frac{n}{p}}$, where $C = C(n,p,q)$.*

Proof. Let r be an arbitrary positive real number. Note that, for every $t > 0$, we have

$$|\{y \in B(0,r) : |f(y)| > t\}| = \left|\left\{y \in B(0,r) : |y| < t^{-\frac{1}{\gamma}}\right\}\right|$$
$$= \left|B(0,r) \cap B(0,t^{-\frac{1}{\gamma}})\right|. \tag{2}$$

We now define $\varphi : (0,\infty) \longrightarrow [0,\infty)$ by the formula

$$\varphi(t) = t\,|\{y \in B(0,r) : |f(y)| > t\}|^{\frac{1}{p}}. \tag{3}$$

For every $t > r^{-\gamma}$, we obtain $t^{-\frac{1}{\gamma}} < r$. Then

$$|\{y \in B(0,r) : |f(y)| > t\}| = \left|B(0,t^{-\frac{1}{\gamma}})\right| = Ct^{-\frac{n}{\gamma}},$$

by using (2). This gives us

$$t\,|\{y \in B(0,r) : |f(y)| > t\}|^{\frac{1}{p}} = Ct\left(t^{-\frac{n}{\gamma}}\right)^{\frac{1}{p}} = Ct^{1-\frac{n}{p\gamma}}, \quad \forall t \in (r^{-\gamma},\infty). \tag{4}$$

On the other hand, for every $t \leqslant r^{-\gamma}$, we have $t^{-\frac{1}{\gamma}} \geqslant r$. Hence

$$|\{y \in B(0,r) : |f(y)| > t\}| = |B(0,r)| = Cr^n,$$

which comes from (2). Therefore,

$$t\,|\{y \in B(0,r) : |f(y)| > t\}|^{\frac{1}{p}} = Ct(r^n)^{\frac{1}{p}} = Ctr^{\frac{n}{p}}, \quad \forall t \in \left(0,r^{-\gamma}\right]. \tag{5}$$

We obtain

$$\varphi(t) = \begin{cases} Ct^{1-\frac{n}{p\gamma}}, & \forall t \in (r^{-\gamma},\infty) \\ Ct(r^n)^{\frac{1}{p}} = Ctr^{\frac{n}{p}}, & \forall t \in (0,r^{-\gamma}]. \end{cases}$$

by virtue of (4) and (5). Observing φ non increasing on $(r^{-\gamma},\infty)$ and non decreasing on $(0,r^{-\gamma}]$, since $1-\frac{n}{p\gamma} < 0$ and $\frac{n}{p} > 0$ respectively, then

$$\sup_{r^{-\gamma}<t<\infty} \varphi(t) = Cr^{-\gamma+\frac{n}{p}} = \sup_{0<t\leqslant r^{-\gamma}} \varphi(t). \tag{6}$$

Thus

$$\|f\|_{wL^p(B(0,r))} = \sup_{t>0} \varphi(t) = Cr^{-\gamma+\frac{n}{p}} = Cr^{-\frac{n}{q}+\frac{n}{p}},$$

that is concluded from Lemma 2. □

By taking Lemma 2 and Lemma 3 as the tools, we are ready to state and prove the first main result of this paper.

THEOREM 1. *Let $1 \leqslant p < q_1 < \infty$ and $1 \leqslant p < q_2 < \infty$. If $q_1 \neq q_2$, then $\mathscr{M}^p_{q_1}(\mathbb{R}^n) \nsubseteq w\mathscr{M}^p_{q_2}(\mathbb{R}^n)$ and $\mathscr{M}^p_{q_2}(\mathbb{R}^n) \nsubseteq w\mathscr{M}^p_{q_1}(\mathbb{R}^n)$.*

Proof. We will only prove that $\mathscr{M}^p_{q_1}(\mathbb{R}^n)$ is not contained by $w\mathscr{M}^p_{q_2}(\mathbb{R}^n)$. The proof that $\mathscr{M}^p_{q_2}(\mathbb{R}^n)$ is not contained by $w\mathscr{M}^p_{q_1}(\mathbb{R}^n)$ can be done by similar method.

Let $\gamma_1 = n/q_1$ and $f_1 : \mathbb{R}^n \longrightarrow \mathbb{R}$, defined by the formula

$$f_1(y) = \begin{cases} |y|^{-\gamma_1}, & y \neq 0, \\ 0, & y = 0. \end{cases}$$

We will show that $f_1 \in \mathscr{M}^p_{q_1}(\mathbb{R}^n) \backslash w\mathscr{M}^p_{q_2}(\mathbb{R}^n)$. Let $x \in \mathbb{R}^n$ and $r > 0$ be arbitrarily given. According to Lemma 1, by replacing γ with γ_1, we obtain

$$|B(x,r)|^{\frac{1}{q_1}-\frac{1}{p}} \|f_1\|_{L^p(B(x,r))} \leqslant Cr^{\frac{n}{q_1}-\frac{n}{p}} r^{\frac{n}{p}-\frac{n}{q_1}} = C < \infty.$$

This gives us

$$\|f_1\|_{\mathscr{M}^p_{q_1}} = \sup_{x\in\mathbb{R}^n, r>0} |B(x,r)|^{\frac{1}{q_1}-\frac{1}{p}} \|f_1\|_{L^p(B(x,r))} < \infty,$$

since x and r are arbitrary. Whence $f_1 \in \mathscr{M}^p_{q_1}(\mathbb{R}^n)$. By virtue to Lemma 3, we have

$$\|f_1\|_{w\mathscr{M}^p_{q_2}} \geqslant |B(0,r)|^{\frac{1}{q_2}-\frac{1}{p}} \|f_1\|_{wL^p(B(0,r))} = Cr^{\frac{n}{q_2}-\frac{n}{p}} r^{\frac{n}{p}-\frac{n}{q_1}} = Cr^{n\left(\frac{1}{q_2}-\frac{1}{q_1}\right)}.$$

Hence $\|f_1\|_{w\mathscr{M}^p_{q_2}} = \infty$. This is due to arbitrary r and $q_1 \neq q_2$. We conclude that $f_1 \notin w\mathscr{M}^p_{q_2}(\mathbb{R}^n)$. Thus, we have already proved that $\mathscr{M}^p_{q_1}(\mathbb{R}^n) \nsubseteq w\mathscr{M}^p_{q_2}(\mathbb{R}^n)$. $\square$

As an immediate consequence of Theorem 1, we recover the result from [5] which is stated in the following corollary.

COROLLARY 1. *Let* $1 \leqslant p < q_1 < \infty$ *and* $1 \leqslant p < q_2 < \infty$. *If* $q_1 \neq q_2$, *then* $w\mathscr{M}^p_{q_1}(\mathbb{R}^n) \nsubseteq w\mathscr{M}^p_{q_2}(\mathbb{R}^n)$ *and* $w\mathscr{M}^p_{q_2}(\mathbb{R}^n) \nsubseteq w\mathscr{M}^p_{q_1}(\mathbb{R}^n)$.

3. A note on the inclusion between Morrey spaces and vanishing Morrey spaces

Let $1 \leqslant p < q < \infty$ and $\delta = \exp(\frac{-2q}{np})$. Define a function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ by the formula

$$g(y) = \begin{cases} \left(\frac{\chi_B(y)}{|y|^{\frac{np}{q}}(\ln|y|)^2}\right)^{\frac{1}{p}}, & y \neq 0, \\ 0, & y = 0, \end{cases} \tag{7}$$

where χ_B is a characteristic function defined on $B = B(0,\delta)$.

The function g in the following lemma is defined by (7). This following lemma shows that the vanishing Morrey spaces in a non empty set.

LEMMA 4. $g \in \mathscr{V}\mathscr{M}^p_q(\mathbb{R}^n)$.

Proof. Let $x \in \mathbb{R}^n$ and $r > 0$ be arbitrarily given. Note that

$$\begin{aligned} |B(x,r)|^{\frac{1}{q}-\frac{1}{p}} \|g\|_{L^p(B(x,r))} &\leqslant C\left(\int_{|y|\leqslant|x-y|<r} \frac{\chi_B(y)}{|x-y|^{n-\frac{np}{q}}|y|^{\frac{np}{q}}(\ln|y|)^2} dy\right)^{\frac{1}{p}} \\ &\quad + C\left(\int_{\substack{\{|x-y|<|y|\} \\ \cap\{|x-y|<r\}}} \frac{\chi_B(y)}{|x-y|^{n-\frac{np}{q}}|y|^{\frac{np}{q}}(\ln|y|)^2} dy\right)^{\frac{1}{p}} \\ &= I + II. \end{aligned} \tag{8}$$

Now we have two cases, that is, $\delta \leqslant r$ or $r < \delta$. Assume $\delta \leqslant r$, then we have

$$I \leqslant \int_{|y|<r} \frac{\chi_B(y)}{|y|^n(\ln|y|)^2} dy = \int_{|y|<\delta} \frac{1}{|y|^n(\ln|y|)^2} dy = C\left(\frac{-1}{\ln(\delta)}\right), \tag{9}$$

and

$$II = \int_{\substack{\{|x-y|<|y|<\delta\}\\ \cap\{|x-y|<r\}}} \frac{1}{|x-y|^{n-\frac{np}{q}}|y|^{\frac{np}{q}}(\ln|y|)^2}dy \leqslant \int_{|x-y|<\delta} \frac{1}{|x-y|^n(\ln|x-y|)^2}dy$$

$$= C\left(\frac{-1}{\ln(\delta)}\right), \tag{10}$$

since $1/t^{np/q}(\ln(t))^2$ decreasing on interval $(0,\delta)$. Assume $r<\delta$. We have

$$I \leqslant \int_{|y|<r} \frac{1}{|y|^n(\ln|y|)^2}dy = C\left(\frac{-1}{\ln(r)}\right) \leqslant C\left(\frac{-1}{\ln(\delta)}\right), \tag{11}$$

and

$$II = \int_{\substack{\{|x-y|<|y|<r\}\\ \cap\{|x-y|<r\}}} \frac{1}{|x-y|^{n-\frac{np}{q}}|y|^{\frac{np}{q}}(\ln|y|)^2}dy \leqslant \int_{|x-y|<r} \frac{1}{|x-y|^n(\ln|x-y|)^2}dy$$

$$= C\left(\frac{-1}{\ln(r)}\right) \leqslant C\left(\frac{-1}{\ln(\delta)}\right), \tag{12}$$

since $1/t^{np/q}(\ln(t))^2$ decreasing on interval $(0,r)\subseteq(0,\delta)$. By virtue of (8), (9), (10), (11), and (12), we conclude that

$$|B(x,r)|^{\frac{1}{q}-\frac{1}{p}}\|g\|_{L^p(B(x,r))} \leqslant I+II \leqslant C\left(\frac{-1}{\ln(\delta)}\right)^{\frac{1}{p}},$$

where $C=C(n,p,q)$. This means $g\in\mathscr{M}_q^p(\mathbb{R}^n)$. We remaind to prove

$$\lim_{r\to 0}\mathscr{M}_f(r)=0. \tag{13}$$

For every $0<r<\delta$, we have shown that

$$\mathscr{M}_f(r) \leqslant C\left(\frac{-1}{\ln(r)}\right)^{\frac{1}{p}}.$$

This means (13) holds and the proof is done. □

Now we define a function that will play as an element of Morrey spaces but not in the vanishing Morrey spaces. Let $1\leqslant p<q<\infty$. For every $k\in\mathbb{N}$, with $k\geqslant 3$, we set $x_k=(2^{-k},\ldots,0)\in\mathbb{R}^n$ and

$$u_k(y) = \begin{cases} 8^{\frac{np}{q}k}, & y\in B(x_k,8^{-k}),\\ 0, & y\notin B(x_k,8^{-k}). \end{cases}$$

Define a function $u:\mathbb{R}^n\longrightarrow\mathbb{R}$ by the formula

$$u(y) = \left(\sum_{k=3}^{\infty} u_k(y)\right)^{\frac{1}{p}}. \tag{14}$$

We first claim that u belongs to the Morrey spaces $\mathscr{M}_q^p(\mathbb{R}^n)$.

LEMMA 5. $u \in \mathscr{M}_q^p(\mathbb{R}^n)$.

Proof. Let $x \in \mathbb{R}^n$ and $r > 0$ be arbitrarily given. There are two casses: (i) $x \notin B(x_k, 2(4^{-k}))$ for every $k \geqslant 3$, or, (ii) $x \in B(x_j, 2(4^{-j}))$ for some $j \geqslant 3$. Assume (i) holds. Then

$$2(4^{-k}) \leqslant |x - x_k| \leqslant |x-y| + |y - x_k| < r + 4^{-k},$$

for every $y \in B(x,r) \cap B(x_k, 8^{-k})$. This means $r^{\frac{np}{q}-n} \leqslant 4^{(n-\frac{np}{q})k}$ and

$$\begin{aligned} r^{\frac{np}{q}-n} \int_{B(x,r)} |u(y)|^p dy &\leqslant \sum_{k=3}^{\infty} 4^{(n-\frac{np}{q})k} \int_{B(x,r)\cap B(x_k,8^{-k})} 8^{\frac{np}{q}k} dy \\ &\leqslant C \sum_{k=3}^{\infty} 2^{(\frac{np}{q}-n)k} < \infty, \end{aligned} \tag{15}$$

where C depends on n. Assume (ii) holds. Since $\{B(x_k, 2(4^{-k}))\}_{k \geqslant 3}$ is a disjoint collection, then there is only one $j \geqslant 3$ such that $x \in B(x_j, 2(4^{-j}))$ and $x \notin B(x_k, 2(4^{-k}))$ for every $k \geqslant 3$ with $k \neq j$. Note that

$$r^{\frac{np}{q}-n} \int_{B(x,r)\cap B(x_j,8^{-j})} u_j(y) dy = r^{\frac{np}{q}-n} \int_{B(x,r)\cap B(x_j,8^{-j})} 8^{\frac{np}{q}j} dy \leqslant C < \infty, \tag{16}$$

where C depends on n, p, and q. By virtue of (16) and the computation of (15), we have

$$\begin{aligned} r^{\frac{np}{q}-n} \int_{B(x,r)} |u(y)|^p dy &= r^{\frac{np}{q}-n} \sum_{k=3}^{\infty} \int_{B(x,r)\cap B(x_k,8^{-k})} u_k(y) dy \\ &= r^{\frac{np}{q}-n} \int_{B(x,r)\cap B(x_j,8^{-j})} u_j(y) dy \\ &\quad + r^{\frac{np}{q}-n} \sum_{\substack{k=3 \\ k\neq j}}^{\infty} \int_{B(x,r)\cap B(x_k,8^{-k})} u_k(y) dy \\ &\leqslant C + C \sum_{\substack{k=3 \\ k\neq j}}^{\infty} 2^{(\frac{np}{q}-n)k} < \infty, \end{aligned} \tag{17}$$

where C depends on n, p, and q. Combining (16) and (17), whence

$$|B(x,r)|^{\frac{1}{q}-\frac{1}{p}} \|u\|_{L^p(B(x,r))} = C \left(r^{\frac{np}{q}-n} \int_{B(x,r)} |u(y)|^p dy \right)^{\frac{1}{p}} \leqslant C < \infty,$$

where C depends on n, p, and q. Therefore $u \in \mathscr{M}_q^p(\mathbb{R}^n)$. □

The following theorem states that the vanishing Morrey spaces is a non empty proper subset of the Morrey spaces. This theorem is the second main result in this paper.

THEOREM 2. *Let $1 \leqslant p < q < \infty$. Then $\mathscr{V}\mathscr{M}_q^p(\mathbb{R}^n)$ is a non empty proper subset of $\mathscr{M}_q^p(\mathbb{R}^n)$.*

Proof. According to Lemma 4, $\mathscr{V}\mathscr{M}_q^p(\mathbb{R}^n)$ is non empty, and according to Lemma 5, the function u belongs to $\mathscr{M}_q^p(\mathbb{R}^n)$. Therefore, we need only to show that u does not belong to $\mathscr{V}\mathscr{M}_q^p(\mathbb{R}^n)$. Let $0 < r < 1$. By the Archimedan property, there is an integer $k \geqslant 3$ such that $8^{-k} < r$. Then

$$\begin{aligned}(\mathscr{M}_f(r))^p &\geqslant Cr^{\frac{np}{q}-n}\int_{B(x_k,r)}|u(y)|^p dy \geqslant C\int_{B(x_k,8^{-k})}u_k(y)dy \\ &= C\int_{B(x_k,8^{-k})}8^{\frac{np}{q}k}dy \geqslant C8^{-nk}\int_{B(x_k,8^{-k})}1dy = C > 0,\end{aligned}$$

where C depends on n. This means $\mathscr{M}_f(r)$ is bounded away from zero as r tends to zero. Thus $u \notin \mathscr{V}\mathscr{M}_q^p(\mathbb{R}^n)$. $\square$

REFERENCES

[1] C. B. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43**, (1938), 126–166.

[2] C. VITANZA, *Functions with vanishing Morrey norm and elliptic partial differential equations*, in Proceedings of Methods of Real Analysis and Partial Differential Equations, Capri, (1990).

[3] D. I. HAKIM AND Y. SAWANO, *Complex interpolation of various subspaces of Morrey spaces*, Sci. China Math., **63**, 5 (2020), 937–964.

[4] H. GUNAWAN, D. I. HAKIM, AND M. IDRIS, *Proper inclusions of Morrey spaces*, Glasnik Matematički., **53**, 1 (2018), 143–151.

[5] H. GUNAWAN, D. I. HAKIM, K. M. LIMANTA, AND A. A. MASTA, *Inclusion properties of generalized Morrey spaces*, Math. Nachr., **290**, 1 (2017), 332–340.

[6] H. GUNAWAN, D. I. HAKIM, E. NAKAI, AND Y. SAWANO, *On inclusion relation between weak Morrey spaces and Morrey spaces*, Nonlinear Analysis., **168**, (2018), 27–31.

[7] N. K. TUMALUN AND H. GUNAWAN, *Morrey spaces are embedded between weak Morrey spaces and Stummel classes*, J. Indones. Math. Soc., **25**3, (2019), 203–209.

[8] N. K. TUMALUN, D. I. HAKIM, AND H. GUNAWAN, *Inclusion between generalized Stummel classes and other function spaces*, Math. Inequal. Appl., **23**2, (2020), 547–562.

[9] N. SAMKO, *Maximal, potential and singular operators in vanishing generalized Morrey spaces*, J. Global. Optim., **57**, (2013), 1385–1399.

(Received March 22, 2021)

Philotheus E. A. Tuerah
Mathematics Department, Faculty of Mathematics and Natural Sciences
Universitas Negeri Manado, Tondano Selatan District, Minahasa Regency
Sulawesi Utara Province, Indonesia, Indonesia
e-mail: pheatuerah@unima.ac.id

Nicky K. Tumalun
Mathematics Department, Faculty of Mathematics and Natural Sciences
Universitas Negeri Manado, Tondano Selatan District, Minahasa Regency
Sulawesi Utara Province, Indonesia
e-mail: nickytumalun@unima.ac.id