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ROUGH FRACTIONAL INTEGRAL OPERATORS ON MORREY—ADAMS SPACES

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Abstract. In 1981, Adams introduced another variant of Morrey spaces, and we call it the Morrey–Adams space. In this paper, we investigate the boundedness of rough fractional integral operators on Morrey–Adams spaces under a weaker condition. We compare it with Adams' results. We then refine the results to vanishing Morrey–Adams spaces in the local sense. We also prove the beyond Adams' inequality of rough fractional integral operators on local Morrey–Adams spaces for the radial functions.

1. Introduction

Let $1 \leqslant p < \infty$ and $\lambda \in \mathbb{R}$. It is well known that the Morrey space $L^{p,\lambda}$ is the set of f with

$$\|f\|_{L^{p,\lambda}} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))} < \infty$$

where $B(x,r) = \{y \in \mathbb{R}^n; |y-x| < r\}$. The Morrey space contains some non-trivial functions if $0 \leqslant \lambda \leqslant n$. The simplest example is the characteristic function $\chi_{B(0,r_0)}$ with $\|\chi_{B(0,r_0)}\|_{L^{p,\lambda}} = c_n r_0^{\frac{n-\lambda}{p}}$ and c_n is a positive constant.

Note that, the Morrey norm $\|\cdot\|_{L^{p,\lambda}}$ can be written as

$$\|f\|_{L^{p,\lambda}} = \sup_{x\in\mathbb{R}^n} \left\| |\cdot|^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,\cdot))} \right\|_{L^\infty(0,\infty)}.$$

In 1981, Adams introduced the function space $L^{p,\lambda}_\theta$ by replacing the L^∞-norm $\|\cdot\|_{L^\infty(0,\infty)}$ within the Morrey norm into the L^θ-norm $\|\cdot\|_{L^\theta(0,\infty)}$ for $1 \leqslant \theta < \infty$ [5, 11]. The space $L^{p,\lambda}_\theta$ is the set of f with

$$\|f\|_{L^{p,\lambda}_\theta} = \sup_{x\in\mathbb{R}^n} \left(\int_0^\infty r^{-\frac{\lambda\theta}{p}} \|f\|^\theta_{L^p(B(x,r))} dr \right)^{\frac{1}{\theta}} < \infty.$$

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By [10, Lemma 1], the space $L^{p,\lambda}_\theta$ is non-trivial if $\frac{p}{\theta} < \lambda < n + \frac{p}{\theta}$. For example, $\|\chi_{B(0,r_0)}\|_{L^{p,\lambda}_\theta} = c_n r_0^{\frac{n-\lambda}{p}+\frac{1}{\theta}}$ where c_n is a positive constant. The interesting thing about $L^{p,\lambda}_\theta$ is that the λ can be greater than the dimension of the function's domain.

Similar to the relation between Lebesgue space $L^\theta(0,\infty)$ and $L^\infty(0,\infty)$, there is no inclusion property between $L^{p,\lambda}_\theta$ and $L^{p,\lambda}$. In instance, we have $f(x) = |x|^{-\frac{n-\lambda}{p}}$ in $L^{p,\lambda} \setminus L^{p,\lambda}_\theta$. On the other hand, the function $g(x) = |x|^{-\frac{n-\kappa}{p}} \chi_{B(0,1)}(x)$ with $n > \kappa > \lambda - \frac{p}{\theta}$ is in $L^{p,\lambda}_\theta \setminus L^{p,\lambda}$.

In the last decade, some authors call the general version of $L^{p,\lambda}_\theta$ (which contains some weight function), and they call the space Morrey-type space (see [3, 4, 10, 11]). In this paper, we call $L^{p,\lambda}_\theta$ the Morrey–Adams space.

The aim of this paper is to investigate the behavior of rough fractional integral operators on Morrey–Adams spaces. Let $0 < \alpha < n$. Let Ω be a zero degree homogeneous function in $\mathbb{R}^n$; $\Omega(\tau x) = \Omega(x)$ for any $x \in \mathbb{R}^n$ and $\tau > 0$. The rough fractional integral operator $T_{\Omega,\alpha}$ is defined as

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy.$$

We have the following two boundedness properties of $T_{\Omega,\alpha}$.

THEOREM 1. *Let* $1 < p < \infty$, $0 < \alpha < \min\{n, \frac{n-\lambda}{p} + \frac{1}{\theta}\}$,

$$\frac{n-\mu}{q} + \frac{1}{\varphi} = \frac{n-\lambda}{p} + \frac{1}{\theta} - \alpha, \quad \frac{\theta(n-\lambda)}{\varphi(n-\mu)} = \frac{p}{q}.$$

If $\Omega \in L^s(S^{n-1})$ *with* $s \geqslant p' = \frac{p}{p-1}$ *and* $\lambda \leqslant \mu < n$ *or* $n < \mu \leqslant \lambda$, *then* $T_{\Omega,\alpha}$ *is bounded from* $L^{p,\lambda}_\theta$ *to* $L^{q,\mu}_\varphi$.

THEOREM 2. *Let* $1 < p < \infty$, $0 < \alpha < \min\{n, \frac{1}{\theta}\}$,

$$\frac{1}{\varphi} = \frac{1}{\theta} - \alpha, \quad \frac{\theta}{\varphi} \leqslant \frac{p}{q}.$$

If $\Omega \in L^s(S^{n-1})$ *with* $s \geqslant p' = \frac{p}{p-1}$, *then* $T_{\Omega,\alpha}$ *is bounded from* $L^{p,n}_\theta$ *to* $L^{q,n}_\varphi$.

For the case of $\mu = \lambda$ and $\Omega \equiv 1$, we note that Theorems 1 and 2 are similar to [11, Theorem 3]. We also note that the condition $\frac{\theta}{\varphi} \leqslant \frac{p}{q}$ in Theorem 2 is weaker than [11, Theorem 3]. For the case $\theta = \varphi = \infty$, Theorem 1 reconstructs the boundedness of $T_{\Omega,\alpha}$ on Morrey spaces (see [6, 9]).

Vanishing Morrey Space is one of the trending topics in recent study (see [1, 2]). To the best of our knowledge, there is still no paper discussing vanishing Morrey–Adams spaces. In Section 3, we shall discuss vanishing Morrey–Adams spaces based on the definition of vanishing Morrey space. We also show the invariance of vanishing Morrey–Adams space for $T_{\Omega,\alpha}$ in the local sense.

In [6], Salim et al discussed inequality beyond Adams of $T_{\Omega,\alpha}$ on the classical Morrey spaces. In Section 4, we investigate the beyond Adams' inequalities of $T_{\Omega,\alpha}$ on the Morrey–Adams spaces. Based on [6], in this investigation, we shall restrict the domain of $T_{\Omega,\alpha}$ with radial function.

2. Operator $T_{\Omega,\alpha}$ on Morrey–Adams spaces

One of the main tool to investigate the operator $T_{\Omega,\alpha}$ is the rough maximal operator M_Ω which is given by

$$M_\Omega f(x) = \sup_{r>0} r^{-n} \int_{B(x,r)} |\Omega(x-y)|\,|f(y)|dy.$$

It is well known that M_Ω is bounded on L^p for $p>1$ if $\Omega \in L^1(S^{n-1})$ with $S^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}$.

We write $a \lesssim b$ if there exists constant $c>0$ such that $a \leqslant cb$. We also write $a \lesssim_\Omega b$ if there exists constant $c_\Omega > 0$ depending on Ω such that $a \leqslant c_\Omega b$.

THEOREM 3. *The operator M_Ω is bounded $L^{p,\lambda}_\theta$ for $p>1$ if $\Omega \in L^s(S^{n-1})$ with $s \geqslant p' = \frac{p}{p-1}$.*

Proof. Fix $z \in \mathbb{R}^n$, and decompose $f = f_1 + f_2$ with $f_1 = f\chi_{B(z,2r)}$. Note that, the decomposition of f depends on $r>0$. By the boundedness of M_Ω on L^p, we obtain

$$\left(\int_0^\infty r^{-\frac{\lambda\theta}{p}} \|M_\Omega f_1\|^\theta_{L^p(B(z,r))} dr\right)^{\frac{1}{\theta}} \lesssim_\Omega \left(\int_0^\infty r^{-\frac{\lambda\theta}{p}} \|f\|^\theta_{L^p(B(z,2r))} dr\right)^{\frac{1}{\theta}} \lesssim \|f\|_{L^{p,\lambda}_\theta}.$$

For $x \in B(z,r)$, we have $B^c(z,2r) \subset B^c(x,r)$. Hence, by Hölder's inequality,

$$\begin{aligned} M_\Omega f_2(x) &\lesssim \sum_{j=1}^\infty \int_{B(x,2^j r)\setminus B(x,2^{j-1}r)} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)|dy \\ &\lesssim_\Omega \sum_{j=1}^\infty (2^j r)^{-\frac{n}{p}} \|f\|_{L^p(B(z,2^{j+1}r))} \end{aligned}$$

where the RHS is independent of x. By Minkowski's inequality and by substituting $t = 2^{j+1}r$, we can proceed as follows.

$$\begin{aligned} \left(\int_0^\infty r^{-\frac{\lambda\theta}{p}} \|M_\Omega f_2\|^\theta_{L^p(B(z,r))} dr\right)^{\frac{1}{\theta}} &\lesssim_\Omega \sum_{j=1}^\infty 2^{-\frac{jn}{p}} \left(\int_0^\infty r^{\frac{-\lambda\theta}{p}} \|f\|^\theta_{L^p(B(z,2^{j+1}r))} dr\right)^{\frac{1}{\theta}} \\ &\lesssim \sum_{j=1}^\infty 2^{\frac{j(\lambda-n)}{p}} \left(\int_0^\infty t^{\frac{-\lambda\theta}{p}} \|f\|^\theta_{L^p(B(z,t))} \frac{dt}{2^j}\right)^{\frac{1}{\theta}} \\ &\lesssim \|f\|_{L^{p,\lambda}_\theta} \sum_{j=1}^\infty 2^{\frac{j(\lambda-n)}{p} - \frac{j}{\theta}} \end{aligned}$$

with the convergent sums due to $\lambda < n + \frac{p}{\theta}$. Following by sublinearity of M_Ω, Theorem 3 is verified. □

In order to use Theorem 3 in the investigation of $T_{\Omega,\alpha}$, we need the following pointwise estimate property.

LEMMA 1. *Let* $f \in L_\theta^{p,\lambda}$, $\Omega \in L^s(S^{n-1})$ *with* $s \geqslant p'$, *and* $\alpha < \frac{n-\lambda}{p} + \frac{1}{\theta}$,

$$|T_{\Omega,\alpha} f(x)| \lesssim_\Omega M_\Omega f(x)^u \|f\|_{L_\theta^{p,\lambda}}^{1-u}.$$

for almost every $x \in \mathbb{R}^n$ *where* $u = 1 - \frac{\alpha p \theta}{(n-\lambda)\theta + p}$.

Proof. Let f be a non-trivial function. Fix $x \in \mathbb{R}^n$, $R > 0$, and we decompose $f = f_1 + f_2$ with $f_1 = f\chi_{B(x,R)}$. We have

$$\begin{aligned} |T_{\Omega,\alpha} f_1(x)| &\leqslant \sum_{j=1}^{\infty} \int_{B(x,2^{-j+1}R)\setminus B(x,2^{-j}R)} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| dy \\ &\lesssim R^\alpha M_\Omega f(x). \end{aligned}$$

Since $\alpha < \frac{n-\lambda}{p} + \frac{1}{\theta}$, by Fubini's Theorem and Hölder's inequality

$$\begin{aligned} |T_{\Omega,\alpha} f_2(x)| &\lesssim \int_{B^c(x,R)} |\Omega(x-y)||f(y)| \int_{|x-y|}^{\infty} t^{\alpha-n-1} dt dy \\ &\lesssim \|\Omega\|_{L^s(S^{n-1})} \int_R^\infty t^{\alpha - \frac{n}{p} - 1} \|f\|_{L^p(B(x,t))} dt \\ &\lesssim R^{\alpha - \frac{n-\lambda}{p} - \frac{1}{\theta}} \|\Omega\|_{L^s(S^{n-1})} \|f\|_{L_\theta^{p,\lambda}}. \end{aligned}$$

The proof then completes once we apply the linearity of $T_{\Omega,\alpha}$

$$|T_{\Omega,\alpha} f(x)| \lesssim R^\alpha M_\Omega f(x) + R^{\alpha - \frac{n-\lambda}{p} - \frac{1}{\theta}} \|\Omega\|_{L^s(S^{n-1})} \|f\|_{L_\theta^{p,\lambda}}$$

and minimize the right-hand-side over $R > 0$. □

The proof of Lemma 1 and Lemma 2 (see Section 4) are motivated by the work of Hedberg in 1972 (see [7]). We now can prove Theorems 1 and 2 as follows.

Proof. Fix $z \in \mathbb{R}^n$. Follow by Lemma 1,

$$\left(\int_0^\infty r^{-\frac{\mu\varphi}{q}} \|T_{\Omega,\alpha} f\|_{L^q(B(z,r))}^\varphi dr \right)^{\frac{1}{\varphi}} \lesssim_\Omega \|f\|_{L_\theta^{p,\lambda}}^{1-u} \left(\int_0^\infty r^{-\frac{\mu\varphi}{q}} \|M_\Omega f^u\|_{L^q(B(z,r))}^\varphi dr \right)^{\frac{1}{\varphi}}$$

For Theorem 1, we have $\frac{n-\mu}{q} + \frac{1}{\varphi} = \frac{n-\lambda}{p} + \frac{1}{\theta} - \alpha$ and $\frac{\theta(n-\lambda)}{\varphi(n-\mu)} = \frac{p}{q}$. Therefore,

$$u = 1 - \frac{\alpha p \theta}{(n-\lambda)\theta + p} = \frac{\frac{n-\lambda}{p} + \frac{1}{\theta} - \alpha}{\frac{n-\lambda}{p} + \frac{1}{\theta}} = \frac{\frac{n-\mu}{q} + \frac{1}{\varphi}}{\frac{n-\lambda}{p} + \frac{1}{\theta}} = \frac{\theta}{\varphi}.$$

Moreover, the conditions $\lambda \leqslant \mu < n$ or $n < \mu \leqslant \lambda$, and $\frac{\theta(n-\lambda)}{\varphi(n-\mu)} = \frac{p}{q}$ imply $u \leqslant \frac{p}{q}$. For Theorem 2, with $\lambda = \mu = n$, $\frac{1}{\varphi} = \frac{1}{\theta} - \alpha$, and $\frac{\theta}{\varphi} \leqslant \frac{p}{q}$, we have

$$u = 1 - \alpha\theta = \theta\left(\frac{1}{\theta} - \alpha\right) = \frac{\theta}{\varphi} \leqslant \frac{p}{q}.$$

Therefore, for both Theorems 1 and 2, we have $u = \frac{\theta}{\varphi}$ and $uq \leqslant p$. We then can use Hölder's inequality with order p/uq. Follow by Theorem 3,

$$\begin{aligned}\left(\int_0^\infty r^{-\frac{\mu\varphi}{q}} \|T_{\Omega,\alpha} f\|^{\varphi}_{L^q(B(z,r))} dr\right)^{\frac{1}{\varphi}} &\lesssim_\Omega \|f\|^{1-u}_{L^{p,\lambda}_\theta} \left(\int_0^\infty r^{\frac{(n-\lambda)\theta}{p} - \frac{n\theta}{p}} \|M_\Omega f\|^{\theta}_{L^p(B(z,r))} dr\right)^{\frac{u}{\theta}} \\ &\lesssim \|f\|^{1-u}_{L^{p,\lambda}_\theta} \|M_\Omega f\|^{u}_{L^{p,\lambda}_\theta} \lesssim_\Omega \|f\|_{L^{p,\lambda}_\theta}\end{aligned}$$

which completes the proof of both Theorems 1 and 2. □

In fact, the parameter condition

$$\frac{n-\mu}{q} + \frac{1}{\varphi} = \frac{n-\lambda}{p} + \frac{1}{\theta} - \alpha \tag{1}$$

is necessary for the boundedness of $T_{\Omega,\alpha}$ from $L^{p,\lambda}_\theta$ to $L^{q,\mu}_\varphi$.

THEOREM 4. *If $T_{\Omega,\alpha}$ is bounded from $L^{p,\lambda}_\theta$ to $L^{q,\mu}_\varphi$, then identity (1) holds.*

Proof. Define $\delta_t f(x) = f(tx)$ for $t > 0$. Let $f \in L^{p,\lambda}_\theta$ be a non-trivial function. Then, $\delta_t f \in L^{p,\lambda}_\theta$ since

$$\|\delta_t f\|_{L^{p,\lambda}_\theta} = t^{-\frac{n-\lambda}{p} - \frac{1}{\theta}} \|f\|_{L^{p,\lambda}_\theta}.$$

We also have

$$T_{\Omega,\alpha} f(x) = t^\alpha T_{\Omega,\alpha} \delta_t f\left(\frac{x}{t}\right), \quad \|T_{\Omega,\alpha} f\|_{L^{q,\mu}_\varphi} = t^{\alpha + \frac{n-\mu}{q} + \frac{1}{\varphi}} \|T_{\Omega,\alpha} \delta_t f\|_{L^{q,\mu}_\varphi}.$$

By the boundedness of $T_{\Omega,\alpha}$, we obtain

$$\begin{aligned}\|T_{\Omega,\alpha} f\|_{L^{q,\mu}_\varphi} &= t^{\alpha + \frac{n-\mu}{q} + \frac{1}{\varphi}} \|T_{\Omega,\alpha} \delta_t f\|_{L^{q,\mu}_\varphi} \\ &\leqslant C t^{\alpha + \frac{n-\mu}{q} + \frac{1}{\varphi}} \|\delta_t f\|_{L^{p,\lambda}_\theta} \\ &= C t^{\alpha + \frac{n-\mu}{q} + \frac{1}{\varphi} - \frac{n-\lambda}{p} - \frac{1}{\theta}} \|f\|_{L^{p,\lambda}_\theta}.\end{aligned}$$

Since t is an arbitrary positive number and f is a non-trivial function, the identity (1) holds. □

3. Vanishing local Morrey–Adams spaces and operator $T_{\Omega,\alpha}$

Let us rewrite $\|f\|_{L_\theta^{p,\lambda}} = \sup_{x\in\mathbb{R}^n} \|\mathfrak{M}_{p,\lambda}(f;x,\cdot)\|_{L^\theta(0,\infty)}$ where

$$\mathfrak{M}_{p,\lambda}(f;x,r) = r^{-\frac{\lambda}{p}}\|f\|_{L^p(B(x,r))}.$$

Let us first recall the definition of vanishing Morrey spaces [1].

$$V_0L^{p,\lambda} = \{f \in L^{p,\lambda} : \lim_{r\to 0}\sup_{x\in\mathbb{R}^n} \mathfrak{M}_{p,\lambda}(f;x,r) = 0\}, \text{ and}$$

$$V_\infty L^{p,\lambda} = \{f \in L^{p,\lambda} : \lim_{r\to \infty}\sup_{x\in\mathbb{R}^n} \mathfrak{M}_{p,\lambda}(f;x,r) = 0\}.$$

Note that the function $\mathfrak{M}_{p,\lambda}(f;x,\cdot) : (0,\infty) \to [0,\infty)$ is a continuous function. If $f \in L_\theta^{p,\lambda}$, then $\mathfrak{M}_{p,\lambda}(f;x,\cdot) \in L^\theta(0,\infty)$ for any $x \in \mathbb{R}^n$ and as the consequence

$$\lim_{r\to\infty} \mathfrak{M}_{p,\lambda}(f;x,r) = 0.$$

Therefore, if we define $V_\infty L_\theta^{p,\lambda}$ in the similar sense as the definition of $V_\infty L^{p,\lambda}$, then we have $L_\theta^{p,\lambda} = V_\infty L_\theta^{p,\lambda}$.

We now define $V_0 L_\theta^{p,\lambda}$ in the similar sense as the definition of $V_0 L^{p,\lambda}$.

DEFINITION 1. Let $1 \leqslant p,\theta < \infty$, and $\frac{p}{\theta} < \lambda < n + \frac{p}{\theta}$. The vanishing Morrey–Adams space $V_0 L_\theta^{p,\lambda}$ is the set of $f \in L_\theta^{p,\lambda}$ with

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n} \mathfrak{M}_{p,\lambda}(f;x,r) = 0.$$

If $f \in V_0 L_\theta^{p,\lambda}$, then $\mathfrak{M}_{p,\lambda}(f;x,r) \to 0$ as $r \to 0$ for any $x \in \mathbb{R}^n$. By continuity of $\mathfrak{M}_{p,\lambda}(f;x,\cdot)$ and vanishing property for $r \to 0$ and $r \to \infty$, we have $\mathfrak{M}_{p,\lambda}(f;x,\cdot) \in L^\infty(0,\infty)$ for almost every $x \in \mathbb{R}^n$. Therefore, the function f is an element of local Morrey space $L^{p,\lambda}(z)$ with the norm

$$\|f\|_{L^{p,\lambda}(z)} = \sup_{r>0} \mathfrak{M}_{p,\lambda}(f;z,r)$$

for almost every $z \in \mathbb{R}^n$.

Let $1 \leqslant p,\theta < \infty$, $\frac{p}{\theta} < \lambda < n + \frac{p}{\theta}$, and $z \in \mathbb{R}^n$. The local Morrey–Adams space $L_\theta^{p,\lambda}(z)$ is a space of function f where

$$\|f\|_{L_\theta^{p,\lambda}(z)} = \|\mathfrak{M}_{p,\lambda}(f;z,\cdot)\|_{L^\theta(0,\infty)} < \infty.$$

Following from the proof of Theorems 1-3, we immediately have two following corollaries.

COROLLARY 1. *Under the same conditions as in Theorem 3, M_Ω is bounded on $L_\theta^{p,\lambda}(z)$ for almost every $z \in \mathbb{R}^n$.*

COROLLARY 2. *Under the same conditions as in Theorem 1 or Theorem 2, $T_{\Omega,\alpha}$ is bounded from $L_\theta^{p,\lambda}(z)$ to $L_\varphi^{q,\mu}(z)$ for almost every $z \in \mathbb{R}^n$.*

We now define the vanishing Morrey–Adams space in the local sense as follows.

DEFINITION 2. Let $1 \leqslant p, \theta < \infty$, $\frac{p}{\theta} < \lambda < n + \frac{p}{\theta}$, and $z \in \mathbb{R}^n$. The vanishing local Morrey–Adams space $V_0 L_\theta^{p,\lambda}(z)$ is the set of $f \in L_\theta^{p,\lambda}(z)$ with

$$\lim_{r\to 0} \mathfrak{M}_{p,\lambda}(f; z, r) = 0.$$

With the same argument as vanishing Morrey–Adams spaces, we note that for any $f \in V_0 L_\theta^{p,\lambda}(z)$, then $f \in L^{p,\lambda}(z)$. Hence, we can refine Corollary 1 and Corollary 2 as follows.

THEOREM 5. *Under the same conditions as in Theorem 3 and $\lambda < n$, for almost every $z \in \mathbb{R}^n$, $M_\Omega(V_0 L_\theta^{p,\lambda}(z)) \subset V_0 L_\theta^{p,\lambda}(z)$.*

THEOREM 6. *Under the same conditions as in Theorem 1 and $\lambda \leqslant \mu < n$, for almost every $z \in \mathbb{R}^n$, $T_{\Omega,\alpha}(V_0 L_\theta^{p,\lambda}(z)) \subset V_0 L_\varphi^{q,\mu}(z)$.*

REMARK 1. In Theorems 1–3, Corollary 1, and Corollary 2, we can discuss the case of $n \leqslant \mu \leqslant \lambda$ (for example $f = \chi_{B(0,1)}$). However, since $f \in V_0 L_\theta^{p,\lambda}(z)$ implies $f \in L^{p,\lambda}(z)$, we should add the conditions $\lambda < n$ for Theorems 5 and 6.

By Lemma 1 and Hölder's inequality, we note that for any $z \in \mathbb{R}^n$ and $r > 0$,

$$\mathfrak{M}_{q,\mu}(T_{\Omega,\alpha} f; z, r) \lesssim_\Omega \|f\|_{L_\theta^{p,\lambda}}^{1-\frac{\theta}{\varphi}} \left(\mathfrak{M}_{p,\lambda}(M_\Omega f; z, r)\right)^{\frac{\theta}{\varphi}}.$$

Therefore, Theorem 6 is immediately proven once we verify Theorem 5. Let us prove Theorem 5.

Proof. Let $f = f_1 + f_2$ with $f_1 = f\chi_{B(z,2r)}$. By boundedness of M_Ω on L^p,

$$\mathfrak{M}_{p,\lambda}(M_\Omega f_1; z, r) \lesssim_\Omega \mathfrak{M}_{p,\lambda}(f; z, 2r).$$

Since $f \in V_0 L_\theta^{p,\lambda}(z)$, we have $\mathfrak{M}_{p,\lambda}(M_\Omega f_1; z, r) \to 0$ as $r \to 0$.

For $x \in B(z,r)$, by substituting $t = 2\tau$, we have

$$\begin{aligned} M_\Omega f_2(x) &\lesssim \int_{B^c(z,2r)} |\Omega(x-y)||f(y)| \int_{|x-y|}^\infty \tau^{-n-1} d\tau dy \\ &\lesssim_\Omega \int_r^\infty \tau^{-\frac{n}{p}-1} \|f\|_{L^p(B(z,2\tau))} d\tau \\ &\lesssim \int_r^\infty t^{-\frac{n-\lambda}{p}-1} \mathfrak{M}_{p,\lambda}(f; z, t) dt \end{aligned}$$

Since $f \in V_0 L_\theta^{p,\lambda}(z)$, we have $f \in L^{p,\lambda}(z)$, which means $\lambda < n$. Since $f \in V_0 L_\theta^{p,\lambda}(z)$, we can find δ such that for any $r < \delta$, $\mathfrak{M}_{p,\lambda}(f; z, r) < \varepsilon$. Hence, $\mathfrak{M}_{p,\lambda}(M_\Omega f_2; z, r)$ is

$$\begin{aligned} &\lesssim_\Omega r^{\frac{n-\lambda}{p}} \left[\int_r^\delta t^{-\frac{n-\lambda}{p}-1} \mathfrak{M}_{p,\lambda}(f; z, t) dt + \int_\delta^\infty t^{-\frac{n-\lambda}{p}-1} \mathfrak{M}_{p,\lambda}(f; z, t) dt\right] \\ &\leqslant \varepsilon + \|f\|_{L^{p,\lambda}(z)} \left(\frac{r}{\delta}\right)^{\frac{n-\lambda}{p}}. \end{aligned}$$

Hence, as r goes to 0, $\mathfrak{M}_{p,\lambda}(M_\Omega f_2; z, r)$ goes to 0. Proof of Theorem 5 completes by applying the sublinearity of M_Ω. $\square$

4. Beyond Adams' inequality

We have proved the boundedness of $T_{\Omega,\alpha}$ from $L^{p,\lambda}_{\theta}$ to $L^{q,\mu}_{\varphi}$ for $\lambda \leqslant \mu < n$ and $n < \mu \leqslant \lambda$ in Theorem 1. It is natural for us to consider the case of $\mu < \lambda < n$ or $n < \lambda < \mu$. In [6], Salim et al investigate $T_{\Omega,\alpha}$ from $L^{p,\lambda}$ to $L^{q,\mu}$ for the case of $\mu < \lambda < n$ and called the results as beyond Adams' inequality. As in [6], in this investigation, we shall restrict the domain of $T_{\Omega,\alpha}$ onto collection of radial functions f in $L^{p,\lambda}_{\theta}(0)$, and prove that the maps $T_{\Omega,\alpha} f$ is in $L^{q,\mu}_{\varphi}(0)$ for the case of $\mu < \lambda < n-1$.

For the case $\mu < \lambda < n-1$, we can't use the same method as in the proof of Theorem 1, since $uq > p$. In order to be able to use Hölder's inequality, we need a new pointwise estimation of $T_{\Omega,\alpha} f$ as follows.

LEMMA 2. *Let $f \in L^{p,\lambda}_{\theta}(0)$ be a radial function. Let $\Omega \in L^s(S^{n-1})$ for $s \geqslant p'$, $\lambda < \gamma < n-1$, and $0 < \alpha < \min\{n, \frac{n-\gamma}{p} + \frac{1}{\theta}\}$,*

$$|T_{\Omega,\alpha} f(x)| \lesssim_{\Omega} M_{\Omega} f(x)^{\nu} |x|^{\frac{(\lambda-\gamma)(1-\nu)}{p}} \|f\|^{1-\nu}_{L^{p,\lambda}_{\theta}(0)}$$

for almost every $x \in \mathbb{R}^n$ where $\nu = 1 - \frac{\alpha p \theta}{(n-\gamma)\theta + p}$.

Proof. Let f be a non-trivial function. Fix $x \in \mathbb{R}^n$ and let $R > 0$. We decompose f as $f_1 + f_2$ with $f_1 = f\chi_{B(x,R)}$. As in the proof of Lemma 1, we have

$$|T_{\Omega,\alpha} f_1(x)| \lesssim R^{\alpha} M_{\Omega} f(x)$$

and

$$|T_{\Omega,\alpha} f_2(x)| \lesssim \|\Omega\|_{L^s(S^{n-1})} \int_R^{\infty} t^{\alpha-\frac{n}{p}-1} \|f\|_{L^p(B(x,t))} dt.$$

We then use Hölder's inequality, and we obtain

$$|T_{\Omega,\alpha} f_2(x)| \lesssim \|\Omega\|_{L^s(S^{n-1})} R^{\alpha-\frac{n-\gamma}{p}-\frac{1}{\theta}} \left(\int_R^{\infty} t^{\frac{-\gamma\theta}{p}} \|f\|^{\theta}_{L^p(B(x,t))} dt \right)^{\frac{1}{\theta}}$$

For $R \geqslant \frac{|x|}{2}$, we have $|x| < 2t$ and $|x| + t < 3t$,

$$\begin{aligned}
\int_R^{\infty} t^{\frac{-\gamma\theta}{p}} \|f\|^{\theta}_{L^p(B(x,t))} dt &\lesssim |x|^{\frac{(\lambda-\gamma)\theta}{p}} \int_{\frac{|x|}{2}}^{\infty} (|x|+t)^{\frac{-\lambda\theta}{p}} \|f\|^{\theta}_{L^p(B(0,|x|+t))} dt \\
&\leqslant |x|^{\frac{(\lambda-\gamma)\theta}{p}} \|f\|^{\theta}_{L^{p,\lambda}_{\theta}(0)}.
\end{aligned}$$

For $R < \frac{|x|}{2}$, we write

$$\int_R^{\infty} t^{\frac{-\gamma\theta}{p}} \|f\|^{\theta}_{L^p(B(x,t))} dt = \int_R^{\frac{|x|}{2}} t^{\frac{-\gamma\theta}{p}} \|f\|^{\theta}_{L^p(B(x,t))} dt + \int_{\frac{|x|}{2}}^{\infty} t^{\frac{-\gamma\theta}{p}} \|f\|^{\theta}_{L^p(B(x,t))} dt$$

which by the estimation for the case of $R \geqslant \frac{|x|}{2}$, we have

$$\int_R^{\infty} t^{\frac{-\gamma\theta}{p}} \|f\|^{\theta}_{L^p(B(x,t))} dt \lesssim \int_R^{\frac{|x|}{2}} t^{\frac{-\gamma\theta}{p}} \|f\|^{\theta}_{L^p(B(x,t))} dt + |x|^{\frac{(\lambda-\gamma)\theta}{p}} \|f\|^{\theta}_{L^{p,\lambda}_{\theta}(0)}$$

Since f is a radial function, and $t < |x|$, by [8, Lemma 1.1.], we have

$$\|f\|^p_{L^p(B(x,t))} \lesssim t^{n-1} \int_{|x|-t}^{|x|+t} |f_0(s)|^p ds$$

where $f_0(|x|) = f(x)$. Hence, we obtain

$$\int_R^\infty t^{\frac{-\gamma\theta}{p}} \|f\|^\theta_{L^p(B(x,t))} dt \lesssim \int_R^{\frac{|x|}{2}} t^{\frac{(n-\gamma-1)\theta}{p}} \left(\int_{|x|-t}^{|x|+t} |f_0(s)|^p ds \right)^{\frac{\theta}{p}} dt + |x|^{\frac{(\lambda-\gamma)\theta}{p}} \|f\|^\theta_{L^{p,\lambda}_\theta(0)}.$$

Since $t < |x|$ and $n-\gamma-1 > 0$, then $t^{\frac{(n-\gamma-1)\theta}{p}} < |x|^{\frac{(n-\gamma-1)\theta}{p}}$. We also have $\frac{|x|}{2} < s < \frac{3|x|}{2}$.

$$\begin{aligned} \int_R^\infty t^{\frac{-\gamma\theta}{p}} \|f\|^\theta_{L^p(B(x,t))} dt &\lesssim |x|^{-\frac{\gamma\theta}{p}} \int_R^{\frac{|x|}{2}} \left(\int_{|x|-t}^{|x|+t} |f_0(s)|^p s^{n-1} ds \right)^{\frac{\theta}{p}} dt + |x|^{\frac{(\lambda-\gamma)\theta}{p}} \|f\|^\theta_{L^{p,\lambda}_\theta(0)} \\ &\lesssim |x|^{\frac{(\lambda-\gamma)\theta}{p}} \|f\|^\theta_{L^{p,\lambda}_\theta(0)}. \end{aligned}$$

Therefore, we can conclude that

$$|T_{\Omega,\alpha} f(x)| \lesssim R^\alpha M_\Omega f(x) + R^{\alpha - \frac{n-\gamma}{p} - \frac{1}{\theta}} \|\Omega\|_{L^s(S^{n-1})} |x|^{\frac{(\lambda-\gamma)\theta}{p}} \|f\|^\theta_{L^{p,\lambda}_\theta(0)}$$

and complete the proof by minimizing the right hand side over $R > 0$. □

By the new pointwise estimation, we obtain the beyond Adams' inequality as follows.

THEOREM 7. *Let $f \in L^{p,\lambda}_0$ be a radial function. Let $\Omega \in L^s(S^{n-1})$ for $s \geqslant p'$. Suppose that $\mu < \lambda < \gamma < n-1$, $0 < \alpha < \min\{n, \frac{n-\gamma}{p} + \frac{1}{\theta}\}$,*

$$\frac{n-\mu}{q} + \frac{1}{\varphi} = \frac{n-\lambda}{p} + \frac{1}{\theta} - \alpha, \quad \frac{n-\mu}{q} = \frac{\gamma-\lambda}{p} + \frac{(n-\gamma)\theta}{p\varphi},$$

$$\frac{\theta}{\varphi} \leqslant \frac{\gamma-\lambda}{\gamma-\mu}, \quad \frac{n\varphi}{q} - \frac{n\theta}{p} > \frac{(\gamma-\lambda)(\varphi-\theta)}{p}.$$

Then,

$$\|T_{\Omega,\alpha} f\|_{L^{q,\mu}_\varphi(0)} \lesssim_\Omega \|f\|_{L^{p,\lambda}_\theta(0)}.$$

REMARK 2. Let $n = 6$, $\mu = 2$, $\lambda = 3$, $\gamma = 4$, $\theta = 1$, $\varphi = 5$, $p = \frac{17}{16}$, $q = \frac{85}{28}$, and $\alpha = \frac{196}{85}$. Then all the parameter conditions in Theorem 7 are fulfilled.

Proof. By $\frac{n-\mu}{q} + \frac{1}{\varphi} = \frac{n-\lambda}{p} + \frac{1}{\theta} - \alpha$ and $\frac{n-\mu}{q} = \frac{\gamma-\lambda}{p} + \frac{(n-\gamma)\theta}{p\varphi}$, we can rewrite v in Lemma 2 as

$$v = 1 - \frac{\alpha}{\frac{n-\gamma}{p} + \frac{1}{\theta}} = \frac{\frac{n-\mu}{q} + \frac{\lambda-\gamma}{p} + \frac{1}{\varphi}}{\frac{n-\gamma}{p} + \frac{1}{\theta}} = \frac{\theta}{\varphi} \left(\frac{\frac{n-\mu}{q} + \frac{\lambda-\gamma}{p} + \frac{1}{\varphi}}{\frac{(n-\gamma)\theta}{p\varphi} + \frac{1}{\varphi}} \right) = \frac{\theta}{\varphi}.$$

From $\frac{n-\mu}{q} = \frac{\gamma-\lambda}{p} + \frac{(n-\gamma)\theta}{p\varphi}$, we have

$$p = \frac{(n-\gamma)\theta q}{(n-\mu)\varphi} + \frac{(\gamma-\lambda)q}{n-\mu} = \left(\frac{n-\gamma}{n-\mu} + \frac{(\gamma-\lambda)\varphi}{(n-\mu)\theta}\right) vq.$$

Since $\frac{\theta}{\varphi} \leqslant \frac{\gamma-\lambda}{\gamma-\mu}$, we note that

$$\frac{n-\gamma}{n-\mu} + \frac{(\gamma-\lambda)\varphi}{(n-\mu)\theta} = 1 + \frac{1}{n-\mu}\left(\mu-\gamma+\frac{(\gamma-\lambda)\varphi}{\theta}\right) \geqslant 1.$$

Hence, $p \geqslant vq$ and we can use Hölder's inequality with order $t = \frac{p}{vq}$. By Lemma 2,

$$\begin{aligned}
&\left(\int_0^\infty r^{-\frac{\mu\varphi}{q}} \|T_{\Omega,\alpha} f\|^{\varphi}_{L^q(B(0,r))} dr\right)^{\frac{1}{\varphi}} \\
&\lesssim_\Omega \|f\|^{1-v}_{L^{p,\lambda}_\theta(0)} \left(\int_0^\infty r^{-\frac{\mu\varphi}{q}} \left\|(M_\Omega f)^v |\cdot|^{\frac{(\lambda-\gamma)(1-v)}{p}}\right\|^{\varphi}_{L^q(B(0,r))} dr\right)^{\frac{1}{\varphi}} \\
&\lesssim \|f\|^{1-v}_{L^{p,\lambda}_\theta(0)} \left(\int_0^\infty r^{-\frac{\mu\varphi}{q}} \|M_\Omega f\|^{\theta}_{L^p(B(0,r))} \left\||\cdot|^{\frac{(\lambda-\gamma)(1-v)qt'}{p}}\right\|^{\frac{\varphi}{qt'}}_{L^1(B(0,r))} dr\right)^{\frac{v}{\theta}} \qquad (2)
\end{aligned}$$

Since $\frac{n\varphi}{q} - \frac{n\theta}{p} > \frac{(\gamma-\lambda)(\varphi-\theta)}{p}$, and $\frac{n\varphi}{qt'} = \frac{n\varphi}{q} - \frac{n\theta}{p}$, we have

$$\frac{(\lambda-\gamma)(1-v)qt'}{p} + n = \frac{qt'}{\varphi}\left(\frac{(\lambda-\gamma)(\varphi-\theta)}{p} + \frac{n\varphi}{q} - \frac{n\theta}{p}\right) > 0.$$

Therefore

$$\left\||\cdot|^{\frac{(\lambda-\gamma)(1-v)qt'}{p}}\right\|^{\frac{\varphi}{qt'}}_{L^1(B(0,r))} \lesssim \left(\int_0^r R^{\frac{(\lambda-\gamma)(1-v)qt'}{p}+n-1} dR\right)^{\frac{\varphi}{qt'}} \lesssim r^{\frac{(\lambda-\gamma)(\varphi-\theta)}{p}+\frac{n\varphi}{q}-\frac{n\theta}{p}}.$$

Since $\frac{(n-\mu)\varphi}{q} = \frac{(\gamma-\lambda)\varphi}{p} + \frac{(n-\gamma)\theta}{p}$, we can confirm that

$$\frac{-\mu\varphi}{q} + \frac{(\lambda-\gamma)(\varphi-\theta)}{p} + \frac{n\varphi}{q} - \frac{n\theta}{p} = -\frac{\lambda\theta}{p}.$$

By (2), and Theorem 3, we can complete the proof as follows.

$$\begin{aligned}
\left(\int_0^\infty r^{-\frac{\mu\varphi}{q}} \|T_{\Omega,\alpha} f\|^{\varphi}_{L^q(B(0,r))} dr\right)^{\frac{1}{\varphi}} &\lesssim \|f\|^{1-v}_{L^{p,\lambda}_\theta(0)} \left(\int_0^\infty r^{-\frac{\lambda\theta}{p}} \|M_\Omega f\|^{\theta}_{L^p(B(0,r))} dr\right)^{\frac{v}{\theta}} \\
&\lesssim \|f\|_{L^{p,\lambda}_\theta(0)}. \quad \square
\end{aligned}$$

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