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A MATRIX INEQUALITY FOR UNITARILY INVARIANT NORMS

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Abstract. In this paper, we present an inequality of matrix norms, which is a generalization of the inequality shown by Zou [Linear Algebra Appl. 562, 154–162].

1. Introduction

As usual, the set of $n\times n$ complex matrices is denoted by M_n. The identity matrix of M_n is denoted by I. For $A\in M_n$, $s_i(A)$ is the $i-th$ largest singular value of A and $s(A)=(s_1(A),\cdots,s_n(A))$. $\lambda_i(A)$ is the $i-th$ largest eigenvalue of A and $\lambda(A)=(\lambda_1(A),\cdots,\lambda_n(A))$. $s_i(A)=\lambda_i(|A|)$ for $|A|=(A^*A)^{\frac{1}{2}}$, where A^* is the conjugate transpose of A. $A\geqslant B$ means that $A-B$ is positive semidefinite. The direct sum of A and B is denoted by $A\oplus B$. The block matrix is presented by $[X_{ij}]$ which X_{ij} is the $i,j-th$ block.

Now we introduce the definition of majorization. Given a real vector $x=(x_1,x_2,\cdots,x_n)\in R^n$, we rearrange its components as $x_{[1]}\geqslant x_{[2]}\geqslant\cdots\geqslant x_{[n]}$.

For $x=(x_1,x_2,\cdots,x_n)$, $y=(y_1,y_2,\cdots,y_n)\in R^n$, if

$$\sum_{i=1}^{k}x_{[i]}\leqslant\sum_{i=1}^{k}y_{[i]},k=1,2,\cdots,n,$$

then we say that x is weakly majorized by y and denote $x\prec_w y$. If $x\prec_w y$ and $\sum_{i=1}^{n}x_i=\sum_{i=1}^{n}y_i$ hold, then we say that x is majorized by y and denote $x\prec y$.

Let $\|\bullet\|$ be any unitary invariant norm. Dou to Ky fan's result (see [2]), it's known that $\|X\|\leqslant\|Y\|$ if and only if $s(X)\prec_w s(Y)$ for $X,Y\in M_n$. Let f be convex and increasing function on $[0,+\infty)$. If $x\prec_w y$, then

$$f(x)\prec_w f(y). \tag{1}$$

For more detail (see [2]).

For $a\in[0,1]$, in [1], Zou proved

$$\left\|\left|AXB^*\right|^{2r}\right\|\leqslant\||aA^*AX+(1-a)XB^*B|^{rp}\|^{\frac{1}{p}}\times\||(1-a)A^*AX+aXB^*B|^{rq}\|^{\frac{1}{q}} \tag{2}$$

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for $r \geqslant \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$. Zou explained that the insertion of X is no idle generalization, a judicious choice can lead to powerful perturbation theorems. Our aim here is to obtain a stronger version of inequality (2) in the same spirit.

In this paper, we present a generalization of inequality (2).

2. Main result

In this section, we first show some Lemmas used in our proof.

LEMMA 1. [3] *Let $A, B \in M_n$ be Hermitian matrices. Then*

$$\lambda(A+B) \prec \lambda(A) + \lambda(B).$$

LEMMA 2. [4] *Let $H = [A_{ij}] \in M_{sn}$ $(s \geqslant 2)$ be positive semidefinite matrix with $A_{ij} = -A_{ij}^*$ $(i \neq j, i, j = 1, 2, \cdots, s)$. Then*

$$\lambda(H) \prec \lambda\left(\sum_{i=1}^{s} A_{ii} \oplus 0\right). \tag{3}$$

THEOREM 1. *Let $A_i, B_i, X \in M_n$, $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $r \geqslant \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$, $a \in [0,1]$. Then*

$$\left\|\left|\sum_{i=1}^{n} A_i X B_i^*\right|^{2r}\right\| \leqslant \left\|\left|\sum_{i=1}^{n} a A_i^* A_i X + \sum_{i=1}^{n} (1-a) X B_i^* B_i\right|^{rp}\right\|^{\frac{1}{p}}$$
$$\times \left\|\left|\sum_{i=1}^{n} (1-a) A_i^* A_i X + \sum_{i=1}^{n} a X B_i^* B_i\right|^{rq}\right\|^{\frac{1}{q}}$$

holds under the conditions that $A_i^* A_j = -A_j^* A_i$, $B_i^* B_j = -B_j^* B_i$ $(i \neq j)$ is skew-Hermitian.

Proof. We first discuss the case when X is positive semidefinite. By inequality (2), we obtain

$$\begin{aligned}
&\left\|\left|\sum_{i=1}^{n} A_i X B_i^* \oplus 0\right|^{2r}\right\| \\
&= \left\|\left|\begin{bmatrix} A_1 X^{\frac{1}{2}} & A_2 X^{\frac{1}{2}} & \cdots & A_n X^{\frac{1}{2}} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} B_1 X^{\frac{1}{2}} & B_2 X^{\frac{1}{2}} & \cdots & B_n X^{\frac{1}{2}} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}^*\right|^{2r}\right\| \\
&\leqslant \left\|\left(a\left[X^{\frac{1}{2}} A_i^* A_j X^{\frac{1}{2}}\right] + (1-a)\left[X^{\frac{1}{2}} B_i^* B_j X^{\frac{1}{2}}\right]\right)^{rp}\right\|^{\frac{1}{p}}
\end{aligned}$$

$$\times \left\| \left((1-a)\left[X^{\frac{1}{2}}A_i^*A_jX^{\frac{1}{2}}\right] + a\left[X^{\frac{1}{2}}B_i^*B_jX^{\frac{1}{2}}\right]\right)^{rq} \right\|^{\frac{1}{q}}$$

$$\leqslant \left\| \left(a\sum_{i=1}^{n} X^{\frac{1}{2}}A_i^*A_iX^{\frac{1}{2}} + (1-a)\sum_{i=1}^{n} X^{\frac{1}{2}}B_i^*B_iX^{\frac{1}{2}}\right)^{rp} \oplus 0 \right\|^{\frac{1}{p}}$$

$$\times \left\| \left((1-a)\sum_{i=1}^{n} X^{\frac{1}{2}}A_i^*A_iX^{\frac{1}{2}} + a\sum_{i=1}^{n} X^{\frac{1}{2}}B_i^*B_iX^{\frac{1}{2}}\right)^{rq} \oplus 0 \right\|^{\frac{1}{q}}$$

Hence,

$$\left\| \left| \sum_{i=1}^{n} A_iXB_i^* \right|^{2r} \right\| \leqslant \left\| \left(\sum_{i=1}^{n} aX^{\frac{1}{2}}A_i^*A_iX^{\frac{1}{2}} + \sum_{i=1}^{n} (1-a)X^{\frac{1}{2}}B_i^*B_iX^{\frac{1}{2}}\right)^{rp} \right\|^{\frac{1}{p}}$$

$$\times \left\| \left(\sum_{i=1}^{n} (1-a)X^{\frac{1}{2}}A_i^*A_iX^{\frac{1}{2}} + \sum_{i=1}^{n} aX^{\frac{1}{2}}B_i^*B_iX^{\frac{1}{2}}\right)^{rq} \right\|^{\frac{1}{q}}.$$

By Proposition 9.1.2 in [2] if a product AB is Hermitian, then $\|AB\| \leqslant \|\mathrm{Re}\, BA\|$. Using this we obtain

$$\sum_{j=1}^{k} s_j \left(\sum_{i=1}^{n} aX^{\frac{1}{2}}A_i^*A_iX^{\frac{1}{2}} + \sum_{i=1}^{n} (1-a)X^{\frac{1}{2}}B_i^*B_iX^{\frac{1}{2}} \right)$$

$$\leqslant \sum_{j=1}^{k} s_j \left(\mathrm{Re} \sum_{i=1}^{n} aA_i^*A_iX + \sum_{i=1}^{n} (1-a)B_i^*B_iX \right)$$

$$\leqslant \sum_{j=1}^{k} s_j \left(\left| \sum_{i=1}^{n} aA_i^*A_iX + \sum_{i=1}^{n} (1-a)XB_i^*B_i \right| \right)$$

Since $r \geqslant \frac{1}{p}$, by inequality (1), we obtain

$$\left\| \left(\sum_{i=1}^{n} aX^{\frac{1}{2}}A_i^*A_iX^{\frac{1}{2}} + \sum_{i=1}^{n} (1-a)X^{\frac{1}{2}}B_i^*B_iX^{\frac{1}{2}} \right)^{rp} \right\| \leqslant \left\| \left| \sum_{i=1}^{n} aA_i^*A_iX + \sum_{i=1}^{n} (1-a)XB_i^*B_i \right|^{rp} \right\|.$$

Hence,

$$\left\| \left| \sum_{i=1}^{n} A_iXB_i^* \right|^{2r} \right\| \leqslant \left\| \left| \sum_{i=1}^{n} aA_i^*A_iX + \sum_{i=1}^{n} (1-a)XB_i^*B_i \right|^{rp} \right\|^{\frac{1}{p}}$$

$$\times \left\| \left| \sum_{i=1}^{n} (1-a)A_i^*A_iX + \sum_{i=1}^{n} aXB_i^*B_i \right|^{rq} \right\|^{\frac{1}{q}}.$$

Next we consider the case when X is any matrix. By the singular value decomposition we know there exist unitary matrices U,V such that $X=UDV^*$, then

$$\begin{aligned}
&\left\|\left|\sum_{i=1}^{n} A_i X B_i^*\right|^{2r}\right\| \\
&= \left\|\left|\sum_{i=1}^{n} A_i UD\left(B_i V\right)^*\right|^{2r}\right\| \\
&\leqslant \left\|\left|\sum_{i=1}^{n} aU^*A_i^*A_iUD + \sum_{i=1}^{n}(1-a)DV^*B_i^*B_iV\right|^{rp}\right\|^{\frac{1}{p}} \\
&\quad \times \left\|\left|\sum_{i=1}^{n}(1-a)U^*A_i^*A_iUD + \sum_{i=1}^{n} aDV^*B_i^*B_iV\right|^{rq}\right\|^{\frac{1}{q}} \\
&= \left\|\left|\sum_{i=1}^{n} aA_i^*A_iUDV^* + \sum_{i=1}^{n}(1-a)UDV^*B_i^*B_i\right|^{rp}\right\|^{\frac{1}{p}} \\
&\quad \times \left\|\left|\sum_{i=1}^{n}(1-a)A_i^*A_iUDV^* + \sum_{i=1}^{n} aUDV^*B_i^*B_i\right|^{rq}\right\|^{\frac{1}{q}} \\
&\leqslant \left\|\left|\sum_{i=1}^{n} aA_i^*A_iX + \sum_{i=1}^{n}(1-a)XB_i^*B_i\right|^{rp}\right\|^{\frac{1}{p}} \times \left\|\left|\sum_{i=1}^{n}(1-a)A_i^*A_iX + \sum_{i=1}^{n} aXB_i^*B_i\right|^{rq}\right\|^{\frac{1}{q}}. \qquad \square
\end{aligned}$$

REMARK 1. Let $n=2$, $A_1=B_1=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $A_2=B_2=\begin{bmatrix}1&1\\0&0\end{bmatrix}$.Then

$$\begin{bmatrix}1&1\\0&0\end{bmatrix} = A_1^*A_2 \neq A_2^*A_1 = \begin{bmatrix}1&0\\1&0\end{bmatrix},\ B_1^*B_2 \neq B_2^*B_1$$

and

$$A_1A_1^*+A_2A_2^* = \begin{bmatrix}3&0\\0&0\end{bmatrix},\ A_1^*A_1+A_2^*A_2 = \begin{bmatrix}2&1\\1&1\end{bmatrix}.$$

A trivial verification shows that $\lambda_1\left(A_1A_1^*+A_2A_2^*\right) > \lambda_1\left(A_1^*A_1+A_2^*A_2\right)$.

REMARK 2. For $n=2$, let $A_1=B_1=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $A_2=B_2=\begin{bmatrix}1&1\\0&0\end{bmatrix}$, $X_1=I$, $X_2=\begin{bmatrix}0&0\\0&1\end{bmatrix}$. Then

$$\lambda_1\left(A_1X_1A_1^*+A_2X_2A_2^*\right) > \lambda_1\left(A_1^*A_1+\frac{A_2^*A_2X_2+X_2A_2^*A_2}{2}\right). \tag{4}$$

Inequality (4) implies that

$$\left\|\left|\sum_{i=1}^{n} A_i X_i B_i^*\right|^{2r}\right\| \leqslant \left\|\left|\sum_{i=1}^{n} aA_i^*A_iX_i + \sum_{i=1}^{n}(1-a)X_iB_i^*B_i\right|^{rp}\right\|^{\frac{1}{p}} \times \left\|\left|\sum_{i=1}^{n}(1-a)A_i^*A_iX_i + \sum_{i=1}^{n} aX_iB_i^*B_i\right|^{rq}\right\|^{\frac{1}{q}}$$

isn't always true without $X_i = X_j \ (i \neq j)$.

COROLLARY 1. *Let* $A,B,X \in M_n$, $p>1$, $q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, $r \geqslant \max\left\{\frac{1}{p},\frac{1}{q}\right\}$, $a\in[0,1]$. *Then*

$$\left\||AXB^*|^{2r}\right\| \leqslant \||aA^*AX+(1-a)XB^*B|^{rp}\|^{\frac{1}{p}} \times \||(1-a)A^*AX+aXB^*B|^{rq}\|^{\frac{1}{q}}.$$

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