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# BOUNDARY SCHWARZ LEMMA FOR HARMONIC AND PLURIHARMONIC MAPPINGS IN THE UNIT BALL

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*Abstract.* In this paper, we consider pluriharmonic and harmonic mappings $f$ defined on the unit ball $\mathbb{B}^n$, $n \geqslant 2$, differentiable at a point $a$ on the boundary of $\mathbb{B}^n$, and $f(\mathbb{B})$ satisfies some convexity hypothesis at $f(a)$. For those mappings $f$, we obtain versions of its boundary Schwarz lemma and the sharp estimate of the eigenvalue related to its Jacobian at $a$. In particular, Theorem 1.4 below, solves the corresponding extremal problems concerning the magnitude of the radial derivative of $f$ at the direction $a$ and improves the main estimates given in [7] and [12]. Moreover, we partly generalized the corresponding results given in [8] and [24].

## 1. Introduction

Let $\mathbb{C}$ be the complex plane, and $\mathbb{C}^n$ the complex Euclidean $n$-space. In this paper, we write a point $z \in \mathbb{C}^n$ as a column vector of the following $n \times 1$ matrix form

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

For two points $z = (z_1, \cdots, z_n)^T$ and $w = (w_1, \cdots, w_n)^T$ in $\mathbb{C}^n$, where the symbol $T$ stands for the transposition, the standard Hermitian scalar product on $\mathbb{C}^n$ is given by $\langle z, w \rangle = \sum_{k=1}^{n} z_k \overline{w_k}$ and the Euclidean norm of $z$ is given by $|z| = \langle z, z \rangle^{\frac{1}{2}}$.

Throughout this paper, we use $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$ to denote the unit ball of the $n$-dimensional complex plane $\mathbb{C}^n$ (the "*complex unit ball*"), and let $\mathbb{S}^{n-1}$ be the boundary of $\mathbb{B}^n$. Moreover, when $n = 1$, the complex plane case, we use $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to denote the unit disk and let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle.

For each $z = x + iy \in \mathbb{C}^n$, its "*real version*" is given by $z' = (x, y)^T \in \mathbb{R}^{2n}$. Note that in the literature, the same notation for "*real unit ball*" is used frequently. To

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avoid the possible confusion, in this paper we use $\mathbf{B}^n$ to denote the unit ball of the $n$-dimensional real Euclidean space, i.e., $\mathbf{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ , and let $\mathbf{S}^{n-1}$ be the boundary of $\mathbf{B}^n$.

The conjugate transpose (or Hermitian transpose) of an $m \times n$ matrix $A_{m,n}$ with complex entries is the $n \times m$ matrix obtained from $A$ by taking the transpose and then taking the complex conjugate of each entry. It is often denoted as $A^H$ (or $A^*$). For real matrices, the conjugate transpose is just the transpose, that is $A^H = A^T$. Furthermore, since $A_{m,n}$ is a linear operator from $\mathbb{R}^m$ into $\mathbb{R}^n$, we can extend the above definition and define the transpose $A^T$ by $\langle Ax, y\rangle = \langle x, A^T y\rangle$, where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. Here $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product on the corresponding spaces.

### Operator norm and the tangent space

For an $n \times n$ complex matrix $A$, we recall the operator norm

$$\|A\| = \sup_{z \neq 0} \frac{\|Az\|}{\|z\|} = \max\{\|A\theta\| : \theta \in \mathbb{S}^{n-1}\}. \tag{1.1}$$

For $p \in \mathbb{R}^n$, we define the tangent space at $p$ as $T_p(\mathbb{R}^n) = \{v_p = (v, p) : v \in \mathbb{R}^n\}$, and we frequently write simply $v \in T_p(\mathbb{R}^n)$ instead of $v_p$, where $v$ denotes a vector with initial point $p$. The tangent space of a manifold can be considered as the generalization of notation of vectors from affine spaces to general manifolds. For example, we can define the tangent space of a sphere, $T_p(\mathbf{S}^{n-1}) = \{v \in \mathbb{R}^n : \langle p, v\rangle = 0\}$, etc. For each $z_0 = x + iy \in \mathbb{S}^{n-1}$, let $z_0' = (x, y)^T \in \mathbf{S}^{2n-1}$ be its real version. Then the tangent space $T_{z_0'}\left(\mathbf{S}^{2n-1}\right)$ is defined by

$$T_{z_0'}\left(\mathbf{S}^{2n-1}\right) = \{\beta \in \mathbb{R}^{2n} : {z_o'}^T \beta = 0\}. \tag{1.2}$$

For the reader's convenience, note here that ${z_o'}^T \beta$ is the product of the row matrix ${z_o'}^T$ and the column matrix $\beta$.

### Schwarz lemma and Schwarz-pick lemma

The classical Schwarz lemma states that an analytic function $f$ maps $\mathbb{D}$ into itself, with $f(0) = 0$, satisfies $|f(z)| \leqslant |z|$ in $\mathbb{D}$.

The classical Schwarz-Pick lemma states that an analytic function $f$ of $\mathbb{D}$ into itself is a contraction in the hyperbolic metric, i.e. for $z, w \in \mathbb{D}$,

$$\frac{|f(z) - f(w)|}{|1 - f(z)\overline{f(w)}|} \leqslant \frac{|z - w|}{|1 - z\overline{w}|}.$$

Letting $z \to w$ on the above inequality, one can get the following classical form:

$$|f'(z)| \leqslant \frac{1 - |f(z)|^2}{1 - |z|^2}. \tag{1.3}$$

These two classical results are well-known and had been extended by many mathematicians.

However, for harmonic functions, people can not expect the classical form of the Schwarz-pick lemma (1.3). Only for some special cases, for example harmonic functions from $\mathbb{D}$ into the interval $(0,1)$, can have such similar form (see [8]). Establishing various versions of the Schwarz lemma and Schwarz-pick lemma have attracted the attention of many mathematicians (see [7]–[12], [15]–[17], [26]–[33]).

**Boundary Schwarz lemma**

The classical Schwarz lemma at the boundary is as follows:

THEOREM A. ([5, Page 42]) *Suppose $f:\mathbb{D}\to\mathbb{D}$ is an analytic function with $f(0)=0$, and, further, $f$ is analytic at $z=1$ with $f(1)=1$. Then, the following two conclusions hold:*

1. $f'(1)\geqslant 1$.
2. $f'(1)=1$ *if and only if* $f(z)\equiv z$.

Theorem A has the following generalization.

THEOREM B. ([17, Theorem 1.1′]) *Suppose $f:\mathbb{D}\to\mathbb{D}$ is an analytic function with $f(0)=0$, and, further, $f$ is analytic at $z=\alpha\in\mathbb{T}$ with $f(\alpha)=\beta\in\mathbb{T}$. Then, the following two conclusions hold:*

1. $\overline{\beta}f'(\alpha)\alpha\geqslant 1$.
2. $\overline{\beta}f'(\alpha)\alpha=1$ *if and only if* $f(z)\equiv e^{i\theta}z$, *where* $e^{i\theta}=\beta\alpha^{-1}$ *and* $\theta\in\mathbb{R}$.

We remark that, when $\alpha=\beta=1$, Theorem B coincides with Theorem A. The interested reader can refer to [27, Lemma 6.1] and [34, 35, 36] for more general result of Theorem B. The Schwarz lemma at the boundary plays an important role in the classical complex analysis and serval complex variables. In 2015, Liu et al. improved Theorem A to higher dimensions ([15, Theorem 3.1]). They also achieved breakthroughs on the growth, covering and distortion results for normalized biholomorphic mappings and pseudconvex domains in $\mathbb{C}^m$ ([16]). Recently, Hamada proved the boundary Schwarz lemma for a $C^1$ mapping of the unit ball $\mathbb{B}^n$ into $\mathbb{B}^m$, where $n,m\geqslant 1$ ([7, Theorem 1.2]).

## 1.1. Motivation

The main purpose of this paper is to improve the corresponding results about boundary Schwarz lemma for pluriharmonic mappings in the unit ball $\mathbb{B}^n\subseteq\mathbb{C}^n$ and for harmonic mappings. It should be noted that, our Theorem 1.4 below, solves the corresponding extremal problems and improves Hamada's estimate [7]. The main techniques of proving these results are the generalized Schwarz lemma and Claim 1.1 below. See for example [18, Theorem 6], for the generalized Schwarz lemma of harmonic mappings. Moreover, in [20], Khalfalah and the second author of this paper establish some Schwarz type inequalities for mappings with bounded Laplacian. These mentioned results are useful and will be used in proving our main theorems.

**A new approach in proving our results**

In this paper, we consider the coordinate space $\mathbb{R}^m$ with the standard basis:

$$\begin{aligned} e_1 &= (1,0,\dots,0)^T \\ e_2 &= (0,1,\dots,0)^T \\ &\vdots \\ e_m &= (0,0,\dots,1)^T \end{aligned}$$

To see that this is a basis, note that an arbitrary vector in $\mathbb{R}^m$ can be written uniquely in the form $x = \sum_{i=1}^m x_i e_i$. The isometries associated with the Euclidean metric, are called Euclidean motions.

Let $D \subset \mathbb{R}^n$, and let $f : D \to D'$ be a function of $D$ into $D' \subset \mathbb{R}^m$, and assume that $a \in \partial D$, the boundary of $D$. If

$$f(x) - f(a) = f'(a)(x-a) + o(x-a)$$

when $x$ through the domain $D$ and tends to $a$, then we say that $f$ is differentiable at $a$ with respect to $D$.

For $a \in \mathbb{R}^m$ and $v \in T_a(\mathbb{R}^m)$ (the tangent space at the point $a$), we define the half space $H(a,v) = \{y \in \mathbb{R}^m : \langle y-a, v\rangle < 0\}$. If $v$ is a unit vector, then we use notation $n_a$ for it and write simply $H_a$ instead of $H(a,v)$. We also notice that in our approach the following simple result is useful:

CLAIM 1.1. Assume that $f$ is differentiable at a point $a \in \mathbb{R}^n$ and let $b = f(a) \in \mathbb{R}^m$. Then by the definition of transpose $T$, we have

$$\langle f'(a)Z, n_b\rangle = \langle Z, f'(a)^T n_b\rangle,$$

for any $Z \in T_a(\mathbb{R}^n)$. The following statements hold:

(i) If $f'(a)$ maps $H_a$ into $H_b$, then $f'(a)^T n_b = \lambda n_a$, where $\lambda > 0$.

(ii) If further, $f$ maps $H_a$ into $H_b$, then $f'(a)^T n_b = \lambda n_a$, where $\lambda \geqslant 0$. In particular if $f'(a)^T n_b \neq 0$, then $\lambda > 0$.

(iii) In both cases (i) and (ii), we have $\lambda = |f'(a)^T n_b| = \langle f'(a) n_a, n_b\rangle$, and $\lambda \leqslant |f'(a) n_a|$.

We remark here that in general $f'(a)n_a$ is not equal to $sn_b$ for some $s\in\mathbb{R}$.

EXAMPLE 1.1. $a=b=0$, $n_a=e_n$, $n_b=e_m$ and $f(x_1,\dots,x_n)=(0,\dots,0,x_n^3)$, shows that $f$ maps $H_{e_n}$ into $H_{e_m}$, and therefore it satisfies all the hypothesis of (ii). Here in addition $f'(0)=0$, and therefore $f'(0)^T=0$ and $\lambda=0$.

*Proof.* Here it is convenient to identify $H_a$ and $H_b$ with subspaces of $T_a(\mathbb{R}^n)$ and $T_b(\mathbb{R}^m)$, respectively. Let $e_1,e_2,\dots,e_n$ be orthogonal basis such that $e_1,e_2,\dots,e_{n-1}\in T_a(H_a)$ (the tangent space of $H_a$), and $e_n=n_a$. By hypothesis $f'(a)$ maps $H_a$ into $H_b$, and therefore we have

$$0=\langle f'(a)X,n_b\rangle=\langle X,f'(a)^Tn_b\rangle \tag{1.4}$$

for all $X\in T_a(H_a)$. This shows that $X_0=f'(a)^Tn_b$ is orthogonal on $T_a(H_a)$. In our setting, it means that it equals to $\lambda e_n$. Then by definition of the transpose, one has

$$\langle f'(a)e_n,n_b\rangle=\langle e_n,f'(a)^Tn_b\rangle=\langle e_n,\lambda e_n\rangle=\lambda.$$

Since $n_a\in H_a$, $f'(a)e_n\in H_b$, by the definition of $H_b$, we first conclude that $\langle f'(a)e_n,n_b\rangle>0$, and hence, $\lambda>0$. This completes the proof of (i).

For the proof of (ii), which is similar to (i), we leave it to the interested reader by considering two cases: $X_0=0$ and $X_0\neq 0$.

(iii) is an immediate corollary of (i) and (ii). □

REMARK 1.1. The following example illustrates the situation concerning the part (i) of Claim 1.1. Let $a=b=0\in\mathbb{R}^3$, $n_a=n_b=e_3$, $A=f'(0)$ be defined by $A(e_1)=A(e_2)=e_1$ and $A(e_3)=e_2+e_3$. Then $\lambda=\langle f'(a)e_n,n_b\rangle=\langle Ae_3,e_3\rangle=1$ and $A^T(e_3)=e_3$. If $h=e_1e_1+x_2e_2+x_3e_3$, $x_3>0$, then $A(h)=A(x_1e_1+x_2e_2+x_3e_3)=(x_1+x_2)e_1+x_3(e_2+e_3)$ and $\langle A(h),e_3\rangle=x_3$.

Thus concerning the part (i) of Claim 1.1, note that in general it is possible that $f'(a)$ does not map $H_a$ onto $H_b$, but even in this case $f'(a)^Tn_b$ exists and equation (1.4) shows that it is orthogonal on $T_a(H_a)$.

**Pluriharmonic mappings**

Define the formal derivatives operators as follows:

$$\partial:=\frac{1}{2}(\partial_x-i\partial_y)\quad\text{and}\quad\overline{\partial}:=\frac{1}{2}(\partial_x+i\partial_y).$$

A twice continuously differentiable, complex-valued function $f$ defined on $\Omega\subseteq\mathbb{C}$ is harmonic on $\Omega$ if

$$\Delta f=\overline{\partial}(\partial f)\equiv 0.$$

A vector-valued mapping $f=(f_1,\cdots,f_n)$ of $\mathbb{C}$ into $\mathbb{C}^n$ is said to be harmonic, if each component $f_j$ $(1\leqslant j\leqslant n)$ is a harmonic mapping in $\mathbb{C}$.

It follows from [4] that each harmonic mapping $w(z)$ in $\mathbb{D}$ has the canonical decomposition $w = h + \overline{g}$. It is clearly that if $f = (f_1, \cdots, f_n)$ is harmonic in $\mathbb{D}$, then each component $f_j$ has the decomposition $f_j = h_j + \overline{g_j}$. Therefore, we have $f = h + \overline{g}$ where $h = (h_1, \cdots, h_n)$ and $g = (g_1, \cdots, g_n)$ are holomorphic mappings of $\mathbb{D}$.

A continuous complex-valued function $f$ defined on a domain $G \subseteq \mathbb{C}^n$ is said to be pluriharmonic if for each fixed $z \in G$ and $\theta \in \mathbb{S}^{n-1}$, the function $f(z + \theta\zeta)$ is harmonic in $\{\zeta : |\zeta| < d_G(z)\}$ where $d_G(z)$ denotes the distance from $z$ to the boundary $\partial G$ of $G$. If $G$ is simply connected, then a real-valued function $u$ defined on $G$ is pluriharmonic if and only if $u$ is the real part of a holomorphic function on $G$.

Clearly, a mapping $f : \mathbb{B}^n \to \mathbb{C}$ is pluriharmonic if and only if $f$ has a representation $f = h + \overline{g}$, where $h$ and $g$ are holomorphic in $\mathbb{B}^n$. We refer to [8] and [11] for more details on pluriharmonic mappings.

**Jacobian**

Let $G$ be a domain in $\mathbb{R}^n$. For a $C^1$ mapping $f : G \to \mathbb{R}^m$, the Jacobian of $f$ at $x$, denoted by $J_f(x)$, is defined as the following $m \times n$ matrix:

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}, & \cdots, & \frac{\partial f_1}{\partial x_n} \\ & \vdots & \\ \frac{\partial f_m}{\partial x_1}, & \cdots, & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

This matrix can be consider as the corresponding linear operator $f'(x)$ which acts on the tangent space $T_x(\mathbb{R}^n)$.

For a complex-valued and differentiable function $f$ from $\mathbb{B}^n$ into $\mathbb{C}$, we introduce the following derivatives:

$$f_z = \left(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}\right) \quad \text{and} \quad f_{\overline{z}} = \left(\frac{\partial f}{\partial \overline{z_1}}, \cdots, \frac{\partial f}{\partial \overline{z_n}}\right).$$

If $f : \mathbb{B}^n \to \mathbb{C}^m$ is differentiable, then we introduce

$$f_z = \left(\frac{\partial f_j}{\partial z_k}\right)_{m\times n} \quad \text{and} \quad f_{\overline{z}} = \left(\frac{\partial f_j}{\partial \overline{z_k}}\right)_{m\times n}.$$

In the literature the $2m \times 2n$ Jacobian matrix of $f$ at $z_0$ in terms of real coordinates is also denoted by $J_f(z_0')$.

Let $f = h + \overline{g}$ be a pluriharmonic mapping from $\mathbb{B}^n$ into $\mathbb{C}^n$. Then, the real Jacobian determinant of $f$ can be written in the following form

$$\det J_f = \det D_f = \det \begin{pmatrix} \partial h & \overline{\partial g} \\ \partial g & \overline{\partial h} \end{pmatrix}$$

and if $h$ is locally biholomorphic, then the determinant of $J_f$ can be written as follows

$$\det J_f = |\det \partial h|^2 \det(I_n - \partial g[\partial h]^{-1} \overline{\partial g[\partial h]^{-1}}).$$

When $n = 2$, i.e., the complex plane, and $f$ is harmonic in $\mathbb{D}$, then its determinant of Jacobian is given as follows

$$\det J_f = |f_z|^2 - |f_{\bar{z}}|^2 = |h'|^2 - |g'|^2.$$

**Boundary Schwarz lemma for harmonic mappings**

In [12], the authors obtained a Schwarz lemma for pluriharmonic mappings between the unit balls of any dimensions, which generalizes the classical Schwarz lemma for bounded harmonic functions to higher dimensions. As an application of this result, the authors also established a boundary Schwarz lemma for pluriharmonic mappings between unit balls with any dimensions. Later, Kalaj [11] extended and simplified the main result of [12].

**Main results**

The main purpose of this paper is to establish a new version of the boundary Schwarz lemma for pluriharmonic mappings (harmonic mappings and $C^1$ mappings) $f$, and obtain the sharp estimate of the eigenvalue $\lambda$ related to the Jacobian of $f$. Our main results are as follows.

THEOREM 1.2. *Let $f$ be a pluriharmonic self-mapping of $\mathbb{B}^n$ having differentiable extension to a boundary point $a \in \mathbb{S}^{n-1}$ such that $f(0) = 0$ and $b = f(a) \in \mathbb{S}^{n-1}$. Assume that $a'$ and $b'$ are the real version of $a$ and $b$, respectively.*

*(i) Then there exists a positive number $\lambda \in \mathbb{R}$ such that $J_f(a')^T b' = \lambda a'$ and*

*(ii)*

$$\lambda \geqslant 2/\pi \geqslant C_{2n},$$

*where $C_{2n}$ is given by (2.3).*

*(iii) In particularly if $n = 1$, we have $\lambda \geqslant \frac{2}{\pi}$. This is sharp.*

We refer to [9, Remark 2.7] for more numerical values of $C_m$ for some small $m$. The part (i) of Theorem 1.1 is proved in [7]. Hamada[1] observed that the function $f = (f_1, f_2, \ldots, f_n)$, i.e. $f(z)$ where $z = (z_1, \ldots, z_n)$, given by $f_1(z_1, \ldots, z_n) = \frac{2}{\pi} \arctan \frac{2x}{1-x^2-y^2}$, $z_1 = x + iy$, and $f_2(z) = \ldots = f_n(z) = 0$ is an extremal for the estimate in the part (ii). It can be concluded from the proof of Theorem 1.2 below, but we leave the straightforward details to the interested reader.

REMARK 1.2. After writing a final version of this paper in order to summarize our results we discovered Claim 1.1. We also realized that the proof of the part (i) in Theorem 1.2 can be based on Claim 1.1, the proof of the part (iii) can be derived from Proposition 4.3, and from Remark 1.4, one can obtain the part (ii). Bearing in mind that some readers (for pedagogical and methodological reasons) may find it interesting to see different evidence, we decided to keep the original proof of this theorem.

[1] in communication with the second author

REMARK 1.3. 1. Hamada proved in [7] that if $f:\mathbb{B}^n\to\mathbb{B}^m$ is a $C^1$ mapping, and suppose $f$ is $C^1$ at $a$ and $b=f(a)\in\mathbb{S}^{m-1}$, then there exists a nonnegative number $\lambda\in\mathbb{R}$ such that $J_f(a')^Tb'=\lambda a'$ and $J_f(a')$ maps $T_{a'}(\mathbf{S}^{2n-1})$ into $T_{b'}(\mathbf{S}^{2m-1})$, where $a'$ and $b'$ are the real version of $a$ and $b$, respectively.

2. If in addition $f$ is pluriharmnic on $\mathbb{B}^n$, then

$$\lambda\geqslant\frac{1-(f(0)')^Tb'}{2}\geqslant\frac{1-|f(0)|}{2},$$

where $(f(0)')^Tb'$ is the product of the row matrix $(f(0)')^T$ and the column matrix $b'$ which also denotes the corresponding scalar product.

3. If further $f(0)=0$, then the lower bound of such eigenvalue $\lambda$ is an absolutely constant $1/2$.

4. In Theorem 1.2, we find a new lower bound which is related to the dimensions $n$. Furthermore, if $n=1$, then

$$C_2=\frac{2}{\pi}>\frac{1}{2}.$$

5. It should be pointed out that, we give Theorem 1.4 below, which shows that if $n>1$ then $\lambda\geqslant\frac{2}{\pi}>C_{2n}$. This shows that if $n>1$, then the estimate in Theorem 1.2 is not sharp for pluriharmonic mappings.

Let $\varphi_\xi(z)=A\frac{\xi-z}{1-\overline{\xi}z}$ be the holomorphic automorphism of $\mathbb{B}^n$, where $A=s_\xi I_n+\frac{\xi\overline{\xi}'}{1+s_\xi}$, $s_\xi=\sqrt{1-|\xi|^2}$. We have the following theorem.

THEOREM 1.3. *Let $f$ be a pluriharmonic self-mapping of $\mathbb{B}^n$ having differentiable extension to a boundary point $a\in\mathbb{S}^{n-1}$ such that $f(\xi)=0$, $\xi\in\mathbb{B}^n$, $b=f(a)\in\mathbb{S}^{n-1}$ and $p=\varphi_\xi(a)$. Assume that $a'$, $b'$, $\xi'$ and $p'$ are the real version of $a$, $b$, $\xi$ and $p$, respectively. Then there exists a positive number $\lambda$ such that*

$$J_f(a')^Tb'=\lambda J^T_{\varphi_\xi}(a')p'$$

*and*

$$\lambda\geqslant\frac{2}{\pi}\geqslant C_{2n},$$

*where $C_{2n}$ is given by (2.3). If in particular $\xi=0$, then Theorem 1.3 coincides with Theorem 1.2. Moreover, there exists a positive number $\lambda=\lambda(f,a,b)$ such that $J_f(a')^Tb'=\lambda a'$.*

Let

$$s^-(x)=\cot\left(\frac{\pi}{4}(x+1)\right),\quad x\in(-1,1).$$

The readers can verify directly that $s^-$ is decreasing function on $(-1,1)$ and $\frac{2}{\pi}s^-(x)\geqslant\frac{1-x}{2}$, for all $x\in[0,1)$. Thus, the following theorem shows that our estimate is better than the corresponding result given by [7].

THEOREM 1.4. *Let $f:\mathbb{B}^n \to \mathbb{B}^m$ be a pluriharmonic function having differentiable extension to a boundary point $a \in \mathbb{S}^{n-1}$ such that $b = f(a) \in \mathbb{S}^{n-1}$. Then*

$$|D_r f(a)| \geqslant \frac{2}{\pi} s^-, \quad where \quad s^- = s^-(|P_b(f(0))|).$$

Here by $D_r f(a)$, we denote the radial derivative of $f$ at the direction $a$.

If $c \in \mathbb{C}^n$ we define the space spanned by $c$ as $\{zc : z \in \mathbb{C}\}$ and denote it with $[c]$.

REMARK 1.4. The projection $P_c$ of $\mathbb{C}^n$ onto $[c]$ is given as follows: $P_c z = \frac{\langle z,c\rangle}{\langle c,c\rangle} c$. Let $f^b$ be defined by $f^b(z) = \langle f(z), b\rangle$. In our approach we use crucial property of pluriharmonic functions: $f^b$ is a complex-valued harmonic on $[a] \cap \mathbb{B}^n$ under hypothesis of the theorem. We remark here in addition that if one set $f_b = P_b \circ f$, then $|f_b(0)| \leqslant |f(0)|$ and therefore $s^-(|f(0)|) \leqslant s = s^-(|f_b(0)|)$. Given $|f(0)|$ be known, the above inequality is sharp. Hence $C_{2n} < \frac{2}{\pi} = C_2$, where $n > 1$. Furthermore, Theorem 1.4 shows that: If $\lambda_0 =: \min \lambda(f,a,b)$ under the family of all pluriharmonic mappings which satisfying the hypothesis of Theorem 1.3, then $\lambda_0 = \frac{2}{\pi} s^-(|f(0)|)$. It is easy to see that a pluriharmonic mapping $f:\mathbb{B}^n \to \mathbb{B}^m$ is harmonic between $\mathbf{B}^{2n}$ and $\mathbf{B}^{2m}$. Therefore, one can easily deduce from Theorem 1.3 and Theorem 1.4 that [12, Theorem 1.1 ] holds true.

The following results generalized the corresponding results given in [12], [24]and [30].

THEOREM 1.5. *Suppose $f:\mathbf{B}^n \to \mathbf{B}^m$ is differentiable at the point $a \in \mathbf{S}^{n-1}$ and let $b = f(a) \in \mathbf{S}^{m-1}$. Then the following results hold:*

*(i) There exists a nonnegative $\lambda \geqslant 0$ such that $f'(a)^T b = \lambda a$;*

*(ii) If $Z \in T_a(\mathbb{R}^n)$, then $|f'(a)Z| \geqslant \lambda |\langle Z,a\rangle|$.*

*If in addition $f$ is a harmonic function, then*

*(iii) $\lambda > c_0$, where $c_0 = \frac{1-|f(0)|}{2^{n-1}}$;*

*(iv) for $Z \in T_a(\mathbb{R}^n)$, we have $|f'(a)Z| \geqslant c_0 |\langle Z,a\rangle|$.*

Let $D$ be a domain of $\mathbb{R}^n$ and $a \in \partial D$, the boundary of $D$. If there is a half space $H_a = \{y : \langle y-a, n_a\rangle < 0\}$, where $n_a$ is a unit vector such that $D \subset H_a$, then we say that $D$ touches $H_a$ at $a$.

THEOREM 1.6. *Let $D$ be a domain in $\mathbb{R}^n$, and let $f : D \to D' \subset \mathbb{R}^m$ be differentiable at a point $a \in \partial D$.*

*(I) Suppose $D$ touches $H_a$ at $a$ and $b = f(a)$ and $D' = f(D)$ touches $H_b$ at $b$. Then there exists a nonnegative $\lambda \geqslant 0$ such that $f'(a)^T n_b = \lambda n_a$.*

*(II) If in particular, $D$ is the unit ball $\mathbf{B}^n$ and that $f$ is harmonic in $\mathbf{B}^n$, then we have $\lambda \geqslant c_0$ and for $Z \in T_a(\mathbb{R}^n)$,*

$$|f'(a)Z| \geqslant c_0|\langle Z, a\rangle|,$$

*where $c_0 = \frac{d(f(0))}{2^{n-1}}$ and $d(f(0)) = dist(f(0), bD')$. If $D'$ is also the unit ball, then $c_0 = \frac{1-|f(0)|}{2^{n-1}}$.*

REMARK 1.5. According to Claim 1.1, it is easy to see that

$$\lambda = |f'(a)^T n_b| = \langle f'(a)n_a, n_b\rangle,$$

and thus, $\lambda \leqslant |f'(a)n_a|$.

Concerning the proof of Theorem 1.6, we give the following comments:

REMARK 1.6. The statement (I) is based on Claim 1.1. Moreover, in the proof of (II) we use that fact: $u(x) = \langle f(x) - b, n_b\rangle$ is a non-negative harmonic function, on which we apply the Harnack's inequality.

The proof of Theorems 1.2–1.6 will be given in Section 3.

### 1.2. Note added in proof

After writing a final version of our manuscript Hamada turned our attention on the arxiv paper [3]. He made great effort in considering several versions of manuscript and gave very useful comments which improved exposition. In [3], the authors generalize the classical Schwarz lemmas of planar harmonic mappings into the sharp forms of Banach spaces, and present some applications to sharp boundary Schwarz type lemmas for pluriharmonic mappings in Banach spaces (cf. also Section 5).

### 1.3. Boundary Schwarz lemma on complex Hilbert balls

Let $B_j$ be the unit ball of a complex Hilbert space $H_j$ for $j = 1, 2$, respectively. Note that if $f$ is $C^1$ at $z_0 \in bB_1$ with values in $H_2$, then the adjoint operator $Df(z_0)^*$ is defined by

$$\mathrm{Re}\Big(\langle Df(z_0)^* w, z\rangle_{H_1}\Big) = \mathrm{Re}\Big(\langle w, Df(z_0)z\rangle_{H_2}\Big) \quad \text{for} \quad z \in H_1, w \in H_2,$$

where $\langle\cdot,\cdot\rangle_{H_j}$ is the inner product of $H_j$, $j = 1, 2$. The following result was obtained in [6, Proposition 1.8].

PROPOSITION 1.7. *Let $B_j$ be the unit ball of a complex Hilbert space $H_j$, for $j = 1, 2$, respectively. Let $f : B_1 \to B_2$ be a pluriharmonic mapping. Assume that $f$ is of class $C^1$ at some point $z_0 \in B_1$ and $f(z_0) = w_0 \in B_2$. Then there exists a constant $\lambda \in R$ such that $Df(z_0)^* w_0 = \lambda z_0$. Moreover,*

$$\lambda \geqslant \frac{1 - \mathrm{Re}\big(\langle f(0), w_0\rangle\big)}{2} > 0.$$

By using this Proposition and the arguments similar to those in the proof of [3, Theorem 3.3], the authors obtain a better estimate:

PROPOSITION 1.8.

$$\lambda \geqslant \max\left\{\frac{2}{\pi} - ||f(0)||, \frac{1-\mathrm{Re}(\langle f(0), w_0\rangle)}{2}\right\}.$$

*In particular if* $f(0)=0$*, then* $\lambda \geqslant 2/\pi$.

Recall the conclusion of (ii) and (iii) in Theorem 1.2, we also have $\lambda \geqslant 2/\pi$ holds true.

THEOREM 1.9. ([3, Theorem 3.3]) *Suppose that* $B_X$ *and* $B_Y$ *are the unit balls of the complex Banach spaces* $X$ *and* $Y$*, respectively, and* $f: B_X \to B_Y$ *is a pluriharmonic mapping. In addition, let* $f$ *be differentiable at* $b \in bB_X$ *with* $\|f(b)\|_Y = 1$*. Then we have*

$$\|Df(b)b\|_Y \geqslant \max\left\{\frac{2}{\pi} - ||f(0)||, \frac{1-||f(0)||}{2}\right\}. \tag{1.5}$$

Our Theorem 1.9 in more specific setting yields a better estimate. We leave it to the interested reader to check it.

Further, Mutavdžić and the second author combining the method developed in the arxiv paper [3], and slightly improved the estimates given in Proposition 1.7, and Theorem 1.9, and showed

$$\lambda \geqslant \frac{2}{\pi}s^{-}(b) = \frac{2}{\pi}\cot\left(\frac{\pi}{4}(1+b)\right) > 0, \quad \text{where} \quad b = \mathrm{Re}\Big(\langle f(0), w_0\rangle\Big),$$

and $\|Df(b)b\|_Y \geqslant 2s^{-}(\|f(0)\|/\pi)$, respectively. They also communicated at Belgrade seminar [37] that the corresponding version of Theorem 1.2 holds for harmonic functions with an optimal estimate $\lambda \geqslant c_n$.

## 2. Preliminaries

It is well-known that a harmonic function $u \in L^{\infty}(\mathbf{B}^n)$ has the following integral representation

$$u(x) = \mathscr{P}[f](x) = \int_{\mathbf{S}^{n-1}} P(x,\zeta) f(\zeta) d\sigma(\zeta),$$

where $f$ is the boundary function of $\mathbb{S}^{n-1}$, and

$$P[x,\zeta] = \frac{1-\|x\|^2}{\|x-\zeta\|^n}, \quad \zeta \in \mathbf{S}^{n-1}$$

is the Poisson kernel and $\sigma$ is the unique normalized rotation invariant Borel measure on $\mathbf{S}^{n-1}$. According to [9], we know that if $u$ is a harmonic self-mapping of $\mathbf{B}^n$ such that $u(0)=0$, then

$$\|u(x)\| \leqslant U(rN), \tag{2.1}$$

where $r = \|x\|$, $N = \{0,\cdots,0,1\}$ and $U$ is a harmonic function of $\mathbf{B}^n$ into $[-1,1]$ defined by

$$U(x) = P[\chi_{S^+} - \chi_{S^-}](x) \tag{2.2}$$

where $\chi$ is the indicator function and $S^+ = \{x \in \mathbf{S}^{n-1} : x_n \geqslant 0\}$, $S^- = \{x \in \mathbf{S}^{n-1} : x_n \leqslant 0\}$. We refer to [1, Chapter 6] for more details.

Recall that the *hypergeometric function* ${}_pF_q$ is defined for $|x| < 1$ by the power series ([14, (2.1.2)])

$${}_pF_q[a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;x] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}.$$

Here $(a)_n$ is the *Pochhammer symbol* and given as follows $(a)_n = \frac{\Gamma(n+a)}{\Gamma(a)}$.

The following result is the so-called *Heinz-Schwarz inequalities*.

LEMMA C. [9, Lemma 2.3] *The function* $V(r) = \frac{\partial U(rN)}{\partial r}$, $0 \leqslant r \leqslant 1$ *is decreasing on the interval* $[0,1]$*, and we have*

$$V(r) \geqslant V(1) = C_m =: \frac{n!\left(1+m-(m-2)_2F_1\left[\frac{1}{2},1;\frac{3+m}{2};-1\right]\right)}{2^{3m/2}\Gamma\left[\frac{1+m}{2}\right]\Gamma\left[\frac{3+m}{2}\right]}. \tag{2.3}$$

We refer the readers to [9, Remark 2.7] for more details on the constant $C_m$ and related functions, when $m = 2,3,4$.

The following lemma is useful and will be used in the proof of our main results.

LEMMA D. ([25, Theorem 2.2]) *For given* $a \in \mathbb{B}^n$*, let* $A = sI_n + \frac{a\overline{a}^T}{1+s}$*, where* $s = \sqrt{1-|a|^2}$ *and* $I_n$ *is the unit square matrix of order* $n$*. Then*

$$\varphi_a(z) = A\frac{a-z}{1-\overline{a}^T z}$$

*is a biholomorphic automorphism of* $\mathbb{B}^n$ *which interchanges* $0$ *and* $a$*. Moreover,* $\varphi_a$ *is biholomorphic in a neighborhood of* $\overline{\mathbb{B}^n}$*, and*

$$A^2 = s^2 I_n + a\overline{a}^T, \quad Aa = a, \quad \varphi_a^{-1} = \varphi_a, \quad J_{\varphi_a}(z) = A\left[-\frac{I_n}{1-\overline{a}^T z} + \frac{(a-z)\overline{a}^T}{(1-\overline{a}^T z)^2}\right].$$

## 3. Proof of the main results

**Proof of Theorem 1.2**

It follows from [7, Theorem 1.2] that $J_f(a')\beta \in T_{b'}(\mathbf{S}^{2n-1})$ for any $\beta \in T_{a'}(\mathbf{S}^{2n-1})$. Assume that $J_f(a')^T b' = \lambda a' + \beta$ for some $\lambda \in \mathbb{R}$ and $\beta \in T_{a'}(\mathbf{S}^{2n-1})$, where $a'$ and $b'$ are the real version of $a$ and $b$. Then

$$|\beta|^2 = (\lambda a' + \beta)^T \beta = (J_f(a')^T b')^T \beta = b'^T J_f(a')\beta = 0$$

by the above argument. Therefore, we have

$$J_f(a')^T b'^T = \lambda a'$$

for some $\lambda \in \mathbb{R}$.

Now we show that $\lambda \geqslant C_{2n}$, where $C_{2n}$ is a constant depending only on the dimensions $n$.

In view of invariance property of harmonic function, see [1, Chapter 1], by using unitary transformation, if needed we can assume that $a' = e_1 = b'$, where $e_1 \in \mathbb{R}^{2n}$.

For $\zeta \in \mathbb{D}$ define $v(\zeta) = \mathrm{Re}\langle f(\zeta, 0, \cdots, 0), e_1\rangle$. Then $v$ maps $\mathbb{D}$ into $(-1,1)$ and $D_r v(1) = \mathrm{Re}\langle f'(e_1)e_1, e_1\rangle = \mathrm{Re}\langle e_1, f'^T(e_1)e_1\rangle = \mathrm{Re}\lambda = \lambda$.

Recall hence a proof of the part (iii) can be derived from Proposition 4.3, and from Remark 1.4 the conclusion (ii) follows. But we prefer here to give our first approach.

For $x \in \mathbf{B}^{2n}$ define $u(x) = \mathrm{Re}\langle f(x), e_1\rangle$. Then $u$ maps $\mathbf{B}^{2n}$ into $(-1,1)$ and $D_r u(1) = \mathrm{Re}\langle f'(e_1)e_1, e_1\rangle = \mathrm{Re}\langle e_1, f'^T(e_1)e_1\rangle = \mathrm{Re}\lambda = \lambda$, where here for simplicity reason, by abusing notation, we write $e_1$ instead of real version $e_1'$.

Now letting $w(x) = 1 - u(x)$, $x \in \mathbf{B}^{2n}$. Since $u$ is harmonic and it maps $\mathbf{B}^{2n}$ into $(-1,1)$, $u(0) = 0$ and $u(e_1) = 1$, using (2.1) we have

$$w(x) = 1 - u(x) \geqslant 1 - U(rN), \quad \text{for} \quad r = |x| < 1.$$

Next, using (2.2), Lemma 3, and [9, Lemma 2.4], we find

$$\lim_{|x| \to 1^-} \frac{1-u(x)}{1-|x|} \geqslant \lim_{r \to 1^-} \frac{1-U(rN)}{1-r} = \left.\frac{\partial U(rN)}{\partial r}\right|_{r=1} \geqslant C_{2n}.$$

Thus $\frac{\partial u}{\partial x_1}(e_1) = \lambda \geqslant C_{2n}$. If $n = 1$, then $U(rN) = 4\arctan(r)/\pi$. Therefore

$$\left.\frac{\partial U(rN)}{\partial r}\right|_{r=1} = \left.\frac{4}{\pi(1+r^2)}\right|_{r=1} = \frac{2}{\pi}.$$

To show this estimate is sharp for $n = 1$, consider the function

$$f(z) = u(z) + iv(z) = \frac{2}{\pi} \arctan \frac{2x}{1 - x^2 - y^2},$$

where $z = x + iy \in \mathbb{D}$. Then $f$ is harmonic and maps $\mathbb{D}$ into itself such that $f(1) = 1$, $v = \mathrm{Im}(f) = 0$ and

$$u_x(1) = f_z(1) + f_{\bar{z}}(1) = \frac{2}{\pi},$$

$$u_y(1) = \frac{1}{i}(f_z(1) - f_{\bar{z}}(1)) = 0.$$

Thus

$$J_f(1)^T = \begin{pmatrix} u_x(1), \; v_x(1) \\ u_y(1), \; v_y(1) \end{pmatrix}.$$

For $a' = (1,0)^T$ and $b' = (1,0)^T \in \mathbb{T}$, we have

$$J_f(1)^T b' = f_x(1)a' = \frac{2}{\pi}a'.$$

This shows that $\lambda = \frac{2}{\pi}$, and the estimate is sharp.

The proof is completed. □

**Proof of Theorem 1.3**

Recall that we assume $\varphi_\xi(a) = p \in \mathbb{S}^{n-1}$. Let $g(z) = f \circ \varphi_\xi(z)$. Then $g$ is a pluriharmonic self-mapping of $\mathbb{B}^n$ satisfying

$$g(0) = f \circ \varphi_\xi(0) = f(\xi) = 0,$$

and

$$g(p) = f \circ \varphi_\xi(p) = f(a) = b \in \mathbb{S}^{n-1}.$$

According to Theorem 1.2, we know that there exists a nonnegative number $\lambda \in \mathbb{R}$ such that

$$J_g(p')^T b' = \lambda p'. \tag{3.1}$$

Since $J_g(p') = J_f(a')J_{\varphi_\xi}(p')$, by the property of transpose operation we have

(1) $J_g^T(p') = (J_f(a')J_{\varphi_\xi}(p'))^T = J_{\varphi_\xi}^T(p')J_f^T(a')$.

Hence by (3.1) we find

(2) $J_{\varphi_\xi}^T(p')J_f^T(a')b' = \lambda p'$.

Since $a = \varphi_\xi(p)$ and the automorphism $\varphi_\xi$ of $\mathbb{B}^n$ has property $\varphi_\xi \circ \varphi_\xi = Id$, we have $p = \varphi_\xi(a)$, and $J_{\varphi_\xi}(p')J_{\varphi_\xi}(a') = Id$. Then

(3) $J_{\varphi_\xi}^T(a')J_{\varphi_\xi}^T(p') = Id$.

According to (2) and (3), we see that $J_f^T(a')b' = \lambda J_{\varphi_\xi}^T(a')p'$.

This completes the proof. □

We advise the readers to recall the projection of $P_a$ in Remark 1.4, and recall Remark 1.6 regarding the following proofs.

**Proof of Theorem 1.4**

Recall that in Remark 1.4, we set $f_b = P_b \circ f$ and let $f^b$ be defined by $f^b(z) = \langle f(z), b\rangle$. Then $f_b(z) = \langle f(z), b\rangle b$ is a vector-valued harmonic on the unit disk $U_a := [a] \cap \mathbb{B}^n$ in $[a]$, and it maps $U_a$ in the unit disk $U_b$ in $[b]$.

Now, let $F^b(z) = \langle f(az), b\rangle$ and set $F_b(z) = \langle f(az), b\rangle b$. Then $F_b$ is a vector-valued harmonic on the unit disk $\mathbb{D}$ and it maps $\mathbb{D}$ into $U_b = [b] \cap \mathbb{B}^n$. Since $F^b$ maps $\mathbb{D}$ into itself and it is a complex-valued harmonic function on $\mathbb{D}$, by the planar version, i.e., Proposition 4.4 below, we find first $|D_r F^b(1)| \geqslant \frac{2}{\pi} s^-$, where $s^- = s^-(|F_b(0)|)$. Next, since the projection decreases the distances

$$|f(ar) - f(a)| \geqslant |F^b(r) - F^b(1)|$$

letting $r \to 1$, we have

$$|D_r f(a)| \geqslant \frac{2}{\pi} s^-.$$

Using Example 4.2 below, we show that this estimate is sharp.

Set $R_a(\lambda a) = \lambda$ and let

$$f_{a,b} = R_b^{-1} \circ v^c \circ R_a \circ P_a,$$

where $c \in (-1,1)$, $|c| = |R_b(P_b f(0))|$, and $v^c$ is defined in Example 4.2. Then $f^0 = f_{a,b}$ maps $\mathbb{B}^n$ onto $(-b,b)$. Moreover, we have $f^0$ is pluriharmonic on $\mathbb{B}^n$ and $f^0(a) = b$. Since $|D_r(f^0)'_a(a)| = \frac{2}{\pi} s^-(c)$, we see that it is the extremal function. □

**Proof of Theorem 1.5**

For $z \in \mathbb{R}^n, z \neq 0$, let $H_z$ be a hyper plane throughout $z$ and orthogonal on $z$.

Let $L$ be a linear functional on the tangent space $T_a(\mathbb{R}^n)$ defined by $L(X) = \langle f'(a)X, b\rangle$. Then there exists a vector $X_0$ such that $L(X) = \langle X, X_0\rangle$.

According to the assumption, we see that $f'(a)$ maps $T_a(H_a)$ into $T_b(H_b)$, and therefore $L$ is zero on $T_a(\mathbf{S}^{n-1})$, and $X_0$ is orthogonal on $T_a(\mathbf{S}^{n-1})$. Hence $X_0 = \lambda a$, where $\lambda > 0$. Moreover,

$$L(X) = \langle f'(a)X, b\rangle = \langle X, \lambda a\rangle. \tag{3.2}$$

By the definition of adjoint operator, i.e.,

$$L(X) = \langle f'(a)X, b\rangle = \langle X, f'(a)^T b\rangle$$

and by (3.2), we get

$$\langle X, f'(a)^T b\rangle = \langle X, \lambda a\rangle,$$

where $X \in T_a(H_a)$. Hence (i) holds true.

We now prove (ii) as follows: Using Euclidean motions, we can assume that $a = b = e_1$. It follows from (i) that

$$f'(e_1)^T e_1 = \lambda e_1. \tag{3.3}$$

Therefore

$$\langle f'(e_1)e_1, e_1\rangle = \langle e_1, f'(e_1)^T e_1\rangle = \langle e_1, \lambda e_1\rangle = \lambda \tag{3.4}$$

and consequently $|f'(e_1)e_1| \geqslant \lambda$.

If $Z \in T_{e_1}(\mathbb{R}^n)$, then we have the following representation $Z = Z_1 + Z_2$, where $Z_1 \in T_{e_1}(H_{e_1})$, and $Z_2 = (\cos\alpha)e_1$ and $\cos\alpha = \langle Z, e_1\rangle$. Then

$$f'(e_1)Z = f'(e_1)Z_1 + f'(e_1)Z_2$$

and therefore by (3.4), we have

$$\langle f'(e_1)Z, e_1\rangle = \langle f'(e_1)Z_2, e_1\rangle = \langle f'(e_1)(\cos\alpha\, e_1), e_1\rangle = \lambda\langle Z, e_1\rangle.$$

Hence $|f'(e_1)Z| \geqslant \lambda|\langle Z, e_1\rangle|$ and this completes the proof of (ii).

Assume further that $f$ is harmonic in $\mathbb{B}^n$. Then the proof of (iii) was already given in [24] and [30]. The proof of (iv) now directly follows from (ii) and (iii). □

**Proof of Theorem 1.6**

For $Y, Y_0 \in T_b(\mathbb{R}^m)$, set $L(Y) = \langle Y, Y_0\rangle$. If $A$ is a linear operator from $T_a(\mathbb{R}^n)$ into $T_b(\mathbb{R}^m)$, then the following function

$$L_1(X) = \langle A(X), Y_0\rangle = L(A(X))$$

is a bounded liner function on $T_a(\mathbb{R}^n)$, and thus, there is a point $X_0 \in T_a(\mathbb{R}^n)$ such that $L_1(X) = \langle X, X_0\rangle$, where $X \in T_a(\mathbb{R}^n)$.

It is easy to see that $X_0 = A^T Y_0$. Next, we specify that $Y_0 = n_b$ and $A = f'(a)$. Using Euclidean motions, we can assume that $a = b = 0$, $T_a(\mathbb{R}^n) = \mathbb{R}^{n-1}$, and

$$T_b(\mathbb{R}^m) = \mathbb{H}^m = \{u = (u_1, \ldots, u_m) \in \mathbb{R}^m : u_m > 0\}.$$

Here we can chose that $n_a = e_n$ and $n_b = e_m$.

According to the assumption of (I), we see that the $m$-th coordinate function $f_m(x)$ has minimum $0$ at $x = 0$ on $D$, and hence, for $X \in \mathbb{R}^{n-1}$, we have

$$0 = \langle f'(a)X, e_m\rangle = \langle X, f'(a)^T e_m\rangle = \langle X, X_0\rangle.$$

This shows that $X_0 = f'(a)^T e_m$ is orthogonal on $\mathbb{R}^{n-1}$. In our setting, it equals to $\lambda e_n$, where $\lambda > 0$.

In the general case, we have

$$0 = \langle f'(a)X, n_b\rangle = \langle X, f'(a)^T n_b\rangle = \langle X, X_0\rangle,$$

which shows that $X_0 = f'(a)^T n_b$ is orthogonal on $T_a(\partial D)$. Therefore, in the general case it also equals to $\lambda n_a$, where $\lambda > 0$. This proves (I).

Before we prove (II), we need the following theorem.

THEOREM 3.1. *Suppose $f$ is a Euclidean harmonic mapping from the unit ball $\mathbf{B}^n \subset \mathbb{R}^n$ onto a bounded domain $D = f(\mathbf{B}^n)$, which contains the ball $\mathbf{B}^n(f(0); R_0)$ and there is a half space $H_b$ which touches a point $b \in \partial D$ such that $D = f(\mathbb{B}) \subset H_b$. Then*

*(i1)* $|f(z) - b| \geqslant (1 - |z|)\widetilde{c}_n R_0$, $z \in \mathbf{B}^n$, *where* $\widetilde{c}_n = \frac{1}{2^{n-1}}$.

*(i2) if in addition $f$ is differentiable at the point $a \in \mathbb{S}^{n-1}$ and $b = f(a)$, then*

$$-\langle f'(a)a, n_b\rangle \geqslant \widetilde{c}_n R_0.$$

This result has been announced in [24, Theorem 1.3]. Sometimes, we refer to this result as a version of Harnack's lemma. In [24] the second author of this paper stated this result under the condition that $D = h(\mathbb{B})$ is convex. But, modification of the proof of [24, Theorem 1.1] (planar case) shows that (i1) also holds under the hypothesis of Theorem 3.1.

*Proof of* (i1). Consider harmonic function $u(x) = \langle f(x) - b, n_b \rangle$. Since $u \in L^{\infty}(\mathbf{B}^n)$, we see that $u$ has the following Poisson integral representation:

$$u(x) = \mathscr{P}[f](x) = \int_{\mathbf{S}^{n-1}} P(x,\zeta) f(\zeta) d\sigma(\zeta),$$

where $f$ is the boundary function of $\mathbf{S}^{n-1}$, and $P[x,\zeta]$ is the Poisson kernel. Since $u$ is a nonnegative harmonic function on $\mathbb{B}$, it follows from the Harnack's inequality that:

$$u(x) \geqslant \frac{1-r}{(1+r)^{n-1}} u(0), \tag{3.5}$$

where $r = |x|$. Then $|f(x) - b| \geqslant u(x)$. Let $l(t) = f(0) + tn_b$, $t \geqslant 0$, be a half line which intersects the sphere $\mathbf{S}(f(0), R_0)$ and $\partial H_b$, respectively at the points $x_0$ and $x_1$. Then by hypothesis $x_0 \in D$ and $x_1 \notin D$, and note that

$$u(0) = \langle f(0) - b, n_b \rangle = |f(0) - x_1| \geqslant |f(0) - x_0| = R_0,$$

we get that $u(0) \geqslant d_D(f(0))$, and hence

$$u(x) \geqslant \frac{1-r}{(1+r)^{n-1}} d_D(f(0)) \geqslant (1-r) d_D(f(0)) \widetilde{c}_n, \tag{3.6}$$

and

$$|f(x) - b| \geqslant u(x) \geqslant \frac{1-r}{(1+r)^{n-1}} d_D(f(0)) \geqslant (1-r) d_D(f(0)) \widetilde{c}_n.$$

*proof of* (i2). Since $f$ is differentiable at the point $a$, we have

$$f(ra) - f(a) = f'(a)(ra - a) + o(1-r) = (r-1) f'(a) a + o(1-r).$$

Then

$$\frac{f(ra) - f(a)}{1-r} = -f'(a)a + o(1).$$

By using (3.6) and the definition of $u$, we have $\langle -f'(a)a + o(1), n_b \rangle \geqslant \widetilde{c}_n R_0$. If $r$ tends to $1$, then $-\langle f'(a)a, n_b \rangle \geqslant \widetilde{c}_n R_0$.

We are now ready to prove the part (II).

According to the assumption, $D$ is the unit ball $\mathbf{B}^n$, we have $H_a$ is defined by $n_a = -a$. By (I) there exists a positive $\lambda > 0$ such that $f'(a)^T n_b = \lambda n_a$ and in particular,

$$\lambda = |f'(a)^T n_b| = \langle f'(a) n_a, n_b \rangle = -\langle f'(a) a, n_b \rangle.$$

Hence by (i2), $\lambda \geqslant \widetilde{c}_n R_0$. Using a similar approach as in the proof of Theorem 1.5, we can prove that:

$$|f'(a) Z| \geqslant c_0 |\langle Z, a \rangle|,$$

where $c_0 = \frac{d(f(0))}{2^{n-1}}$ and $d_{D'}(f(0)) = dist(f(0), bD')$. This completes the proof. □

## 4. Appendix 1

In this section, we outline some results in the planar case related to a boundary version of Schwarz lemma (see for example [21]). For further results of this type see also the paper [10, Theorem 6], in which the authors have considered a boundary version of Schwarz lemma for $\alpha$-harmonic functions, and in particular the special case $\alpha = 0$.

Recall that is convenient to use notation $s = s(b) = \tan(\frac{\pi}{4}(b+1))$, $s^+ = s^+(b) = \tan(\frac{\pi}{4}(|b|+1))$. Note if $a = \tan\frac{b\pi}{4}$, then $s = s(b) = \frac{1+a}{1-a}$.

PROPOSITION 4.1. *Suppose $f : \mathbb{D} \to \mathbb{D}$ is a differentiable function. Fix $b \in \mathbb{T}$, assume that $f$ has a continuous extension at $b$, such that $f(b) = c \in \mathbb{T}$. If $f$ is differentiable at $b$, and if there is a real function $\varphi : [0,1] \to [0,1]$, such that $\varphi'(1)$ exists and $|f(rb)| \leqslant \varphi(r)$ for $r$ near $1$, then*

$$|\Lambda_f(b)| \geqslant |D_r f(1)| \geqslant \varphi'(1),$$

*where $\Lambda_f = |f_z| + |f_{\bar{z}}|$, and $D_r f$ is the radial derivative of $f$.*

We remark here that since $f$ has a continuously extension at $b$, we see that $|f(rb)| \to 1$ as $r \to 1$, and by the hypothesis $\varphi : [0,1] \to [0,1]$, we have $\varphi(1) = 1$.

*Proof.* By using rotations, it is not loss of generalities to assume that $b = c = 1$. According to the triangle inequality

$$|f(r) - f(1)| = |f(r) - 1| \geqslant 1 - \varphi(r) = \varphi(1) - \varphi(r)$$

one has

$$\frac{|f(r) - f(1)|}{1-r} \geqslant \frac{\varphi(1) - \varphi(r)}{1-r}.$$

Hence when $r \to 1$, we get $|\Lambda_f(b)| \geqslant |D_r f(1)| \geqslant \varphi'(1)$. □

Fix $b \in (-1,1)$ and let $a = \tan\frac{b\pi}{4}$. For $r \in (0,1)$, let

$$M_b(r) = \frac{4}{\pi}\arctan\frac{a+r}{1+ar} \quad \text{and} \quad m_b(r) = \frac{4}{\pi}\arctan\frac{a-r}{1-ar}.$$

The following theorem was given in [18].

THEOREM E. ([18, Theorem 6]) *Let $u : \mathbb{D} \to (-1,1)$ be a harmonic function such that $u(0) = b$. Then*

$$m_b(|z|) \leqslant u(z) \leqslant M_b(|z|), \quad \textit{for all} \quad z \in \mathbb{D}, \tag{4.1}$$

*and the above inequalities are sharp.*

EXAMPLE 4.2. Set $\phi^0 = i\frac{2}{\pi}\ln A_0$, where $A_0 = \frac{1+z}{1-z}$, and let $T_a(z) = \frac{z+a}{1+\overline{a}z}$ be the Möbius transformation. Suppose $\phi^a = \phi^0 \circ T_a$ and set $u^a = \mathrm{Re}\phi^a$.

Let

$$\Theta = \{z:\ z = iy,\ y \in [-1,1]\}$$

be an "interval" of the $y$-axis. It is easy to see that $\phi^0$ maps $\Theta$ onto the interval $[-1,1]$. Elementary calculations show that $(\phi^0)'(z) = i\frac{4}{\pi}\frac{1}{1-z^2}$, and thus, $(\phi^a)'(z) = -(\phi^0)'(T_a)\frac{1-|a|^2}{(1-\overline{a}z)^2}$. Moreover, $|(\phi^0)'(z)|$ attains its minimum at the points $\pm i$ on $\mathbb{T}$ and is equal to $\frac{2}{\pi}$, while the minimum $|T_a'(z)|$ on $\mathbb{T}$ equals to $e^-(a) = |T_a'(1)| = \frac{1-|a|}{1+|a|}$. Furthermore, if $a \in \Theta$, then $\min|D_r u^a|$ on $\mathbb{T}$ equals to $\frac{2}{\pi}e^-(a)$.

We remark here that if we fix $a \in (-1,1)$, and set $v^a(z) = -\mathrm{Re}\phi^0(iT_a)$, then $v^a$ maps the closed unit disk $\overline{\mathbb{D}}$ onto the interval $[-1,1]$. It also maps the interval $[-1,1]$ onto itself. Furthermore, it fixes the points $-1, 0$ and $1$.

PROPOSITION 4.3. *Let $u : \mathbb{D} \to (-1,1)$ be a harmonic function such that $u(0) = b$ and $a = \tan\frac{b\pi}{4}$. Assume that $u$ has a continuously extension to the boundary point $z_0 \in \mathbb{T}$, and $u(z_0) = \pm 1$. If $u$ is differentiable at $z_0$, then $|D_r u(z_0)| \geqslant \frac{2}{\pi}\frac{1-|a|}{1+|a|}$.*

*Proof.* Case 1. Suppose first that $b \in [0,1)$. If we set $T_a(r) = \frac{a+r}{1+ar}$, then $\varphi(r) := A(r) = \frac{4}{\pi}\arctan T_a(r)$ and therefore

$$A'(r) = \frac{4}{\pi}\frac{1-a^2}{(1+ar)^2+(a+r)^2}.$$

If $u(z_0) = 1$, then $u(rz_0) > 0$ for $r$ near 1, and by Theorem 5, we have $u(rz_0) \leqslant A(r) =: M_b(r)$. Since in particular $A'(1) = \frac{2}{\pi}\frac{1-a}{1+a}$, by Proposition 4.1, we see that

$$|D_r u(z_0)| \geqslant A'(1) = \frac{2}{\pi}\frac{1-a}{1+a}.$$

Case 2. If $b \in (-1,0)$, then we consider $v = -u$, and by Case 1, we have again $|D_r u(z_0)| \geqslant \frac{2}{\pi}\frac{1-|a}{1+|a|}$. $\square$

PROPOSITION 4.4. *Let $u : \mathbb{D} \to \mathbb{D}$ be a harmonic function such that $u(0) = b$. Assume that $u$ has a continuously extension to the boundary point $z_0 \in \mathbb{T}$, $u(z_0) = c \in \mathbb{T}$ and $a = \tan\frac{|\mathrm{Re}(\overline{c}b)|\pi}{4}$. If $u$ is differentiable at $z_0$, then $|D_r u(z_0)| \geqslant \frac{2}{\pi}\frac{1-|a|}{1+|a|}$.*

*Proof.* Consider the function $v = \mathrm{Re}(\overline{c}u)$. Then function $v$ is real-valued harmonic, $v(z_0) = 1$ and $v(0) = \mathrm{Re}(\overline{c}b)$. Now we can apply Proposition 4.3 on $v$. $\square$

Note here that $a = |a|$ and $P_c b = \langle b, c\rangle c = b$, $|v(0)| \leqslant |b|$ and therefore $s^-(|P_c b|) \leqslant s^-(v(0)) = \frac{1-|a|}{1+|a|} = e^-(|a|)$.

## 5. Appendix 2

In this section we give some additional explanations for the convenience of the reader.

Let $G \subset \mathbb{C}^n$, $f = (f_1, \dots f_m) : G \to \mathbb{C}^m$, $f_i = u_i + iv_i$, $i = 1, \dots, m$, and $G' = \{z' : z \in G\}$. The "Real version" of $f$, denoted by $f_{re}$, is defined on $G'$ by $f_{re}(z') = f(z)' = (u_1, \dots, u_m, v_1, \dots, v_m)(z')$. In the literature the $2m \times 2n$ Jacobian matrix of $f$ at $z_0$ in terms of real coordinates is also denoted by $J_f(z_0')$. Thus the linear operator $f_{re}'(z_0)$ can identified by the matrix $J_f(z_0')$.

The conjugate transpose of an $m \times n$ matrix $\boldsymbol{A}$ is formally defined by

$$\left(\boldsymbol{A}^{\mathrm{H}}\right)_{ij} = \overline{\boldsymbol{A}_{ji}},$$

where the subscript $(i,j)$-th entry, for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, and the overbar denotes a scalar complex conjugate. This definition can also be written as

$$\boldsymbol{A}^{\mathrm{H}} = \left(\overline{\boldsymbol{A}}\right)^{\mathsf{T}} = \overline{\boldsymbol{A}^{\mathsf{T}}}\boldsymbol{A}^{\mathrm{H}} = \left(\overline{\boldsymbol{A}}\right)^{\mathsf{T}} = \overline{\boldsymbol{A}^{\mathsf{T}}},$$

where $\boldsymbol{A}^{\mathsf{T}}$ denotes the transpose and $\overline{\boldsymbol{A}}$ denotes the matrix with complex conjugated entries. Note that $\langle \boldsymbol{A}x, y \rangle_m = \left\langle x, \boldsymbol{A}^{\mathrm{H}} y \right\rangle_n$ holds for any m-by-n matrix $\boldsymbol{A}$, any vector $x \in \mathbb{C}^n$, and any vector $y \in \mathbb{C}^m$. Here, $\langle \cdot, \cdot \rangle_m$ denotes the standard complex inner product on $\mathbb{C}^m$, and similarly for $\langle \cdot, \cdot \rangle_n$.

Each complex $m \times n$ matrix $A$ determines a linear map of $C^n$ to $C^m$. The adjoint of this map corresponds to the conjugate transpose of A: $A^* = \overline{A^T}$, which is called the Hermitian of $A$ (sometimes denoted by $A^H$).

### Trigonometric identities and $s^-$

Tangents and cotangents of sums are given by

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}, \tag{5.1}$$

$$\cot(\alpha \pm \beta) = \frac{\cot\alpha \cot\beta \mp 1}{\cot\beta \pm \cot\alpha} \tag{5.2}$$

Hence $\cot(\alpha \pm \beta) = \dfrac{1 \mp \tan\alpha \tan\beta}{\tan\alpha \pm \tan\beta}$ and in particular since $\tan(\pi/4) = 1$, we find $\cot(\pi(1+\beta)/4) = \dfrac{1 - \tan(\beta\pi/4)}{1 + \tan(\beta\pi/4)}$, and $\tan(\pi(1-\alpha)/4) = \cot(\pi(1+\alpha)/4)$.

Recall that we use notation

$$s^-(x) = \cot\left(\frac{\pi}{4}(x+1)\right), \quad x \in (-1,1).$$

If we set $y = \tan(x\pi/4)$, we have $s^-(x) = \frac{1-y}{1+y}$.

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