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INEQUALITIES AND NUMERICAL RESULTS OF APPROXIMATION FOR BIVARIATE q–BASKAKOV–DURRMEYER TYPE OPERATORS INCLUDING q–IMPROPER INTEGRAL

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Abstract. In this study, we investigate inequalities estimating the error of the approximation of bivariate q-Baskakov-Durrmeyer type operators including q-improper integral. We firstly introduce bivariate q-Baskakov-Durrmeyer type operators including the q-improper integral. We obtain inequalities estimating the error of the approximation for these operators. Later, we introduce generalized Boolean sum (GBS) operators associated to the bivariate q-Baskakov-Durrmeyer type operators including the q-improper integral, and we give an inequality estimating the error of the approximation for the GBS operators. Lastly, we present numerical results of error estimations for certain functions with the help of maple software.

1. Introduction

Agrawal and Thamer [1] defined the following Baskakov-Durrmeyer type operators including the improper integral with the help of Lebesgue integrable functions g defined on $[0,\infty)$:

$$M_n(g;x) = (n-1)\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k-1}(\rho)\, g(\rho)\, d\rho + (1+x)^{-n} g(0). \tag{1}$$

Here

$$p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-(n+k)}.$$

The rate of convergence of these operators was studied by Gupta in [5]. Later, Agrawal and Kumar in [2] introduced a q-type generalization of these operators.

We firstly give the certain notations of the q-calculus before we present this q-type generalization of the Baskakov-Durrmeyer type operators including the q-improper integral. For non-negative integer k, the q-integer $[k]_q$ is defined as follows:

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1 \\ k & q = 1. \end{cases} \tag{2}$$

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Paper JMI-16-36

The q-factorial of $[k]_q$ is defined by

$$[k]_q! = \begin{cases} \prod_{r=1}^{k} [r]_q, & k = 1,2,\ldots, \\ 1, & k = 0. \end{cases} \tag{3}$$

Let n,k be non-negative integers such that $0 \leqslant k \leqslant n$. The q-binomial coefficients are defined as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}. \tag{4}$$

For $A > 0$, let be considered the following q-improper integral depending on A:

$$\int_0^{\infty/A} g(\rho)\, d_q\rho = (1-q) \sum_{n=-\infty}^{\infty} g\left(\frac{q^n}{A}\right) \frac{q^n}{A},$$

which exists finitely when the series $\sum_{n=-\infty}^{\infty} g\left(\frac{q^n}{A}\right)\frac{q^n}{A}$ is convergent. The q-beta function including the q-improper integral depending on A is defined by

$$B_q(t,s) = K(A,t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}} d_q x. \tag{5}$$

Here, the notation of the q-Pochhammer symbol is denoted by

$$(a+b)_q^n = \prod_{j=0}^{n-1} \left(a + q^j b\right). \tag{6}$$

Moreover,

$$K(x,t) = \frac{1}{x+1} x^t \left(1 + \frac{1}{x}\right)_q^t (1+x)_q^{1-t}, \tag{7}$$

and the following recurrence relation [6] holds:

$$K(A,t+1) = q^t K(A,t). \tag{8}$$

For $t,s > 0$, another definition of the q-beta function is defined as follows:

$$B_q(t,s) = \int_0^1 x^{t-1} (1-qx)_q^{s-1} d_q x. \tag{9}$$

The relation between the q-beta function and q-gamma function is known with the following equality:

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}. \tag{10}$$

The comprehensive details about the q-calculus can be found in the reference [13].

Agrawal and Kumar defined in [2] the following q-Baskakov-Durrmeyer type operators including the q-improper integral by

$$M_n^q(g;x) = [n-1]_q \sum_{k=1}^{\infty} p_{n,k}^q(x) \int_0^{\infty/A} q^{k-1} p_{n,k-1}^q(\rho) g(\rho) d_q\rho + p_{n.0}^q(x)\, g(0). \tag{11}$$

Here, g is q-improper integrable function, $x \in [0,\infty)$, $q \in (0,1)$ and

$$p_{n,k}^q(x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{k(k-1)/2} \frac{x^k}{(1+x)_q^{(n+k)}} \tag{12}$$

such that

$$\sum_{k=0}^{\infty} p_{n,k}^q(x) = 1. \tag{13}$$

They proved a basic convergence theorem by obtaining a recurrence relation for these operators. Moreover, they estimated the rate of convergence of these operators, and they obtained the weighted approximation results.

In the recent years, many researchers have contributed with the investigations on the varied onevariate and bivariate operators, for instance, readers can see the references [3, 4, 7, 11, 12, 14, 15, 16, 17, 18].

In this study, we purpose to investigate the error estimations of bivariate q-Baskakov-Durrmeyer type operators including the q-improper integral. In Section 2, we initially introduce bivariate q-Baskakov-Durrmeyer type operators including the q-improper integral, and we give some auxiliary results for these operators. In Section 3, we present inequalities estimating the error of the approximation of these operators by means of the complete modulus of continuity, the partial modulus of continuity and the Lipschitz class functions. In Section 4, we additionaly construct generalized Boolean sum operators of these operators and give an inequality estimating the error of the approximation of them by means of the mixed modulus of smoothness of Bögel continuous functions. In Section 5, we present the applications including some numerical results of error estimations for certain functions. In the last section, we discuss the conclusions.

2. Definition of bivariate operators

In this section, we introduce bivariate q-Baskakov-Durrmeyer type operators including the q-improper integral, and give some auxiliary results for these operators.

DEFINITION 1. Let $\mathbb{R}_+ = [0,\infty)$, $A_1, A_2 > 0$, $q_1, q_2 \in (0,1)$ and g be a bivariate continuous and bounded function defined on $\mathbb{R}_+ \times \mathbb{R}_+$. For all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$ and

$n_1, n_2 \in \mathbb{N}$, we define bivariate q-Baskakov-Durrmeyer type operators including the q-improper integral as follows:

$$M_{n_1,n_2}^{q_1,q_2}(g;x,y) = [n_1-1]_{q_1}[n_2-1]_{q_2}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}P_{n_1,k}^{q_1}(x)P_{n_2,j}^{q_2}(y)$$
$$\times\int_0^{\frac{\infty}{A_1}}\int_0^{\frac{\infty}{A_2}} q_1^{k-1}q_2^{j-1}P_{n_1,k-1}^{q_1}(\rho)P_{n_2,j-1}^{q_2}(\sigma)g(\rho,\sigma)\,d_{q_2}\sigma\, d_{q_1}\rho$$
$$+P_{n_1,0}^{q_1}(x)P_{n_2,0}^{q_2}(y)g(0,0).$$

Here,

$$P_{n_1,k}^{q_1}(x) = \begin{bmatrix} n_1+k-1 \\ k \end{bmatrix}_{q_1} q_1^{\frac{k(k-1)}{2}}\frac{x^k}{(1+x)_{q_1}^{n_1+k}},$$
$$P_{n_2,j}^{q_2}(y) = \begin{bmatrix} n_2+j-1 \\ j \end{bmatrix}_{q_2} q_2^{\frac{j(j-1)}{2}}\frac{y^j}{(1+y)_{q_2}^{n_2+j}}.$$

The bivariate q-Baskakov-Durrmeyer type operators including the q-improper integral are tensor product kind linear positive operators.

In the following lemma, we give equalities equalvalent to the definition of the bivariate operators $M_{n_1,n_2}^{q_1,q_2}$.

LEMMA 1. *We have the following equalities for the bivariate operators* $M_{n_1,n_2}^{q_1,q_2}$:

$$i)\ M_{n_1,n_2}^{q_1,q_2}(g;x,y) = M_{n_1}^{x}\left(M_{n_2}^{y}(g;q_2);q_1\right) + P_{n_1,0}^{q_1}(x)P_{n_2,0}^{q_2}(y)g(0,0),$$

$$ii)\ M_{n_1,n_2}^{q_1,q_2}(g;x,y) = M_{n_2}^{y}(M_{n_1}^{x}(g;q_1);q_2) + P_{n_1,0}^{q_1}(x)P_{n_2,0}^{q_2}(y)g(0,0).$$

Here

$$M_{n_1}^{x}(g;q_1) := [n_1-1]_{q_1}\sum_{k=1}^{\infty}P_{n_1,k}^{q_1}(x)\int_0^{\frac{\infty}{A_1}} q_1^{k-1}P_{n_1,k-1}^{q_1}(\rho)g(\rho,\sigma)\,d_{q_1}\rho,$$

$$M_{n_2}^{y}(g;q_2) := [n_2-1]_{q_2}\sum_{j=1}^{\infty}P_{n_2,j}^{q_2}(y)\int_0^{\frac{\infty}{A_2}} q_2{}^{j-1}P_{n_2,j-1}^{q_2}(\sigma)g(\rho,\sigma)\,d_{q_2}\sigma.$$

LEMMA 2. *We have the following images at the bivariate test functions for the bivariate operators* $M_{n_1,n_2}^{q_1,q_2}$:

$$i)\ M_{n_1,n_2}^{q_1,q_2}(1;x,y) = 1,$$

$$ii)\ M_{n_1,n_2}^{q_1,q_2}(\rho;x,y) = q_1x + \frac{x}{[n_1-2]_{q_1}} + \frac{x}{q_1[n_1-2]_{q_1}},$$

$$iii)\ M_{n_1,n_2}^{q_1,q_2}(\sigma;x,y) = q_2y + \frac{y}{[n_2-2]_{q_2}} + \frac{y}{q_2[n_2-2]_{q_2}},$$

$$iv)\ M_{n_1,n_2}^{q_1,q_2}\left(\rho^2;x,y\right) = q_1^2x^2 + \frac{\left(q_1^2+q_1+1\right)x^2}{q_1\left[n_1-3\right]_{q_1}} + \frac{\left(q_1^2+q_1+1\right)x^2}{q_1\left[n_1-2\right]_{q_1}} + \frac{\left(q_1^2+q_1+1\right)^2x^2}{\left[n_1-2\right]_{q_1}\left[n_1-3\right]_{q_1}} + \frac{\left(q_1+1\right)x}{q_1^2\left[n_1-3\right]_{q_1}} + \frac{\left(q_1+1\right)x}{q_1^4\left[n_1-2\right]_{q_1}\left[n_1-3\right]_{q_1}},$$

$$v)\ M_{n_1,n_2}^{q_1,q_2}\left(\sigma^2;x,y\right) = q_2^2y^2 + \frac{\left(q_2^2+q_2+1\right)y^2}{q_2\left[n_2-3\right]_{q_2}} + \frac{\left(q_2^2+q_2+1\right)y^2}{q_2\left[n_2-2\right]_{q_2}} + \frac{\left(q_2^2+q_2+1\right)^2y^2}{\left[n_2-2\right]_{q_2}\left[n_2-3\right]_{q_2}} + \frac{\left(q_2+1\right)y}{q_2^2\left[n_2-3\right]_{q_2}} + \frac{\left(q_2+1\right)y}{q_2^4\left[n_2-2\right]_{q_2}\left[n_2-3\right]_{q_2}}.$$

Proof. (i) By (13) and Lemma 1, (i) is obvious.

(ii) By Lemma 1 (i), we can write

$$\begin{aligned} M_{n_1,n_2}^{q_1,q_2}\left(\rho;x,y\right) &= M_{n_1}^{x}\left(\rho M_{n_2}^{y}(1;q_2);q_1\right) \\ &= \left[n_1-1\right]_{q_1}\sum_{k=1}^{\infty}P_{n_1,k}^{q_1}(x)\int_0^{\frac{\infty}{A_1}} q_1^{k-1}P_{n_1,k-1}^{q_1}(\rho)\rho\, d_{q_1}\rho \\ &\quad\times\left[n_2-1\right]_{q_2}\sum_{j=1}^{\infty}P_{n_2,j}^{q_2}(y)\int_0^{\frac{\infty}{A_2}} q_2{}^{j-1}P_{n_2,j-1}^{q_2}(\sigma)\, d_{q_2}\sigma \\ &= \frac{\left[n_1\right]_{q_1}x}{q_1\left[n_1-2\right]_{q_1}}\sum_{k=0}^{\infty}P_{n_1+1,k}^{q_1}(x). \end{aligned}$$

In the last equality, by (13) and considering that $\left[n_1\right]_{q_1} = q_1^2\left[n_1-2\right]_{q_1} + q_1 + 1$, the proof of (ii) is completed.

(iii) By Lemma 1 (i), (5) and (8), after simple calculation, we can write

$$\begin{aligned} M_{n_1,n_2}^{q_1,q_2}\left(\rho;x,y\right) &= M_{n_1}^{x}\left(\rho^2 M_{n_2}^{y}(1;q_2);q_1\right) \\ &= \left[n_1-1\right]_{q_1}\sum_{k=1}^{\infty}P_{n_1,k}^{q_1}(x)\int_0^{\frac{\infty}{A_1}} q_1^{k-1}P_{n_1,k-1}^{q_1}(\rho)\rho^2\, d_{q_1}\rho \\ &\quad\times\left[n_2-1\right]_{q_2}\sum_{j=1}^{\infty}P_{n_2,j}^{q_2}(y)\int_0^{\frac{\infty}{A_2}} q_2{}^{j-1}P_{n_2,j-1}^{q_2}(\sigma)\, d_{q_2}\sigma \end{aligned}$$

$$= \frac{[n_1]_{q_1}[n_1+1]_{q_1}x^2}{q_1^4[n_1-2]_{q_1}[n_1-3]_{q_1}}\sum_{k=0}^{\infty}P_{n_1+2,k}^{q_1}(x)$$
$$+\frac{[2]_{q_1}[n_1]_{q_1}x}{q_1^3[n_1-2]_{q_1}[n_1-3]_{q_1}}\sum_{k=0}^{\infty}P_{n_1,k}^{q_1}(x).$$

In the last equality, by (13) and considering that

$$[n_1]_{q_1} = q_1^2[n_1-2]_{q_1}+q_1+1$$

and

$$\begin{aligned}[n_1]_{q_1}[n_1+1]_{q_1} &= q_1^6[n_1-2]_{q_1}[n_1-3]_{q_1}\\ &+q_1^3\left(q_1^2+q_1+1\right)[n_1-2]_{q_1}\\ &+q_1^3\left(q_1^2+q_1+1\right)[n_1-3]_{q_1}\\ &+\left(q_1^2+q_1+1\right)^2,\end{aligned}$$

the proof of (iii) is completed.

The proof of (iv) and (v) are similarly obtained to (ii) and (iii) by replacing q_1 and n_1 with q_2 and n_2 and by considering Lemma 1 (ii), respectively. □

3. Some inequalities

In this part, we estimate rate of convergence of the bivariate q-Baskakov-Durrmeyer type operators including the q-improper inregral.

Let A be compact subset of $\mathbb{R}^2$ and $C(A)$ denote the space of all real-valued continuous functions defined on A. For any $g \in C(A)$, $\delta_1 > 0$, $\delta_2 > 0$ and for each (ρ,σ), $(x,y) \in A$, the complete modulus of continuity of g is defined by

$$\omega(g;\delta_1,\delta_2) = sup\{|g(\rho,\sigma)-g(x,y)| : |\rho-x| \leqslant \delta_1, |\sigma-y| \leqslant \delta_2\},$$

which satisfies the following limit:

$$\lim_{\delta_1,\delta_2\to 0^+}\omega(g;\delta_1,\delta_2) = 0,$$

and the following inequality:

$$|g(\rho,\sigma)-g(x,y)| \leqslant \omega(g;\delta_1,\delta_2)\left(1+\frac{|\rho-x|}{\delta_1}\right)\left(1+\frac{|\sigma-y|}{\delta_2}\right). \tag{14}$$

Let $I = I_1 \times I_2$ be a rectangular region of $\mathbb{R}^2$ such that $I_i = [0,r_i]$, $i = 1,2$.

THEOREM 1. *Let* $\{q_{1,n_1}\}$ *and* $\{q_{2,n_2}\}$ *be any sequences such that* $q_{1,n_1}, q_{2,n_2} \in (0,1)$ *satisfying*

$$\lim_{n_1\to\infty} q_{1,n_1} = 1 = \lim_{n_2\to\infty} q_{2,n_2}. \tag{15}$$

If $g \in C(I)$ *then the following inequality holds:*

$$\left|M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(g;x,y)-g(x,y)\right| \leqslant 4\omega\left(g;\sqrt{\gamma_{n_1}^{x,q_{1,n_1}}},\sqrt{\gamma_{n_2}^{y,q_{2,n_2}}}\right).$$

Here

$$\begin{aligned}\gamma_{n_1}^{x,q_{1,n_1}} &= (q_{1,n_1}-1)^2x^2+\frac{\left(q_{1,n_1}^2+q_{1,n_1}+1\right)x^2}{q_{1,n_1}[n_1-3]_{q_{1,n_1}}}\\ &+\frac{\left(q_{1,n_1}^2-q_{1,n_1}-1\right)x^2}{q_{1,n_1}[n_1-2]_{q_{1,n_1}}}+\frac{\left(q_{1,n_1}^2+q_{1,n_1}+1\right)^2x^2}{[n_1-2]_{q_{1,n_1}}[n_1-3]_{q_{1,n_1}}}\\ &+\frac{(q_{1,n_1}+1)x}{q_{1,n_1}^2[n_1-3]_{q_{1,n_1}}}+\frac{(q_{1,n_1}+1)x}{q_{1,n_1}^4[n_1-2]_{q_{1,n_1}}[n_1-3]_{q_{1,n_1}}},\end{aligned} \tag{16}$$

$$\begin{aligned}\gamma_{n_2}^{y,q_{2,n_2}} &= (q_{2,n_2}-1)^2y^2+\frac{\left(q_{2,n_2}^2+q_{2,n_2}+1\right)y^2}{q_{2,n_2}[n_2-3]_{q_{2,n_2}}}\\ &+\frac{\left(q_{2,n_2}^2-q_{2,n_2}-1\right)y^2}{q_{2,n_2}[n_2-2]_{q_{2,n_2}}}+\frac{\left(q_{2,n_2}^2+q_{2,n_2}+1\right)^2y^2}{[n_2-2]_{q_{2,n_2}}[n_2-3]_{q_{2,n_2}}}\\ &+\frac{(q_{2,n_2}+1)y}{q_{2,n_2}^2[n_2-3]_{q_{2,n_2}}}+\frac{(q_{2,n_2}+1)y}{q_{2,n_2}^4[n_2-2]_{q_{2,n_2}}[n_2-3]_{q_{2,n_2}}}.\end{aligned} \tag{17}$$

Proof. Applying the operators $M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}$ to (14) and considering the linearity and the positivity of these operators, we can write

$$\begin{aligned}\left|M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(g;x,y)-g(x,y)\right| &\leqslant M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(|g(\rho,\sigma)-g(x,y)|;x,y)\\ &\leqslant \omega(g;\delta_1,\delta_2)\Big\{M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)\\ &\quad+\frac{1}{\delta_1}M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(|\rho-x|;x,y)\\ &\quad+\frac{1}{\delta_2}M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(|\sigma-y|;x,y)\\ &\quad+\frac{1}{\delta_1\delta_2}M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(|\rho-x||\sigma-y|;x,y)\Big\}.\end{aligned}$$

By Lemma 1 and applying Cauchy-Schwarz inequality to the last inequality, we obtain

$$\begin{aligned}\left|M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(g;x,y)-g(x,y)\right| &\leqslant \omega(g;\delta_1,\delta_2)\Big\{M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)\\ &\quad+\frac{1}{\delta_1}\left(M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\rho-x)^2;x,y\right)\right)^{1/2}\end{aligned}$$

$$+\frac{1}{\delta_2}\left(M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\sigma-y)^2;x,y\right)\right)^{1/2}$$
$$\left.+\frac{1}{\delta_1\delta_2}\left(M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\rho-x)^2(\sigma-y)^2;x,y\right)\right)^{1/2}\right\}$$

Taking Lemma 2 into account, after simple calculation, selecting $\delta_1=\sqrt{\gamma_{n_1}^{x,q_{1,n_1}}}$ and $\delta_2=\sqrt{\gamma_{n_2}^{y,q_{2,n_2}}}$, we complete the proof of the theorem. □

For any $g\in C(A)$, $\delta_1>0$, $\delta_2>0$ and for each (ρ,σ), (x,σ), $(\rho,y)\in A$, the partial modulus of continuity with respect to x and y are defined by

$$\omega_1(g;\delta_1)=sup\{|g(\rho,\sigma)-g(x,\sigma)|:|\rho-x|\leqslant\delta_1\},$$

$$\omega_2(g;\delta_2)=sup\{|g(\rho,\sigma)-g(\rho,y)|:|\sigma-y|\leqslant\delta_2\},$$

respectively. It is clear that $\omega_1(g;\delta_1)$, $\omega_2(g;\delta_2)$ satisfy the properties of the usual modulus of continuity, i.e., we have the following inequalities:

$$|g(\rho,\sigma)-g(x,\sigma)|\leqslant\omega_1(g;\delta_1)\left(1+\frac{(\rho-x)^2}{\delta_1}\right),$$

$$|g(\rho,\sigma)-g(\rho,y)|\leqslant\omega_2(g;\delta_2)\left(1+\frac{(\sigma-y)^2}{\delta_2}\right).$$

THEOREM 2. *Let* $\{q_{1,n_1}\}$ *and* $\{q_{2,n_2}\}$ *be any sequences such that* $q_{1,n_1},q_{2,n_2}\in(0,1)$ *satisfying the condition given in* (15). *If any* $g\in C(I)$ *then we have*

$$\left|M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(g;x,y)-f(x,y)\right|\leqslant 2\omega_1\left(g;\gamma_{n_1}^{x,q_{1,n_1}}\right)+2\omega_2\left(g;\gamma_{n_2}^{y,q_{2,n_2}}\right),$$

where $\gamma_{n_1}^{x,q_{1,n_1}}$ *and* $\gamma_{n_2}^{y,q_{2,n_2}}$ *are as in* (16) *and* (17)*, respectively.*

Proof. Considering the definition of the partial modulus of continuity, considering the linearity and the positivity of the operators $M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}$, we can write

$$\begin{aligned}\left|M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(g;x,y)-g(x,y)\right| &\leqslant M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(|g(\rho,\sigma)-g(x,y)|;x,y)\\ &\leqslant M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(|g(\rho,\sigma)-g(x,\sigma)|;x,y)\\ &\quad+M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(|g(x,\sigma)-g(x,y)|;x,y)\\ &\leqslant\omega_1(g;\delta_1)\left(1+\frac{1}{\delta_1}M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\rho-x)^2;x,y\right)\right)\\ &\quad+\omega_2(g;\delta_1)\left(1+\frac{1}{\delta_2}M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\sigma-y)^2;x,y\right)\right).\end{aligned}$$

By considering Lemma 2 and by selecting $\delta_1 = \gamma_{n_1}^{x,q_{1,n_1}}$ and $\delta_2 = \gamma_{n_2}^{y,q_{2,n_2}}$, the proof of the theorem is completed. □

For any function $g \in C(A)$, for each $(\rho,\sigma),(x,y) \in A$ and $0 < \theta_1, \theta_2 \leqslant 1$, if there exists a real number $M_g > 0$ such that g satisfies the following inequality:

$$|g(\rho,\sigma) - g(x,y)| \leqslant M_g |\rho - x|^{\theta_1} |\sigma - y|^{\theta_1},$$

then g is called a function of the Lipschitz class denoted by $Lip_{M_g}(\theta_1,\theta_2)$

THEOREM 3. *Let $\{q_{1,n_1}\}$ and $\{q_{2,n_2}\}$ be any sequences such that $q_{1,n_1}, q_{2,n_2} \in (0,1)$ satisfying the condition given in (15). If any $g \in Lip_{M_g}(\theta_1,\theta_2)$ then the following inequality holds:*

$$\left|M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(g;x,y) - g(x,y)\right| \leqslant M_g \left(\sqrt{\gamma_{n_1}^{x,q_{1,n_1}}}\right)^{\theta_1} \left(\sqrt{\gamma_{n_2}^{y,q_{2,n_2}}}\right)^{\theta_2}, \; M_g > 0.$$

Here $\gamma_{n_1}^{x,q_{1,n_1}}$ and $\gamma_{n_2}^{y,q_{2,n_2}}$ are as in (16) and (17), respectively.

Proof. Since $g \in Lip_{M_g}(\theta_1,\theta_2)$, by Lemma 1 and considering the linearity and the positivity of the operators $M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}$, we can write

$$\begin{aligned}\left|M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(g;x,y) - g(x,y)\right| &\leqslant M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(|g(\rho,\sigma) - g(x,y)|;x,y) \\ &\leqslant M_g M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left(|\rho - x|^{\theta_1}|\sigma - y|^{\theta_2};x,y\right) \\ &= M_g M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left(|\rho - x|^{\theta_1};x,y\right) \\ &\quad \times M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left(|\sigma - y|^{\theta_2};x,y\right).\end{aligned}$$

By Lemma 1 and Lemma 2, applying the Hölder inequality for $u_i = \frac{2}{\theta_i}, v_i = \frac{2}{2-\theta_i}$ for $i = 1,2$ to the last inequality, respectively, we obtain

$$\begin{aligned}\left|M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(g;x,y) - g(x,y)\right| &\leqslant M_g \left(M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\rho - x)^2;x,y\right)\right)^{\theta_1/2} \\ &\quad \times \left(M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)\right)^{2/2-\theta_1} \\ &\quad \times \left(M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\sigma - y)^2;x,y\right)\right)^{\theta_2/2} \\ &\quad \times \left(M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)\right)^{2/2-\theta_2} \\ &\leqslant M_g \left(\sqrt{\gamma_{n_1}^{x,q_{1,n_1}}}\right)^{\theta_1} \left(\sqrt{\gamma_{n_2}^{y,q_{2,n_2}}}\right)^{\theta_2},\end{aligned}$$

which completes proof. □

4. GBS operators

Bögel defined Bögel-continuous and Bögel-bounded functions. We recall the basic notations given by Bögel. The details can be found in the references [8, 9, 10].

Let A be compact subset of $\mathbb{R}^2$. A function $g : A \to \mathbb{R}$ is called Bögel-continuous function at $(\rho,\sigma) \in A$ if

$$\Delta_{(x,y)} g\left[\rho,\sigma;x,y\right] = 0,$$

where $\Delta_{(x,y)} g\left[\rho,\sigma;x,y\right]$ denotes the mixed diference defined by

$$\Delta_{(x,y)} g\left[\rho,\sigma;x,y\right] = g\left(x,y\right) - g\left(x,\sigma\right) - g\left(\rho,y\right) + g\left(\rho,\sigma\right).$$

Let A be a subset of $\mathbb{R}^2$. A function $g : A \to \mathbb{R}$ is Bögel-bounded function on A if there exists $M > 0$ such that

$$\left|\Delta_{(x,y)} g\left[\rho,\sigma;x,y\right]\right| \leqslant M,$$

for every $(\rho,\sigma), (x,y) \in A$.

Let A be a compact subset of $\mathbb{R}^2$, then each Bögel-continuous function is a Bögel-bounded function and $C_b(A)$ denote the space of all the real valued Bögel-continuous functions defined on A endowed with the norm

$$\|g\|_B = sup\left\{\left|\Delta_{(x,y)} g\left[\rho,\sigma;x,y\right]\right| : (x,y),(\rho,\sigma) \in A\right\}.$$

It is obvious that $C(A) \subset C_b(A)$.

DEFINITION 2. We define generalized Boolean sum (GBS) operators of the bivariate q-Baskakov-Durrmeyer type operators including q-improper integral by

$$D_{n_1,n_2}^{q_1,q_2}(g;x,y) = M_{n_1,n_2}^{q_1,q_2}\left(g\left(\rho,y\right) + g\left(x,\sigma\right) - g\left(\rho,\sigma\right);x,y\right),$$

for all (ρ,σ), $(x,y) \in I$ and $g \in C(I)$.

The mixed modulus of smoothness of $g \in C_b(A)$ is defined by

$$\omega_{\text{mixed}}(g;\delta_1,\delta_2) = sup\left\{\left|\Delta_{(x,y)} g\left[\rho,\sigma;x,y\right]\right| : |\rho - x| \leqslant \delta_1, |\sigma - y| \leqslant \delta_2\right\},$$

for all (ρ,σ), $(x,y) \in A$ and $\delta_1 > 0$, $\delta_2 > 0$, which satisfies the following inequality:

$$\left|\Delta_{(x,y)} g\left[\rho,\sigma;x,y\right]\right| \leqslant \left(1 + \frac{|\rho - x|}{\delta_1}\right)\left(1 + \frac{|\sigma - y|}{\delta_2}\right)\omega_{\text{mixed}}(g;\delta_1,\delta_2). \tag{18}$$

THEOREM 4. *Let $\{q_{1,n_1}\}$ and $\{q_{2,n_2}\}$ be any sequences such that $q_{1,n_1}, q_{2,n_2} \in (0,1)$ satisfying the condition given in (15). If $g \in C_b(I)$, then, for all $(x,y) \in I$, the following inequality holds:*

$$\left|D_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(g;x,y) - g(x,y)\right| \leqslant 4\omega_{mixed}\left(g;\sqrt{\gamma_{n_1}^{x,q_{1,n_1}}},\sqrt{\gamma_{n_2}^{y,q_{2,n_2}}}\right),$$

where $\gamma_{n_1}^{x,q_{1,n_1}}$ and $\gamma_{n_2}^{y,q_{2,n_2}}$ are as in (16) and (17).

Proof. Considering the definition of the mixed difference $\Delta g[\rho,\sigma;x,y]$ and (18), taking the Cauchy-Schwarz inequality into account, we can write

$$\begin{aligned}\left|D_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(g(\rho,\sigma);x,y)-g(x,y)\right| &\leqslant M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left(\left|\Delta_{(x,y)}g[\rho,\sigma;x,y]\right|;x,y\right)\\ &\leqslant\left\{1+\frac{1}{\delta_1}\left(M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\rho-x)^2;x,y\right)\right)^{1/2}\right.\\ &\quad+\frac{1}{\delta_2}\left(M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\rho-y)^2;x,y\right)\right)^{1/2}\\ &\quad+\frac{1}{\delta_1\delta_2}\left(M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\rho-x)^2;x,y\right)\right)^{1/2}\\ &\quad\left.\times\left(M_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\sigma-y)^2;x,y\right)\right)^{1/2}\right\}\\ &\quad\times\omega_{\text{mixed}}(g;\delta_1,\delta_2)\end{aligned}$$

By considering Lemma 2 and by choosing $\delta_1=\sqrt{\gamma_{n_1}^{x,q_{1,n_1}}}$ and $\delta_2=\sqrt{\gamma_{n_2}^{y,q_{2,n_2}}}$, the proof of the theorem is completed. □

5. Numerical results

In this section, we present applications including some numerical results of the error estimations of the bivariate q-Baskakov-Durrmeyer type operators including the q-improper integral and associated GBS operators for some certain functions with the help of maple software.

Let us choose $q_n=q_{1,n_1}=q_{2,n_2}=\frac{n-1}{n}$ and $\theta=\theta_1=\theta_2$ such that $0<\theta\leqslant 1$ and let $E\left(M_{n_1,n_2}^{q_1,q_2}(g),g\right)$ and $E\left(D_{n_1,n_2}^{q_1,q_2}(g),g\right)$ denote the errors of the approximation of $M_{n_1,n_2}^{q_1,q_2}(g)$ to g and of $D_{n_1,n_2}^{q_1,q_2}(g)$ to g, respectively.

EXAMPLE 1. Let $r_1=r_2=4$ then $I_1=I_2$ and $I=[0,4]\times[0,4]$. We consider $g_1(x,y)=xy$ for each $(x,y)\in[0,4]\times[0,4]$. It is obvious that $g_1\in Lip_{M_g}(\theta,\theta)\subset C(I)$.

In Table 1-5, we have some numerical results of the error estimations for the bivariate q-Baskakov-Durrmeyer type operators and associated GBS operators.

Table 1: The approximation error of $M_{n,n}^{q_n,q_n}(g_1)$ to g_1 by means of the complete modulus of continuity.

Indis Number	$n=1\times10^6$	$n=1\times10^5$	$n=1\times10^4$
$\delta_1=\delta_2$	$.1139029452\times10^{-3}$	$.1139085459\times10^{-2}$	$.1139645674\times10^{-1}$
$\omega(g_1;\delta_1,\delta_2)$	$.9112105877\times10^{-3}$	$.9111386158\times10^{-2}$	$.9104177469\times10^{-1}$
$E_1\left(M_{n,n}^{q_n;q_n}(g_1),g_1\right)$	$.3644842351\times10^{-2}$	$.3644554463\times10^{-1}$	$.3641670988$

Table 2: The approximation error of $M_{n,n}^{q_n,q_n}(g_1)$ to g_1 by means of the partial modulus of continuties.

Indis Number	$n=1\times10^6$	$n=1\times10^5$	$n=1\times10^4$
$\delta_1=\delta_2$	$.1139029452\times10^{-3}$	$.1139085459\times10^{-2}$	$.1139645674\times10^{-1}$
$\omega_1(g_1;\delta_1)=\omega_2(g_1;\delta_2)$	$.1139029452\times10^{-1}$	$.4556341836\times10^{-2}$	$.4558582696\times10^{-1}$
$E_2\left(M_{n,n}^{q_n,q_n}(g_1),g_1\right)$	$.4556117808\times10^{-1}$	$.1822536734\times10^{-1}$	$.1823433078$

Table 3: The approximation error of $M_{n,n}^{q_n,q_n}(g_1)$ to g_1 by means of the Lipschitz functions for $\theta=0.1\times10^{-5}$

Indis Number	$n=1\times10^6$	$n=1\times10^5$	$n=1\times10^4$
$\delta_1=\delta_2$	$.6902146233\times10^{-2}$	$.2182668528\times10^{-1}$	$.6902780895\times10^{-1}$
M_{g_1}	$.5517007928\times10^{-1}$	$.1741384101$	$.5474605603$
$E_3\left(M_{n,n}^{q_n,q_n}(g_1),g_1\right)$	$.5516953024\times10^{-1}$	$.1741370781$	$.5474576333$

Table 4: The approximation error of $M_{n,n}^{q_n,q_n}(g_1)$ to g_1 by means of the Lipschitz functions for $n=1\times10^6$

θ	$\theta=0.1\times10^{-5}$	$\theta=0.1\times10^{-4}$	$\theta=0.1\times10^{-3}$
$\delta_1=\delta_2$	$.6902146233\times10^{-2}$	$.6902146233\times10^{-2}$	$.1139029452\times10^{-3}$
M_{g_1}	$.5517007928\times10^{-1}$	$.5517502090\times10^{-1}$	$.9128668793\times10^{-3}$
$E_4\left(M_{n,n}^{q_n,q_n}(g_1),g_1\right)$	$.5516953024\times10^{-1}$	$.5516953024\times10^{-1}$	$.9112105875\times10^{-3}$

Table 5: The approximation error of $D_{n,n}^{q_n,q_n}(g_1)$ to g_1 by means of the mixed modulus of continuity

Indis Number	$n=1\times10^6$	$n=1\times10^5$	$n=1\times10^4$
$\delta_1=\delta_2$	$.1139029452\times10^{-3}$	$.1139085459\times10^{-2}$	$.1139645674\times10^{-1}$
$\omega_{mixed}(g_1;\delta_1,\delta_2)$	$.1297388093\times10^{-7}$	$.1297515683\times10^{-5}$	$.1298792262\times10^{-3}$
$E_5\left(D_{n,n}^{q_n,q_n}(g_1),g_1\right)$	$.5189552372\times10^{-7}$	$.5190062732\times10^{-5}$	$.5195169048\times10^{-3}$

EXAMPLE 2. Let $r_1=r_2=5$ then $I_1=I_2$ and $I=[0,5]\times[0,5]$. We consider $g_2(x,y)=x\sin y$ for each $(x,y)\in[0,5]\times[0,5]$. It is clear that $g_2\in C(I)$ but $g_2\notin Lip_{M_g}(\theta,\theta)$ for $0<\theta\leqslant1$, i.e., there does not exist $M_{g_2}>0$ such that

$$|g_2(\rho,\sigma)-g_2(x,y)|\leqslant M_{g_2}|\rho-x|^{\theta}|\sigma-y|^{\theta}$$

for each $(\rho,\sigma),(x,y)\in[0,5]\times[0,5]$ and $0<\theta\leqslant1$. The error estimation of q-Baskakov-Durrmeyer type operators for g_2 can not be processed by mean of functions of Lipschitz class.

Therefore, in Table 6-8, we possess some other numerical results of the error estimations of the bivariate q-Baskakov-Durrmeyer type operators associated GBS operators.

Table 6: The approximation error of $M_{n,n}^{q_n,q_n}(g_2)$ to g_2 by means of the complete modulus of continuity

Indis Number	$n = 1\times10^6$	$n = 1\times10^5$	$n = 1\times10^4$
$\delta_1 = \delta_2$	$.1740183984\times10^{-3}$	$.1740270443\times10^{-2}$	$.1741135257\times10^{-1}$
$\omega(g_2;\delta_1,\delta_2)$	$.8873084\times10^{-3}$	$.88721870\times10^{-2}$	$.886309929\times10^{-1}$
$E_6\left(M_{n,n}^{q_n,q_n}(g_2),g_2\right)$	$.35492336\times10^{-2}$	$.354887480\times10^{-1}$	$.3545239716$

Table 7: The approximation error of $M_{n,n}^{q_n,q_n}(g_2)$ to g_2 by means of the partial modulus of continuities

Indis Number	$n = 1\times10^5$	$n = 1\times10^4$	$n = 1\times10^3$
$\delta_1 = \delta_2$	$.1740183984\times10^{-3}$	$.1740270443\times10^{-2}$	$.1741135257\times10^{-1}$
$\omega_1(g_2;\delta_1) = \omega_2(g_2;\delta_2)$	$.1740183984\times10^{-3}$	$.1740270443\times10^{-2}$	$.1741135257\times10^{-1}$
$E_7\left(M_{n,n}^{q_n,q_n}(g_2),g_2\right)$	$.6960735936\times10^{-3}$	$.6961081772\times10^{-2}$	$.6964541028\times10^{-1}$

Table 8: The approximation error of $D_{n,n}^{q_n,q_n}(g_2)$ to g_2 by means of the mixed modulus of continuity

Indis Number	$n = 1\times10^5$	$n = 1\times10^4$	$n = 1\times10^3$
$\delta_1 = \delta_2$	$.1740183984\times10^{-3}$	$.1740270443\times10^{-2}$	$.1741135257\times10^{-1}$
$\omega_{mixed}(g_2;\delta_1,\delta_2)$	$.3028240283\times10^{-7}$	$.3028540062\times10^{-5}$	$.3031513765\times10^{-3}$
$E_8\left(D_{n,n}^{q_n,q_n}(g_2),g_2\right)$	$.1211296113\times10^{-6}$	$.1211416025\times10^{-4}$	$.1212605506\times10^{-2}$

Hereby, Example 1 and 2 concretely illustrate that the approximation of $M_{n,n}^{q_n,q_n}$ and $D_{n,n}^{q_n,q_n}$ becomes better for increasing value of n.

6. Conclusion

Let $\{q_{1,n_1}\}$ and $\{q_{2,n_2}\}$ be any sequences satisfying the condition (15). Then, we have

$$\lim_{n_1\to\infty}\gamma_{n_1}^{x,q_{1,n_1}}=0=\lim_{n_2\to\infty}\gamma_{n_2}^{y,q_{2,n_2}},$$

for each $x\in I_1$ and $y\in I_2$

Therefore, all the results in this study give us the degrees of the approximation of the bivariate q-Baskakov-Durrmeyer operators including the q-improper integral and pertaining to the GBS operators in different respects.

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