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FEKETE–SZEGÖ TYPE INEQUALITIES FOR CLASSES OF ANALYTIC FUNCTIONS DEFINED BY USING THE MODIFIED DZIOK–SRIVASTAVA AND THE OWA–SRIVASTAVA FRACTIONAL CALCULUS OPERATORS

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Abstract. By making use of the operator $\mathscr{K}_{\lambda_1,\lambda_2}^{m,r,s}f(z)$ which was previously defined as a generalization of Dziok-Srivastava operator [19, 17], the new class $S^*(\phi,m,r,s,\lambda_1,\lambda_2)$ was introduced and sharp upper bounds of $|a_3-\mu a_2^2|$ for the functions belonging to it were determined. Furthermore, Fekete-Szegö inequalities for certain classes of functions defined through fractional derivatives were also solved out in the sight of Owa-Srivastava fractional calculus operators.

1. Introduction

Historically, Fekete and Szegö had proposed a new inequality for the coefficients of univalent analytic functions on 1933. Interstingly, several researchers later on have obtained the Fekete-Szegö inequality for functions belonging to several classes of univalent functions [1, 2, 3, 4] and bi-univalent functions [5, 6, 7]. See also [8, 9, 10]. For instance, Srivastava et al. [11] have obtained the Fekete-Szegö inequality for a subclass of q-starlike functions with respect to symmetrical points. See also [12, 13, 14, 15, 16].

Recently, there has been a rising interest in finding the way to correlating Fekete-Szegö inequality with hypergeometric functions, which is the main theme of this study. The hypergeometric functions was found in 1655 by John Wallis and its importance is stemmed from its applications in many subjects such as, numerical analysis, dynamical system and mathematical physics.

Let $\mathscr{A}$ be the class of analytic functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k; \qquad (z\in\mathbb{U}=\{z\in C:|z|<1\}), \tag{1.1}$$

and $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of univalent functions in $\mathbb{U}$. A function $f(z)$ is said to be in the class $\mathscr{S}^*$ of starlike functions of order zero in $\mathbb{U}$, if $\Re\{\frac{zf'(z)}{f(z)}\}>0$ for $z\in\mathbb{U}$.

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For two analytic functions $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$ and $g(z)=z+\sum_{k=2}^{\infty}b_kz^k$ in the open unit disc $\mathbb{U}=\{z\in C:|z|<1\}$, the Hadamard Product (or convolution) $f*g$ of f and g is defined by

$$f(z)*g(z)=(f*g)(z)=z+\sum_{k=2}^{\infty}a_kb_kz^k. \tag{1.2}$$

Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$, then the function $p(z)$ is said to be subordinate to $q(z)$ in $\mathbb{U}$, written by

$$p(z)\prec q(z); \qquad (z\in\mathbb{U}),$$

if there exists a function $w(z)$ which is analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$; $z\in\mathbb{U}$, such that $p(z)=q(w(z))$ for $z\in\mathbb{U}$. From the definition of the subordinations, it is easy to show that the above subordination implies that

$$p(0)=q(0) \qquad \textit{and} \qquad p(\mathbb{U})\subset q(\mathbb{U}).$$

For complex parameters $\alpha_1,\ldots\alpha_r$ and $\beta_1,\ldots\beta_s$ $(\beta_j\neq 0,-1,-2,\ldots;j=1\ldots s)$, Dziok and Srivastava [17] defined the generalized hypergeometric function ${}_rF_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s;z)$ by

$${}_rF_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s;z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_r)_k}{(\beta_1)_k\ldots(\beta_s)_k}\frac{z^k}{k!}; \tag{1.3}$$

$$(r\leqslant s+1;r,s\in\mathbb{N}_0=\mathbb{N}\cup 0;z\in\mathbb{U}),$$

where $(x)_k$ is the Pochhammer symbol defined, in terms of Gamma function Γ, by

$$(x)_k=\frac{\Gamma(x+k)}{\Gamma(x)}=\begin{cases}1 & \text{if} \quad k=0,\\ x(x+1)\ldots(x+k-1) & \text{if} \quad k\in\mathbb{N},\end{cases}$$

Dziok and Srivastava [17, 18] defined also the linear operator

$$H(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s)f(z)=z+\sum_{k=2}^{\infty}\Gamma_ka_kz^k, \tag{1.4}$$

where

$$\Gamma_k=\frac{(\alpha_1)_{k-1}\ldots(\alpha_r)_{k-1}}{(\beta_1)_{k-1}\ldots(\beta_s)_{k-1}(k-1)!}. \tag{1.5}$$

In 2014, Alhindi and Darus [19] generalized Dziok-Srivastava operator by introducing the following operator:

$$\mathcal{K}_{\lambda_1,\lambda_2}^{m,r,s}f(z)=z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^{m}}\Gamma_ka_kz^k, \tag{1.6}$$

where Γ_k is given in (1.5), $\alpha_1,\ldots\alpha_r$ and $\beta_1,\ldots\beta_s$ $(\beta_j\neq 0,-1,-2,\ldots;j=1\ldots s)$ are complex parameters, see also [20, 21]. In 2017, Cang and Liu introduced two subclasses of meromorphically multivalent functions associated with Dziok-Srivastava operator [22]. Later on, Wanas and Majeed defined a new class for higher-order derivatives of multivalent analytic functions associated with Dziok-Srivastava operator [23, 24]. Recently, Yan and Liu derived certain geometric properties of analytic functions associated with the Dziok-Srivastava operator [25].

In this paper, Fekete-Szegö inequality for functions that belong to the new subclass $S^*(\phi,m,r,s,\lambda_1,\lambda_2)$ is determined, which will be introduced in the next section. Moreover, some applications associated with functions defined by fractional derivatives are also discussed. In order to prove our main results, the following lemmas are recalled as follows:

LEMMA 1.1. [26, 27] *If* $p(z) = 1+d_1z+d_2z+d_3z+\dots\ (z\in\mathbb{U})$ *is a function with positive real part, then for any complex number* v,

$$|d_2 - vd_1^2| \leqslant 2\max\{1,|2v-1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \qquad p(z) = \frac{1+z}{1-z}.$$

LEMMA 1.2. [28] *If* $p(z) = 1+d_1z+d_2z^2\dots$ *is an analytic function with positive real part in* $\mathbb{U}$, *then*

$$|d_2 - vd_1^2| \leqslant \begin{cases} -4v+2 & \text{if} \quad v\leqslant 0, \\ 2 & \text{if} \quad 0\leqslant v\leqslant 1, \\ 4v-2 & \text{if} \quad v\geqslant 1. \end{cases}$$

When $v<0$ *or* $v>1$, *the equality holds if and only if p(z) is* $(1+z)/(1-z)$ *or one of its rotations. If* $0<v<1$, *then, the equality holds if and only if p(z) is* $(1+z^2)/(1-z^2)$ *or one of its rotations. If v=0, the equality holds if and only if*

$$p(z) = \left(\frac{1}{2}+\frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2}-\frac{1}{2}\lambda\right)\frac{1-z}{1+z}; \qquad (0\leqslant\lambda\leqslant 1),$$

or one of its rotations. If $v=1$, *the equality holds if and only if*

$$\frac{1}{p(z)} = \left(\frac{1}{2}+\frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2}-\frac{1}{2}\lambda\right)\frac{1-z}{1+z}; \qquad (0\leqslant\lambda\leqslant 1).$$

Also the above upper bound is sharp, and it can be improved as follows when $0<v<1$*:*

$$|d_2 - vd_1^2| + v|d_1|^2 \leqslant 2; \qquad (0<v\leqslant 1/2),$$

and

$$|d_2 - vd_1^2| + (1-v)|d_1|^2 \leqslant 2; \qquad (1/2<v\leqslant 1).$$

2. Main results

By making use of the operator $\mathscr{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)$, the class $S^*(\phi,m,r,s,\lambda_1,\lambda_2)$ is defined as follows:

DEFINITION 2.1. Let $\phi(z) = 1+B_1z+B_2z^2+\ldots$ be univalent starlike function with respect to '1' which maps the unit disk $\mathbb{U}$ onto a region in the right half plane which is symmetric with respect to the real axis, and let $B_1 > 0$. Then the function $f \in \mathscr{A}$ is in the class $S^*(\phi,m,r,s,\lambda_1,\lambda_2)$ if

$$\frac{z(\mathscr{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z))'}{\mathscr{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)} \prec \phi(z); \qquad z \in \mathbb{U}, \tag{2.1}$$

where $\mathscr{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)$ is defined by (1.6).

2.1. Fekete and Szegö inequality

In this section, Fekete-Szegö type inequality for functions $f(z)$ in the class $S^*(\phi,m,r,s,\lambda_1,\lambda_2)$ is investigated in the following theorems:

THEOREM 2.2. *If $f(z)$ given by (1.1) belongs to $S^*(\phi,m,r,s,\lambda_1,\lambda_2)$ then*

$$|a_3-\mu a_2^2| \leqslant \frac{(1+2\lambda_2)^m}{4(1+2\lambda_1)^{(m-1)}}B_1\max\left\{1, \left|\frac{B_2}{B_1}+B_1-\mu 4\frac{(1+2\lambda_1)^{(m-1)}(1+\lambda_2)^{2m}}{(1+2\lambda_2)^m(1+\lambda_1)^{2(m-1)}}B_1\right|\right\}$$

The result is sharp.

Proof. Let $f \in S^*(\phi,m,r,s,\lambda_1,\lambda_2)$, then there exists a Schwarz function $w(z) \in \mathscr{A}$ such that

$$\frac{z(\mathscr{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z))'}{\mathscr{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)} = \phi(w(z)); \qquad (z \in \mathbb{U}). \tag{2.2}$$

If $p_1(z)$ is analytic and has positive real part in $\mathbb{U}$ and $p_1(0) = 1$, then

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1+d_1z+d_2z^2+d_3z^3+\ldots; \qquad z \in \mathbb{U}. \tag{2.3}$$

From (2.3), we obtain

$$w(z) = \frac{d_1}{2}z+\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)z^2+\ldots. \tag{2.4}$$

Let

$$p(z) = \frac{z(\mathscr{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z))'}{\mathscr{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)} = 1+b_1z+b_2z^2+\ldots \qquad (z \in \mathbb{U}), \tag{2.5}$$

which gives

$$b_1 = \frac{(1+\lambda_1)^{m-1}}{(1+\lambda_2)^m} a_2 \qquad \text{and} \qquad b_2 = \frac{4(1+2\lambda_1)^{m-1}}{(1+2\lambda_2)^m} a_3 - \left(\frac{(1+\lambda_1)^{m-1}}{(1+\lambda_2)^m} \right)^2 a_2^2. \quad (2.6)$$

Since $\phi(z)$ is univalent and $p \prec \phi$, therefore using (2.4), we obtain

$$\begin{aligned} p(z) &= \phi(w(z)) \\ &= 1 + \frac{B_1 d_1}{2} z + \left\{ \frac{1}{2} \left(d_2 - \frac{d_1^2}{2} \right) B_1 + \frac{1}{4} d_1^2 B_2 \right\} z^2 + \dots . \end{aligned} \quad (2.7)$$

Then, from (2.5),(2.6) and (2.7), we may write

$$\frac{(1+\lambda_1)^{m-1}}{(1+\lambda_2)^m} a_2 = \frac{B_1 d_1}{2},$$

$$\frac{1}{2} \left(d_2 - \frac{d_1^2}{2} \right) B_1 + \frac{1}{4} d_1^2 B_2 = \frac{4(1+2\lambda_1)^{m-1}}{(1+2\lambda_2)^m} a_3 - \left(\frac{(1+\lambda_1)^{m-1}}{(1+\lambda_2)^m} \right)^2 a_2^2.$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{1}{8} \frac{(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}} B_1 [d_2 - v d_1^2], \quad (2.8)$$

where

$$v = \frac{1}{2} \left[1 - \frac{B_2}{B_1} - B_1 + \mu \frac{4(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}{(1+2\lambda_2)^m (1+\lambda_1)^{2(m-1)}} B_1 \right].$$

Now, our result is followed by an application of Lemma 1.1. Also, the result is sharp for the functions

$$\frac{z(\mathscr{K}_{\lambda_1,\lambda_2}^{m,r,s} f(z))'}{\mathscr{K}_{\lambda_1,\lambda_2}^{m,r,s} f(z)} = \phi(z^2),$$

$$\frac{z(\mathscr{K}_{\lambda_1,\lambda_2}^{m,r,s} f(z))'}{\mathscr{K}_{\lambda_1,\lambda_2}^{m,r,s} f(z)} = \phi(z).$$

This completes the proof of Theorem 2.2. □

Next, by using Lemma 1.2, we can obtain the following theorem.

THEOREM 2.3. *If $f(z)$ given by (1.1) belongs to $S^*(\phi, m, r, s, \lambda_1, \lambda_2)$ then*

$$|a_3 - \mu a_2^2| \leqslant \begin{cases} \frac{1}{4} \frac{(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}} \left[B_2 + B_1^2 - \mu \frac{4(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}{(1+2\lambda_2)^m (1+\lambda_1)^{2(m-1)}} B_1^2 \right] & \text{if } \mu \leqslant \sigma_1, \\ \frac{1}{4} \frac{(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}} B_1 & \text{if } \sigma_1 \leqslant \mu \leqslant \sigma_2, \\ -\frac{1}{4} \frac{(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}} \left[B_2 + B_1^2 - \mu \frac{4(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}{(1+2\lambda_2)^m (1+\lambda_1)^{2(m-1)}} B_1^2 \right] & \text{if } \mu \geqslant \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{1}{4}\frac{(1+2\lambda_2)^m(1+\lambda_1)^{2(m-1)}}{(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}\frac{1}{B_1}\left[-1+\frac{B_2}{B_1}+B_1\right],$$

and

$$\sigma_2 := \frac{1}{4}\frac{(1+2\lambda_2)^m(1+\lambda_1)^{2(m-1)}}{(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}\frac{1}{B_1}\left[1+\frac{B_2}{B_1}+B_1\right].$$

The result is sharp.

Proof. First, let $\mu \leqslant \sigma_1$, then

$$\begin{aligned}|a_3-\mu a_2^2| &\leqslant \frac{1}{4}\frac{(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}}B_1\,[-2v+1]\\ &= \frac{1}{4}\frac{(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}}\left[B_2+B_1^2-\mu\frac{4(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}{(1+2\lambda_2)^m(1+\lambda_1)^{2(m-1)}}B_1^2\right].\end{aligned}$$

When $\sigma_1 \leqslant \mu \leqslant \sigma_2$, using the above calculations, we obtain

$$|a_3-\mu a_2^2| \leqslant \frac{1}{4}\frac{(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}}B_1.$$

Finally, if $\mu \geqslant \sigma_2$, then we have

$$\begin{aligned}|a_3-\mu a_2^2| &\leqslant \frac{1}{4}\frac{(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}}B_1\,[2v-1]\\ &= -\frac{1}{4}\frac{(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}}\left[B_2+B_1^2-\mu\frac{4(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}{(1+2\lambda_2)^m(1+\lambda_1)^{2(m-1)}}B_1^2\right].\end{aligned}$$

To show that these bounds are sharp, we define the functions $K_{\phi_n}(n=2,3\dots)$ by

$$\frac{z(\mathscr{K}^{m,r,s}_{\lambda_1,\lambda_2}L_{\phi_n}(z))'}{\mathscr{K}^{m,r,s}_{\lambda_1,\lambda_2}L_{\phi_n}(z)} = \phi(z^{n-1}); \qquad L_{\phi_n}(0)=0=[L_{\phi_n}]'(0)-1,$$

and the functions F_λ, G_λ by

$$\frac{z(\mathscr{K}^{m,r,s}_{\lambda_1,\lambda_2}F_\lambda(z))'}{\mathscr{K}^{m,r,s}_{\lambda_1,\lambda_2}F_\lambda(z)} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right); \qquad F_\lambda(0)=0=[F_\lambda]'(0)-1,$$

and

$$\frac{z(\mathscr{K}^{m,r,s}_{\lambda_1,\lambda_2}G_\lambda(z))'}{\mathscr{K}^{m,r,s}_{\lambda_1,\lambda_2}G_\lambda(z)} = \phi\left(\frac{-z(z+\lambda)}{1+\lambda z}\right); \qquad G_\lambda(0)=0=[G_\lambda]'(0)-1.$$

It is obvious that the functions $L_{\phi_n}, F_\lambda, G_\lambda \in S^*(\phi,c)$. Also, we write $L_\phi := L_{\phi_2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is L_ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if f is L_{ϕ_3} or one of its rotations. If $\mu = \sigma_1$, then equality holds if and only if f is F_λ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is G_λ or one of its rotations. This completes the proof of Theorem 2.3.

If $\sigma_1 < \mu < \sigma_2$, in view of Lemma 1.2, Theorem 2.3 can be improved. □

THEOREM 2.4. *Let $f(z)$ given by (1.1) belongs to $S^*(\phi,m,r,s,\lambda_1,\lambda_2)$, and σ_3 be given by*

$$\sigma_3 := \frac{1}{4}\frac{(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}}\frac{1}{B_1}\left[\frac{B_2}{B_1}+B_1\right]$$

If $\sigma_1 < \mu \leqslant \sigma_3$, then

$$\left|a_3-\mu a_2^2\right| + \frac{1}{B_1^2}\left[(B_1-B_2)\frac{(1+\lambda_1)^{2(m-1)}(1+2\lambda_2)^m}{4(1+\lambda_2)^{2m}(1+2\lambda_1)^{m-1}}\right.$$
$$\left. - B_1^2\frac{(1+\lambda_1)^{2(m-1)}(1+2\lambda_2)^m}{4(1+\lambda_2)^{2m}(1+2\lambda_1)^{m-1}} + \mu B_1^2\right] \leqslant \frac{(1+2\lambda_2)^m}{4(1+2\lambda_1)^{m-1}}B_1.$$

If $\sigma_3 < \mu \leqslant \sigma_2$, then

$$\left|a_3-\mu a_2^2\right| + \frac{1}{B_1^2}\left[(B_1+B_2)\frac{(1+\lambda_1)^{2(m-1)}(1+2\lambda_2)^m}{4(1+\lambda_2)^{2m}(1+2\lambda_1)^{m-1}}\right.$$
$$\left. + B_1^2\frac{(1+\lambda_1)^{2(m-1)}(1+2\lambda_2)^m}{4(1+\lambda_2)^{2m}(1+2\lambda_1)^{m-1}} - \mu B_1^2\right] \leqslant \frac{(1+2\lambda_2)^m}{4(1+2\lambda_1)^{m-1}}B_1.$$

2.2. Applications to functions defined by fractional derivatives

In this section, a new subclass of $S^*(\phi,m,r,s,\lambda_1,\lambda_2)$ is introduced in order to prove the following theorems.

DEFINITION 2.5. For a fixed $g \in \mathscr{A}$, let $S^g(\phi,m,r,s,\lambda_1,\lambda_2)$ be the class of functions $f \in \mathscr{A}$ for which $(f*g) \in S^*(\phi,m,r,s,\lambda_1,\lambda_2)$.

DEFINITION 2.6. [29] Let $f(z)$ be analytic in a simply connected region of the z-plane containing origin. The fractional derivative of f of order ζ is defined by

$${}_0D_z^{\zeta} f(z) := \frac{1}{\Gamma(1-\zeta)}\frac{d}{dz}\int_0^z (z-\rho)^{-1} f(\rho)d\rho; \qquad (0 \leqslant \zeta < 1),$$

where the multiplicity of $(z-\rho)^{-}\zeta$ is removed by requiring that $log(z-\rho)$ is real for $(z-\rho) > 0$.

Owa and Srivastava [30], [31] used Definition 2.6 to introduce a fractional derivative operator $\Omega^{\zeta} : \mathscr{A} \to \mathscr{A}$ defined by

$$(\Omega^{\zeta} f)(z)) = \Gamma(2-\zeta) z_0^{\zeta} D_z^{\zeta} f(z), \qquad (\zeta \neq 2,3,4,\ldots).$$

The class $S^{\zeta}(\phi,m,r,s,\lambda_1,\lambda_2)$ consists of the functions $f \in \mathscr{A}$ for which $\Omega^{\zeta} f \in S^*(\phi,m,r,s,\lambda_1,\lambda_2)$. The class $S^{\zeta}(\phi,m,r,s,\lambda_1,\lambda_2)$ is a special case of the class $S^g(\phi,m, r,s,\lambda_1,\lambda_2)$ when

$$g(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\zeta)}{\Gamma(k+1-\zeta)} z^k; \qquad (z \in \mathbb{U}).$$

Now, applying Theorem 2.3 for the function $(f*g)=z+g_2a_2z^2+g_3a_3z^3+\dots$, we get the following theorem after changing the parameter μ:

THEOREM 2.7. *Let* $g(z)=z+\sum_{k=0}^{\infty}g_kz^k(g_k>0)$. *If* $f(z)$ *given by* (1.1) *belongs to* $S^g(\phi,m,r,s,\lambda_1,\lambda_2)$, *then*

$$|a_3-\mu a_2^2|\leqslant\begin{cases}\frac{1}{4}\frac{(1+2\lambda_2)^m}{g_3(1+2\lambda_1)^{m-1}}\left[B_2+B_1^2-\mu\frac{g_3}{g_2^2}\frac{4(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}{(1+2\lambda_2)^m(1+\lambda_1)^{2(m-1)}}B_1^2\right] & \text{if } \mu\leqslant\eta_1,\\ \frac{1}{4}\frac{(1+2\lambda_2)^m}{g_3(1+2\lambda_1)^{m-1}}B_1 & \text{if } \eta_1\leqslant\mu\leqslant\eta_2,\\ -\frac{1}{4}\frac{(1+2\lambda_2)^m}{g_3(1+2\lambda_1)^{m-1}}\left[B_2+B_1^2-\mu\frac{g_3}{g_2^2}\frac{4(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}{(1+2\lambda_2)^m(1+\lambda_1)^{2(m-1)}}B_1^2\right] & \text{if } \mu\geqslant\eta_2,\end{cases}$$

where

$$\eta_1:=\frac{1}{4}\frac{g_2^2(1+2\lambda_2)^m(1+\lambda_1)^{2(m-1)}}{g_3(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}\frac{1}{B_1}\left[-1+\frac{B_2}{B_1}+B_1\right],$$

and

$$\eta_2:=\frac{1}{4}\frac{g_2^2(1+2\lambda_2)^m(1+\lambda_1)^{2(m-1)}}{g_3(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}\frac{1}{B_1}\left[1+\frac{B_2}{B_1}+B_1\right].$$

The result is sharp.

Since

$$\Omega^{\zeta}f(z)=z+\sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\zeta)}{\Gamma(k+1-\zeta)}z^k,$$

We have

$$g_2:=\frac{\Gamma(3)\Gamma(2-\zeta)}{\Gamma(3-\zeta)}=\frac{2}{2-\zeta} \tag{2.9}$$

and

$$g_3:=\frac{\Gamma(4)\Gamma(2-\zeta)}{\Gamma(4-\zeta)}=\frac{6}{(2-\zeta)(3-\zeta)}. \tag{2.10}$$

For g_2,g_3 given by (2.9) and (2.10) respectively, Theorem 2.7 is reduced to the following :

THEOREM 2.8. *Let* $\zeta<2$. *If* $f(z)$ *given by* (1.1) *belongs to* $S^*(\phi,m,r,s,\lambda_1,\lambda_2)$. *Then*

$$|a_3-\mu a_2^2|\leqslant\begin{cases}\frac{1}{24}\frac{(2-\zeta)(3-\zeta)(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}}\left[B_2+B_1^2-\frac{3}{2}\mu\left(\frac{2-\zeta}{3-\zeta}\right)\frac{4(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}{(1+2\lambda_2)^m(1+\lambda_1)^{2(m-1)}}B_1^2\right]\\ \qquad \text{if } \mu\leqslant\eta_1,\\ \frac{1}{24}\frac{(2-\zeta)(3-\zeta)(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}}B_1\\ \qquad \text{if } \eta_1\leqslant\mu\leqslant\eta_2,\\ -\frac{1}{24}\frac{(2-\zeta)(3-\zeta)(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}}\left[B_2+B_1^2\frac{3}{2}\mu\left(\frac{2-\zeta}{3-\zeta}\frac{4(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}{(1+2\lambda_2)^m(1+\lambda_1)^{2(m-1)}}B_1^2\right]\\ \qquad \text{if } \mu\geqslant\eta_2,\end{cases}$$

where

$$\eta_1^* := \frac{1}{6}\frac{(3-\zeta)}{(2-\zeta)}\frac{(1+2\lambda_2)^m(1+\lambda_1)^{2(m-1)}}{(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}\frac{1}{B_1}\left[-1+\frac{B_2}{B_1}+B_1\right],$$

and

$$\eta_2^* := \frac{1}{6}\frac{(3-\zeta)}{(2-\zeta)}\frac{(1+2\lambda_2)^m(1+\lambda_1)^{2(m-1)}}{(1+2\lambda_1)^{m-1}(1+\lambda_2)^{2m}}\frac{1}{B_1}\left[1+\frac{B_2}{B_1}+B_1\right].$$

The result is sharp.

Conclusion

In this research study, the Fekete-Szego inequality was correlated with a certain class of hypergeometric function. Thus, based on the generalized hypergeometric operator $\mathscr{K}_{\lambda_1,\lambda_2}^{m,r,s}f(z)$ which was introduced earlier [19], the class $S^*(\phi,m,r,s,\lambda_1,\lambda_2)$ was derived in the light of starlike functions. The Fekete-Szegö inequality was obtained for the first two coefficients of the function $f(z)\in S^*(\phi,m,r,s,\lambda_1,\lambda_2)$. Moreover, the classes $S^g(\phi,m,r,s,\lambda_1,\lambda_2)$ and $S^\zeta(\phi,m,r,s,\lambda_1,\lambda_2)$ were defined as subclasses of $S^*(\phi,m,r,s,\lambda_1,\lambda_2)$. Then, the Fekete-Szegö inequality was investigated again including some fractional derivatives.

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