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SOME GENERALIZED NUMERICAL RADIUS INEQUALITIES FOR HILBERT SPACE OPERATORS

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Abstract. Some generalizations and refinements inequalities for the operator norm and numerical radius of the product and sum of Hilbert space operators are established. Refinements of some famous norm operators and numerical radius inequalities are also pointed out. As shown in this work, these refinements generalize and refine some recent and old results obtained in the literature.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space with norm $\|.\|$ and inner product $\langle .,. \rangle$. Let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$, $\|.\|$ will also denote the norm in $\mathscr{B}(\mathscr{H})$. An operator $A \in \mathscr{B}(\mathscr{H})$ is called positive if $\langle Ax,x \rangle \geqslant 0$ for all $x \in \mathscr{H}$, and we then write $A \geqslant 0$. In addition, we write $A > 0$ if A is a positive invertible operator. Recall that a function $f : J \longrightarrow \mathbb{R}$ is convex if

$$f(\alpha x + (1-\alpha)y) \leqslant \alpha f(x) + (1-\alpha) f(y), \tag{1}$$

for all $\alpha \in [0,1]$ and all $x, y \in J$. It is well known [20] that a continuous function f is convex in a real interval $I \subseteq \mathbb{R}$ if it has the property

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right) \leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i f(a_i), \tag{2}$$

where $a_i \in I$, $1 \leqslant i \leqslant n$ are given data points and $p_1, p_2, \cdots, p_n$ are a set of non-negative real numbers constrained by $\sum_{i=1}^{n} p_i = P_n$. If f is concave, then preceding inequality is reversed.

The Schwarz inequality for positive operators asserts that if A is a positive operator in $\mathscr{B}(\mathscr{H})$, then

$$|\langle Ax,y \rangle|^2 \leqslant \langle Ax,x \rangle \langle Ay,y \rangle \text{ for all } x,y \in \mathscr{H}. \tag{3}$$

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For an arbitrary operator $A \in \mathscr{B}(\mathscr{H})$, a mixed Schwarz inequality has been established in [8]. This inequality asserts that

$$|\langle Ax, y\rangle|^2 \leqslant \left\langle |A|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle \tag{4}$$

for all $x, y \in \mathscr{H}$ and $0 \leqslant \alpha \leqslant 1$. Here $|A| = \sqrt{A^*A}$ and $|A^*| = \sqrt{AA^*}$.

The famous Cauchy-Schwarz inequality which states: if $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathbf{b} = (b_1, \cdots, b_n)$ are two n-tuples of real numbers, then

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 \leqslant \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \tag{5}$$

with equality holding if and only if $\mathbf{a}$ and $\mathbf{b}$ are scalar multiples of each other. The inequality is called the Cauchy-Schwarz-Buniakowski or simply Cauchy.

For a bounded operator $A \in \mathscr{B}(\mathscr{H})$, the numerical range $W(A)$ is the image of the unit sphere under the quadratic form $x \longrightarrow \langle Tx, x\rangle$ associated with the operator. More precisely,

$$W(A) = \{\langle Ax, x\rangle : x \in \mathscr{H}, \|x\| = 1\}.$$

Also, the numerical radius is defined to be

$$w(A) = \sup\{|\lambda| : \lambda \in W(A)\} = \sup_{\|x\|=1} |\langle Ax, x\rangle|.$$

The spectral radius of an operator A is defined to be

$$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\},$$

where $\sigma(A)$ is the spectrum of A.

It is well known that $w(.)$ defines a norm on $\mathscr{B}(\mathscr{H})$, which is equivalent to the usual operator norm $\|.\|$. In fact, for any $A \in \mathscr{B}(\mathscr{H})$,

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\|. \tag{6}$$

Also, if $A \in \mathscr{B}(\mathscr{H})$ is normal, then $w(A) = \|A\|$.

An important inequality for $w(T)$ is the power inequality stating that $w(A^n) \leqslant (w(A))^n$ for every natural number n.

Several numerical radius inequalities improving the inequalities in (6) have been recently given in [3, 7, 9, 16, 18, 19, 22]. For instance, Kittaneh [10, 11] proved that for any $A \in \mathscr{B}(\mathscr{H})$,

$$w(A) \leqslant \frac{1}{2}\left\||A|^2 + |A^*|\right\| \leqslant \frac{1}{2}\left(\|A\| + \left\|A^2\right\|^{1/2}\right), \tag{7}$$

and

$$\frac{1}{4}\|A^*A + AA^*\| \leqslant w^2(A) \leqslant \frac{1}{2}\|A^*A + AA^*\|. \tag{8}$$

Also, in the same paper, it was shown that

$$\|A+B\|^2 \leqslant \left\| |A|^2+|B|^2 \right\| + \left\| |A^*|^2+|B^*|^2 \right\|. \tag{9}$$

Kittaneh and El-Haddad [14] established the generalizations of inequality (7) and the second inequality (8) as follows:

$$w^r(A) \leqslant \frac{1}{2}\left\| |A|^{2r\lambda} + |A^*|^{2r(1-\lambda)} \right\| \tag{10}$$

and

$$w^{2r}(A) \leqslant \left\| \lambda |A|^{2r} + (1-\lambda)|A^*|^{2r} \right\|, \tag{11}$$

where $0<\lambda<1$ and $r \geqslant 1$.

Although some open problems related to the numerical radius inequalities for bounded linear operator still remain open, the investigation to establish numerical radius inequalities for several bounded linear operators has been started, (see for instance [6] and [18]). If $A,B \in \mathscr{B}(\mathscr{H})$, then

$$w(AB) \leqslant 4w(A)w(B).$$

In the case that $AB = BA$, we have

$$w(AB) \leqslant 2w(A)w(B).$$

Moreover, if A and B are normal, then

$$w(AB) \leqslant w(A)w(B).$$

Dragomir [4] proved that if $A,B \in \mathscr{B}(\mathscr{H})$, then

$$w^{2r}(B^*A) \leqslant \left\| \alpha |A|^{\frac{2r}{\alpha}} + (1-\alpha)|B|^{\frac{2r}{1-\alpha}} \right\| \tag{12}$$

for all $\alpha \in (0,1)$ and $r \geqslant 1$. In the same work, he also proved that if $A,B,C,D \in \mathscr{B}(\mathscr{H})$, we have

$$\left\| \frac{AB+CD}{2} \right\|^2 \leqslant \left\| \frac{|A|^{2r}+|C|^{2r}}{2} \right\|^{\frac{1}{r}} \left\| \frac{|B|^{2s}+|D|^{2s}}{2} \right\|^{\frac{1}{s}} \tag{13}$$

for all $s,r \geqslant 1$.

An interesting numerical radius inequality has been established by Sattari et al. [21], it has been shown that if $A,X,B \in \mathscr{B}(\mathscr{H})$ such that A and B are positive, then

$$w^r(A^\alpha X B^\alpha) \leqslant \|X\|^r \left\| \frac{1}{p}A^{pr} + \frac{1}{q}B^{qr} \right\|^\alpha \tag{14}$$

for all $0 \leqslant \alpha \leqslant 1$, $r \geqslant 1$ and $p,q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr, qr \geqslant 2$.

In this paper, We present a generalizations of inequalities (13) and (14). Refinements of some famous the operator norm and numerical radius inequalities are also pointed out. As shown in this work, these refinements generalize and refine some recent and old results obtained in the literature.

2. Norm inequalities for product of operators

In this section, we establish a considerable improvements and generalizations of inequalities (9) and (13).

THEOREM 1. *Let* $A_i, B_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$ *and* $r, s \geqslant 1$ *we have*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} B_i^* A_i \right\|^2 \leqslant \left\| \frac{1}{n} \sum_{i=1}^{n} |A_i|^{2r} \right\|^{\frac{1}{r}} \left\| \frac{1}{n} \sum_{i=1}^{n} |B_i|^{2s} \right\|^{\frac{1}{s}}. \tag{15}$$

Proof. By the Schwarz inequality in the Hilbert space, we have

$$\begin{aligned} \left| \left\langle \left(\sum_{i=1}^{n} B_i^* A_i \right) \right\rangle x, y \right|^2 &= \left| \sum_{i=1}^{n} \langle B_i^* A_i x, y \rangle \right|^2 \leqslant \left(\sum_{i=1}^{n} |\langle B_i^* A_i x, y \rangle| \right)^2 \\ &\leqslant \left(\sum_{i=1}^{n} \langle A_i^* A_i x, x \rangle^{\frac{1}{2}} \langle B_i^* B_i y, y \rangle^{\frac{1}{2}} \right)^2 \end{aligned}$$

for any $x, y \in \mathscr{H}$.

Now, on utilizing the Cauchy inequality (5), we then conclude that

$$\left(\sum_{i=1}^{n} \langle A_i^* A_i x, x \rangle^{\frac{1}{2}} \langle B_i^* B_i y, y \rangle^{\frac{1}{2}} \right)^2 \leqslant \left(\sum_{i=1}^{n} \langle A_i^* A_i x, x \rangle \right) \left(\sum_{i=1}^{n} \langle B_i^* B_i y, y \rangle \right) \tag{16}$$

for any $x, y \in \mathscr{H}$.

Utilizing the arithmetic-geometric mean inequality and then the convexity of the function $f(t) = t^k, k \geqslant 1$ and in view of inequality (2) with $P_n = 1$ and $p_i = \frac{1}{n}$ $(i = 1, \cdots, n)$, we have

$$\begin{aligned} &\left(\sum_{i=1}^{n} \langle A_i^* A_i x, x \rangle \right) \left(\sum_{i=1}^{n} \langle B_i^* B_i y, y \rangle \right) \\ &\leqslant n^2 \left\langle \left[\frac{1}{n} \sum_{i=1}^{n} (A_i^* A_i)^r \right] x, x \right\rangle^{\frac{1}{r}} \left\langle \left[\frac{1}{n} \sum_{i=1}^{n} (B_i^* B_i)^s \right] y, y \right\rangle^{\frac{1}{s}} \end{aligned} \tag{17}$$

for any unit vectors $x, y \in \mathscr{H}$.

Consequently, we have

$$\left| \left\langle \frac{1}{n} \sum_{i=1}^{n} B_i^* A_i x, y \right\rangle \right|^2 \leqslant \left\langle \left[\frac{1}{n} \sum_{i=1}^{n} (A_i^* A_i)^r \right] x, x \right\rangle^{\frac{1}{r}} \left\langle \left[\frac{1}{n} \sum_{i=1}^{n} (B_i^* B_i)^s \right] y, y \right\rangle^{\frac{1}{s}} \tag{18}$$

for any unit vectors $x, y \in \mathscr{H}$.

Taking the supremum over all unit vectors $x, y \in \mathscr{H}$, we deduce the desired result. □

REMARK 1. If $r = s$, then inequality (15) is equivalent with

$$\left\| \frac{1}{n} \sum_{i=1}^{n} B_i^* A_i \right\|^{2r} \leqslant \left\| \frac{1}{n} \sum_{i=1}^{n} |A_i|^{2r} \right\| \left\| \frac{1}{n} \sum_{i=1}^{n} |B_i|^{2r} \right\|. \tag{19}$$

COROLLARY 1. *For any $A_i \in \mathscr{H}$ $(i = 1, \cdots, n)$ we have*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} A_i \right\|^{2r} \leqslant \left\| \frac{1}{n} \sum_{i=1}^{n} |A_i|^{2r} \right\|, \tag{20}$$

for $r \geqslant 1$. Also, we have

$$\left\| \frac{1}{n} \sum_{i=1}^{n} A_i^2 \right\|^{2} \leqslant \left\| \frac{1}{n} \sum_{i=1}^{n} |A_i|^{2r} \right\|^{\frac{1}{r}} \left\| \frac{1}{n} \sum_{i=1}^{n} |A_i|^{2s} \right\|^{\frac{1}{s}}. \tag{21}$$

for all $r, s \geqslant 1$, and in particular

$$\left\| \frac{1}{n} \sum_{i=1}^{n} A_i^2 \right\|^{2r} \leqslant \left\| \frac{1}{n} \sum_{i=1}^{n} |A_i|^{2r} \right\| \left\| \frac{1}{n} \sum_{i=1}^{n} |A_i|^{2r} \right\| \tag{22}$$

for all $r \geqslant 1$

Proof. The inequality (20) follows from (15) for $B_i = I$, while inequality (21) is obtained from the same inequality for $B_i = A_i^*$ and inequality (22) follows from (15) for $B_i = A_i$. □

Another particular result of interest is the following one:

COROLLARY 2. *For any $A_i, B_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$ and $n = 2m$, we have*

$$\left\| \frac{B_1^* A_1 + A_1^* B_1 + \cdots + B_m^* A_m + A_m^* B_m}{n} \right\|^2$$
$$\leqslant \left\| \sum_{i=1}^{m} \frac{|A_i|^{2r}| + |B_i|^{2r}}{n} \right\|^{\frac{1}{r}} \left\| \sum_{i=1}^{m} \frac{|A_i|^{2s}| + |B_i|^{2s}}{n} \right\|^{\frac{1}{s}}$$

for all $r, s \geqslant 1$ and, in particular

$$\left\| \frac{B_1^* A_1 + A_1^* B_1 + \cdots + B_m^* A_m + A_m^* B_m}{n} \right\| \leqslant \left\| \sum_{i=1}^{m} \frac{|A_i|^{2r}| + |B_i|^{2r}}{n} \right\|$$

for all $r \geqslant 1$.

COROLLARY 3. *For any* $C_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, 2m)$ *we have*

$$\left\| \frac{1}{2m} \sum_{i=1}^{2m} C_i \right\|^2 \leqslant \left\| \frac{1}{2m} \sum_{i=1}^{m} \left(|C_{2i-1}|^{2r} + mI \right) \right\|^{\frac{1}{r}} \left\| \frac{1}{2m} \sum_{i=1}^{m} \left(|C_{2i}^*|^{2s} + mI \right) \right\|^{\frac{1}{s}}$$

for all $r, s \geqslant 1$. *In particular*

$$\|C\|^2 \leqslant \left\| \frac{|C|^{2r} + I}{2} \right\|^{\frac{1}{r}} \left\| \frac{|C^*|^{2s} + I}{2} \right\|^{\frac{1}{s}}.$$

Moreover, for every $r \geqslant 1$, *we have*

$$\|C\|^{2r} \leqslant \left\| \frac{|C|^{2r} + I}{2} \right\| \left\| \frac{|C^*|^{2r} + I}{2} \right\|.$$

Proof. The proof is obvious by inequality (15) on choosing

$$C_{2i-1} = A_{2i-1}, C_{2i} = B_{2i}^*, B_{2i-1} = A_{2i} = I \quad (i = 1, \cdots, m). \quad \square$$

The following lemma provides a simple however useful extension for four operators of the Schwarz inequality due to Dragomir [5].

LEMMA 1. *Let* $A, B, C, D \in \mathscr{B}(\mathscr{H})$. *Then for* $x, y \in \mathscr{H}$ *we have the inequality*

$$|\langle DCBAx, y \rangle|^2 \leqslant \left\langle |BA|^2 x, x \right\rangle \left\langle |(DC)^*|^2 y, y \right\rangle. \tag{23}$$

The equality case holds if and only if the vectors BAx *and* $C^* D^* y$ *are linearly dependent in* $\mathscr{H}$.

THEOREM 2. *Let* $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2 \in \mathscr{B}(\mathscr{H})$. *Then*

$$\left\| \frac{D_1 C_1 B_1 A_1 + D_2 C_2 B_2 A_2}{2} \right\|^2 \leqslant \left\| \frac{|B_1 A_1|^{2r} + |B_2 A_2|^{2r}}{2} \right\|^{\frac{1}{r}} \left\| \frac{|(D_1 C_1)^*|^{2s} + |(D_2 C_2)^*|^{2s}}{2} \right\|^{\frac{1}{s}} \tag{24}$$

for all $r, s \geqslant 1$. *In particular,*

$$\left\| \frac{D_1 C_1 B_1 A_1 + D_2 C_2 B_2 A_2}{2} \right\|^{2r} \leqslant \left\| \frac{|B_1 A_1|^{2r} + |B_2 A_2|^{2r}}{2} \right\| \left\| \frac{|(D_1 C_1)^*|^{2r} + |(D_2 C_2)^*|^{2r}}{2} \right\|$$

for all $r \geqslant 1$.

Proof. By the Schwarz inequality in the Hilbert space, we have

$$\begin{aligned}
&|\langle (D_1 C_1 B_1 A_1 + D_2 C_2 B_2 A_2) x, y \rangle|^2 \\
&\leqslant [|\langle (D_1 C_1 B_1 A_1) x, y \rangle| + |\langle (D_2 C_2 B_2 A_2) x, y \rangle|]^2 \\
&\leqslant \left[\left\langle |B_1 A_1|^2 x, x \right\rangle^{\frac{1}{2}} \left\langle |(D_1 C_1)^*|^2 y, y \right\rangle^{\frac{1}{2}} + \left\langle |B_2 A_2|^2 x, x \right\rangle^{\frac{1}{2}} \left\langle |(D_2 C_2)^*|^2 y, y \right\rangle^{\frac{1}{2}} \right]^2
\end{aligned}$$

for any $x,y \in \mathscr{H}$.

Now, by using the elementary inequality

$$(ab+cd)^2 \leqslant (a^2+c^2)(b^2+d^2), \qquad a,b,c,d \in \mathbb{R},$$

we then conclude that

$$\begin{aligned}&= |\langle (D_1C_1B_1A_1 + D_2C_2B_2A_2)x,y\rangle|^2 \\ &\leqslant \left[\left\langle |B_1A_1|^2 x,x\right\rangle + \left\langle |B_2A_2|^2 x,x\right\rangle\right] . \left[\left\langle |(D_1C_1)^*|^2 y,y\right\rangle + \left\langle |(D_2C_2)^*|^2 y,y\right\rangle\right]\end{aligned}$$

for any $x,y \in \mathscr{H}$.

Applying the arithmetic-geometric mean inequality and then the convexity of the function $f(t) = t^q$ $(q \geqslant 1)$, we have for all $r,s \geqslant 1$ that

$$\begin{aligned}&\left[\left\langle |B_1A_1|^2 x,x\right\rangle + \left\langle |B_2A_2|^2 x,x\right\rangle\right] . \left[\left\langle |(D_1C_1)^*|^2 y,y\right\rangle + \left\langle |(D_2C_2)^*|^2 y,y\right\rangle\right] \\ &\leqslant 4\left\langle \left(\frac{|B_1A_1|^{2r} + |B_2A_2|^{2r}}{2}\right)x,x\right\rangle^{\frac{1}{r}} \left\langle \left(\frac{|(D_1C_1)^*|^{2s} + |(D_2C_2)^*|^{2s}}{2}\right)y,y\right\rangle^{\frac{1}{s}}\end{aligned}$$

for all unit vectors $x,y \in \mathscr{H}$.

Consequently, we have

$$\begin{aligned}&\left|\left\langle \left[\frac{D_1C_1B_1A_1 + D_2C_2B_2A_2}{2}\right]x,y\right\rangle\right|^2 \\ &\leqslant \left\langle \left(\frac{|B_1A_1|^{2r} + |B_2A_2|^{2r}}{2}\right)x,x\right\rangle^{\frac{1}{r}} \left\langle \left(\frac{|(D_1C_1)^*|^{2s} + |(D_2C_2)^*|^{2s}}{2}\right)y,y\right\rangle^{\frac{1}{s}}\end{aligned}$$

for all unit vectors $x,y \in \mathscr{H}$.

Taking the supremum all unit vectors $x,y \in \mathscr{H}$, we deduce the desired result. □

Inequality (24) includes several operator norm inequalities as special cases.

COROLLARY 4. *Let $T,S \in \mathscr{B}(\mathscr{H})$, $\alpha+\beta \geqslant 1$ and $\gamma+\delta \geqslant 1$. Then*

$$\left\|\frac{T|T|^{\beta-1}T|T|^{\alpha-1} + S|S|^{\gamma-1}S|S|^{\delta-1}}{2}\right\|^2 \leqslant \left\|\frac{|T|^{2r\alpha} + |S|^{2r\delta}}{2}\right\|^{\frac{1}{r}} \left\|\frac{|T^*|^{2s\beta} + |S^*|^{2s\gamma}}{2}\right\|^{\frac{1}{s}} \tag{25}$$

for all $s,r \geqslant 1$. In particular,

$$\left\|\frac{T|T|^{\beta-1}T|T|^{\alpha-1} + S|S|^{\gamma-1}S|S|^{\delta-1}}{2}\right\|^{2r} \leqslant \left\|\frac{|T|^{2r\alpha} + |S|^{2r\delta}}{2}\right\| \left\|\frac{|T^*|^{2r\beta} + |S^*|^{2r\gamma}}{2}\right\|$$

for all $r \geqslant 1$.

Proof. Let $T = U|T|$ and $S = V|S|$ be the polar decomposition of the operator T and S, where U,V are partial isometry and the kernel $\ker(U) = \ker(|T|)$, $\ker(V) = \ker(|S|)$. If we take $D_1 = U$, $C_1 = |T|^{\beta}$, $B_1 = U$ and $A_1 = |T|^{\alpha}$ and $D_2 = V$, $C_2 = |S|^{\gamma}$, $B_2 = V$ and $A_2 = |S|^{\delta}$, we have

$$\begin{aligned} D_1C_1B_1A_1 &= T|T|^{\beta-1}T|T|^{\alpha-1} \\ D_2C_2B_2A_2 &= S|S|^{\gamma-1}S|S|^{\delta-1} \end{aligned}$$

and

$$\begin{aligned} |B_1A_1|^2 = |T|^{\alpha}U^*U|T|^{\alpha} = |T|^{2\alpha}, \ \ |B_2A_2|^2 = |S|^{2\delta} \\ |(D_1C_1)^*|^2 = U|T|^{2\beta}U^* = |T^*|^{2\beta}, \ \ |(D_2C_2)^*|^2 = |S^*|^{2\gamma}. \end{aligned}$$

Now, the result follows by Theorem 2. $\square$

REMARK 2. In Corollary 4,

(a) if we take $\alpha = \beta = 1 = \gamma = \delta$, we have

$$\left\| \frac{T^2 + S^2}{2} \right\|^2 \leqslant \left\| \frac{|T|^{2r} + |S|^{2r}}{2} \right\|^{\frac{1}{r}} \left\| \frac{|T^*|^{2s} + |S^*|^{2s}}{2} \right\|^{\frac{1}{s}}.$$

for all $r,s \geqslant 1$. In particular,

$$\left\| \frac{T^2 + S^2}{2} \right\|^{2r} \leqslant \left\| \frac{|T|^{2r} + |S|^{2r}}{2} \right\| \left\| \frac{|T^*|^{2r} + |S^*|^{2r}}{2} \right\|$$

for all $r \geqslant 1$.

(b) If we take $T = S$ and $\alpha = \beta = 1 = \gamma = \delta$, we have

$$\left\| T^2 \right\|^2 \leqslant \left\| |T|^{2r} \right\|^{\frac{1}{r}} \left\| |T^*|^{2s} \right\|^{\frac{1}{s}}$$

for all $r,s \geqslant 1$.

COROLLARY 5. *Let $T,S \in \mathscr{B}(\mathscr{H})$, $\alpha,\beta,\gamma,\delta \geqslant 0$ such that $\alpha+\beta \geqslant 2$ and $\gamma+\delta \geqslant 2$. Then*

$$\left\| \frac{T^*|T^*|^{\alpha+\beta-2}T + S^*|S^*|^{\gamma+\delta-2}S}{2} \right\|^2 \leqslant \left\| \frac{|T|^{2r\alpha} + |S|^{2r\delta}}{2} \right\|^{\frac{1}{r}} \left\| \frac{|T|^{2s\beta} + |S|^{2s\gamma}}{2} \right\|^{\frac{1}{s}} \tag{26}$$

for all $r,s \geqslant 1$. In particular,

$$\left\| \frac{T^*|T^*|^{\alpha+\beta-2}T + S^*|S^*|^{\gamma+\delta-2}S}{2} \right\|^{2r} \leqslant \left\| \frac{|T|^{2r\alpha} + |S|^{2r\delta}}{2} \right\| \left\| \frac{|T|^{2r\beta} + |S|^{2r\gamma}}{2} \right\|$$

for $r \geqslant 1$.

Proof. Let $T^* = U|T^*|$, $S^* = V|S^*|$ be the polar decomposition of the operator T^* and S^*, where U,V are the partial isometry and the kernel $\ker(U) = \ker(|T|)$ and $\ker(V) = \ker(|S|)$. Then $T = |T^*|U^*$ and $S = |S^*|V^*$. If we take $D_1 = U$, $D_2 = V$, $C_1 = |T^*|^{\beta}$, $C_2 = |S^*|^{\gamma}$, $B_1 = |T^*|^{\alpha}$, $B_2 = |T^*|^{\delta}$ and $A_1 = U^*$, $A_2 = V^*$, we have

$$\begin{aligned} D_1C_1B_1A_1 &= T^*|T^*|^{\alpha+\beta-2}T = U|T^*|^{\beta+\alpha}U^* \\ D_2C_2B_2A_2 &= S^*|S^*|^{\gamma+\delta-2}S = V|T^*|^{\gamma+\delta}V^*. \end{aligned}$$

Hence

$$\begin{aligned} |B_1A_1|^2 &= |T|^{2\alpha}, \ \ |(D_1C_1)^*|^2 = |T|^{2\beta} \\ |B_2A_2|^2 &= |S|^{2\gamma}, \ \ |(D_2C_2)^*|^2 = |S|^{2\delta}. \end{aligned}$$

Now, the result follows by Theorem 2. □

REMARK 3. In Corollary 5,

(i) If we take $\alpha = \beta = 1 = \gamma = \delta$, we have

$$\left\| \frac{T^*T + S^*S}{2} \right\|^2 \leqslant \left\| \frac{|T|^{2r} + |S|^{2r}}{2} \right\|^{\frac{1}{r}} \left\| \frac{|T|^{2s} + |S|^{2s}}{2} \right\|^{\frac{1}{s}}$$

for all $r,s \geqslant 1$. In particular,

$$\left\| \frac{T^*T + S^*S}{2} \right\|^{2r} \leqslant \left\| \frac{|T|^{2r} + |S|^{2r}}{2} \right\| \left\| \frac{|T|^{2r} + |S|^{2r}}{2} \right\|$$

for all $r \geqslant 1$.

(ii) If we take $\alpha = \beta = 1 = \gamma = \delta$ and $S = T^*$, we have

$$\left\| \frac{T^*T + TT^*}{2} \right\|^2 \leqslant \left\| \frac{|T|^{2r} + |T^*|^{2r}}{2} \right\|^{\frac{1}{r}} \left\| \frac{|T|^{2s} + |T^*|^{2s}}{2} \right\|^{\frac{1}{s}}$$

for all $r,s \geqslant 1$. In particular,

$$\left\| \frac{T^*T + TT^*}{2} \right\|^{r} \leqslant \left\| \frac{|T|^{2r} + |T^*|^{2r}}{2} \right\|$$

for all $r \geqslant 1$.

By the same arguments of Theorem 1, we can generalize Theorem 2 as follows.

THEOREM 3. *Let* $A_i, B_i, C_i, D_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$. *Then*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} D_iC_iB_iA_i \right\|^2 \leqslant \left\| \frac{1}{n} \sum_{i=1}^{n} |B_iA_i|^{2r} \right\|^{\frac{1}{r}} \left\| \frac{1}{n} \sum_{i=1}^{n} |(D_iC_i)^*|^{2s} \right\|^{\frac{1}{s}} \tag{27}$$

for all $r,s \geqslant 1$. *In particular,*

$$\left\|\frac{1}{n}\sum_{i=1}^{n} D_iC_iB_iA_i\right\|^{2r} \leqslant \left\|\frac{1}{n}\sum_{i=1}^{n} |B_iA_i|^{2r}\right\| \left\|\frac{1}{n}\sum_{i=1}^{n} |(D_iC_i)^*|^{2r}\right\|$$

for all $r \geqslant 1$.

The following lemma is very useful in the sequel which is known as the generalized mixed Schwartz inequality.

LEMMA 2. *Let* $T \in \mathscr{B}(\mathscr{H})$ *and* $x,y \in \mathscr{H}$ *be any vectors.*

(i) If $\alpha,\beta \geqslant 0$ *such that* $\alpha+\beta=1$, *then* $|\langle Tx,y\rangle|^2 \leqslant \left\langle |T|^{2\alpha}x,x\right\rangle \left\langle |T|^{2\beta}y,y\right\rangle$,

(ii) If f,g *are non-negative continuous functions on* $[0,\infty)$ *satisfying* $f(t)g(t)=t$ $(t \geqslant 0)$, *then* $|\langle Tx,y\rangle| \leqslant \|f(|T|)x\|\,\|g(|T^*|)y\|$.

THEOREM 4. *Let* $A,B,C,D \in \mathscr{B}(\mathscr{H})$ *such that* $|A|B = B^*|A|$ *and* $|C|D = D^*|C|$. *If* f *and* g *are as in Lemma 2, then for all* $r,s \geqslant 1$, *we have*

$$\|AB+CD\|^2 \leqslant r_0^2 2^{2-m} \left\|f^{2r}(|A|)+f^{2r}(|C|)\right\|^{\frac{1}{r}} \left\|g^{2s}(|A^*|)+g^{2s}(|C^*|)\right\|^{\frac{1}{s}}, \qquad (28)$$

where $r_0 = \max\{r(B), r(D)\}$ *and* $m = \frac{1}{r}+\frac{1}{s}$. *In particular, for all* $r \geqslant 1$, *we have*

$$\|AB+CD\|^{2r} \leqslant r_0^2 2^{2r-2} \left\|f^{2r}(|A|)+f^{2r}(|C|)\right\| \left\|g^{2r}(|A^*|)+g^{2r}(|C^*|)\right\|.$$

To prove Theorem 4, we need the following lemma was established by Kittaneh [9].

LEMMA 3. *Let* $A,B \in \mathscr{B}(\mathscr{H})$ *such that* $|A|B = B^*|A|$. *If* f *and* g *are as in Lemma 2, then*

$$|\langle ABx,y\rangle| \leqslant r(B)\,\|f(|A|)x\|\,\|g(|A^*|)y\| \qquad (29)$$

for all $x,y \in \mathscr{H}$.

Proof of Theorem 4. By the Schwarz inequality in the Hilbert space, we have for all $x,y \in \mathscr{H}$

$$\begin{aligned}
&|\langle (AB+CD)x,y\rangle|^2 \\
&= |\langle ABx,y\rangle + \langle CDx,y\rangle|^2 \leqslant [|\langle ABx,y\rangle| + |\langle CDx,y\rangle|]^2 \\
&\leqslant [r(B)\,\|f(|A|)x\|\,\|g(|A^*|)y\| \\
&\quad + r(D)\,\|f(|C|)x\|\,\|g(|C^*|)y\|]^2 \ \text{(by Lemma 3)} \\
&\leqslant r_0^2\left[\left\langle f^2(|A|)x,x\right\rangle^{\frac{1}{2}}\left\langle g^2(|A^*|)y,y\right\rangle^{\frac{1}{2}} + \left\langle f^2(|C|)x,x\right\rangle^{\frac{1}{2}}\left\langle g^2(|C^*|)y,y\right\rangle^{\frac{1}{2}}\right]^2,
\end{aligned}$$

where $r_0 = \max\{r(B), r(D)\}$.

Now, on utilizing the elementary inequality

$$(ab+cd)^2 \leqslant (a^2+c^2)(b^2+d^2), \qquad a,b,c,d \in \mathbb{R},$$

we then conclude that

$$\begin{aligned}&\left[\left\langle f^2(|A|)x,x\right\rangle^{\frac{1}{2}}\left\langle g^2(|A^*|)y,y\right\rangle^{\frac{1}{2}}+\left\langle f^2(|C|)x,x\right\rangle^{\frac{1}{2}}\left\langle g^2(|C^*|)y,y\right\rangle^{\frac{1}{2}}\right]^2\\ &\leqslant \left[\left\langle f^2(|A|)x,x\right\rangle+\left\langle f^2(|C|)x,x\right\rangle\right]\left[\left\langle g^2(|A^*|)y,y\right\rangle+\left\langle g^2(|C^*|)y,y\right\rangle\right]\end{aligned}$$

for all $x,y \in \mathscr{H}$.

Utilizing the arithmetic-geometric mean inequality and then the convexity of the function $f(t)=t^q$ $(q \geqslant 1)$, we have for all $r,s \geqslant 1$ that

$$\begin{aligned}&\left[\left\langle f^2(|A|)x,x\right\rangle+\left\langle f^2(|C|)x,x\right\rangle\right]\left[\left\langle g^2(|A^*|)y,y\right\rangle+\left\langle g^2(|C^*|)y,y\right\rangle\right]\\ &\leqslant 4\left\langle\left[\frac{f^{2r}(|A|)+f^{2r}(|C|)}{2}\right]x,x\right\rangle^{\frac{1}{r}}\left\langle\left[\frac{g^{2s}(|A^*|)+g^{2s}(|C^*|)}{2}\right]y,y\right\rangle^{\frac{1}{s}}\end{aligned}$$

for all unit vectors $x,y \in \mathscr{H}$. Taking the supremum all unit vectors $x,y \in \mathscr{H}$, we deduce the desired result. □

In Theorem 4, if we take $f(t)=t^{\alpha}$, $g(t)=t^{1-\alpha}$, we have

COROLLARY 6. *Let $A,B,C,D \in \mathscr{B}(\mathscr{H})$ such that $|A|B=B^*|A|$ and $|C|D=D^*|C|$ and $0 \leqslant \alpha \leqslant 1$. Then for all $r,s \geqslant 1$ we have*

$$\|AB+CD\|^2 \leqslant r_0^2 2^{2-m}\left\||A|^{2r\alpha}+|C|^{2r\alpha}\right\|^{\frac{1}{r}}\left\||A^*|^{2s(1-\alpha)}+|C^*|^{2s(1-\alpha)}\right\|^{\frac{1}{s}},$$

where $r_0=\max\{r(B),r(D)\}$ and $m=\frac{1}{r}+\frac{1}{s}$. In particular, for all $r \geqslant 1$, we have

$$\|AB+CD\|^{2r} \leqslant r_0^2 2^{2r-2}\left\||A|^{2r\alpha}+|C|^{2r\alpha}\right\|\left\||A^*|^{2s(1-\alpha)}+|C^*|^{2s(1-\alpha)}\right\|.$$

Note that our inequality in the previous theorem is a generalization of the second inequality (9)when we set $B=D=I$.

3. Generalized numerical radius inequalities

In this section, we will prove several numerical radius inequalities. We have obtained the recently proved numerical radius inequality as a special case, we will the current numerical radius inequality is sharper than the recently proved numerical radius inequality. Our results summarize many results in the literature.

THEOREM 5. *If $A \in \mathscr{B}(\mathscr{H})$, then*

$$w^{2r}(A) \leqslant \left(\frac{1-\alpha}{2}\right)w^r(A^2)+\left(\frac{1+\alpha}{2}\right)\|A\|^{2r} \tag{30}$$

for any $r \geqslant 1$ and $\alpha \in [0,1]$.

To prove Theorem 5, we need the following two lemmas.

LEMMA 4. *For every vectors $a,b,e \in \mathscr{H}$ with $\|e\| = 1$, we have*

$$|\langle a,e\rangle \langle e,b\rangle| \leqslant \left(\frac{1+\alpha}{2}\right) \|a\| \|b\| + \left(\frac{1-\alpha}{2}\right) |\langle a,b\rangle| \leqslant \|a\| \|b\| \tag{31}$$

for every $\alpha \in [0,1]$.

Proof. We recall the following refinement of the Cauchy-Schwartz inequality obtained by Dragomir in [2]. It says that

$$|\langle a,b\rangle| \leqslant |\langle a,b\rangle - \langle a,e\rangle \langle e,b\rangle| + |\langle a,e\rangle \langle e,b\rangle| \leqslant \|a\| \|b\| \tag{32}$$

for all $a,b,e \in \mathscr{H}$ with $\|e\| = 1$.

From inequality (32), we conclude that

$$\begin{aligned}
|\langle a,e\rangle \langle e,b\rangle| &= \alpha |\langle a,e\rangle \langle e,b\rangle| + (1-\alpha)|\langle a,e\rangle \langle e,b\rangle| \\
&\leqslant \alpha \|a\| \|b\| + \left(\frac{1-\alpha}{2}\right) [\|a\| \|b\| + |\langle a,b\rangle|] \\
&= \left(\frac{1+\alpha}{2}\right) \|a\| \|b\| + \left(\frac{1-\alpha}{2}\right) |\langle a,b\rangle| \leqslant \|a\| \|b\| \\
&\leqslant \|a\| \|b\|.
\end{aligned}$$

This completes the proof. □

The second lemma is a simple consequence of the classical Jensen and Young inequalities (see [17]).

LEMMA 5. *Let $a,b \geqslant 0$, $0 \leqslant \alpha \leqslant 1$ and $p,q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

(i) $a^{\alpha} b^{1-\alpha} \leqslant \alpha a + (1-\alpha)b \leqslant (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}$;

(ii) $ab \leqslant \dfrac{a^p}{p} + \dfrac{b^q}{q} \leqslant \left(\dfrac{a^{pr}}{p} + \dfrac{b^{qr}}{q}\right)^{\frac{1}{r}}$;

for every $r \geqslant 1$.

Proof of Theorem 5. In inequality (31), put $e = x$ with $\|x\| = 1$, $a = Ax$, $b = A^*x$ in inequality (31) and use Lemma 5 to get

$$\begin{aligned}
|\langle Ax,x\rangle|^2 &\leqslant \left(\frac{1-\alpha}{2}\right) |\langle Ax, A^*x\rangle| + \left(\frac{1+\alpha}{2}\right) \|Ax\| \|A^*x\| \\
&\leqslant \left(\frac{1-\alpha}{2}\right) \left|\left\langle A^2x,x\right\rangle\right| + \left(\frac{1+\alpha}{2}\right) \|Ax\| \|A^*x\| \\
&\leqslant \left[\left(\frac{1-\alpha}{2}\right) \left|\left\langle A^2x,x\right\rangle\right|^r + \left(\frac{1+\alpha}{2}\right) \|Ax\|^r \|A^*x\|^r\right]^{\frac{1}{r}}.
\end{aligned}$$

Hence

$$|\langle Ax,x\rangle|^{2r} \leqslant \left(\frac{1-\alpha}{2}\right)\left|\langle A^2x,x\rangle\right|^r + \left(\frac{1+\alpha}{2}\right)\|Ax\|^r\|A^*x\|^r. \tag{33}$$

Taking the supremum over all unit vectors $x\in\mathscr{H}$, we deduce the desired result. □

PROPOSITION 1. *Let $A\in\mathscr{B}(\mathscr{H})$ and f and g be nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$ $(t\geqslant 0)$. Then*

$$w^{2r}(A) \leqslant \left(\frac{1-\alpha}{2}\right)\left\|\frac{1}{p}f^{pr}\left(|A|^2\right)+\frac{1}{q}g^{qr}\left(|(A^2)^*|\right)\right\| + \left(\frac{1+\alpha}{2}\right)\|A\|^{2r} \tag{34}$$

for all $r\geqslant 1$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,qr\geqslant 2$.

Proof. Let $x\in\mathscr{H}$ be a unit vector, we have

$$\begin{aligned}
\left|\langle A^2x,x\rangle\right|^r &\leqslant \left\|f\left(|A^2|\right)x\right\|^r\left\|g\left(|(A^*)^2|\right)x\right\|^r \quad \text{(by Lemma 2)}\\
&= \left\langle f^2\left(|A|^2\right)x,x\right\rangle^{\frac{r}{2}}\left\langle g^2\left(|(A^2)^*|\right)x,x\right\rangle^{\frac{r}{2}}\\
&\leqslant \frac{1}{p}\left\langle f^2\left(|A|^2\right)x,x\right\rangle^{\frac{pr}{2}}+\frac{1}{q}\left\langle g^2\left(|(A^2)^*|\right)x,x\right\rangle^{\frac{r}{2}} \quad \text{(by Lemma 5)}\\
&\leqslant \frac{1}{p}\left\langle f^{pr}\left(|A|^2\right)x,x\right\rangle+\frac{1}{q}\left\langle g^{qr}\left(|(A^2)^*|\right)x,x\right\rangle \quad \text{(by Lemma 6)}\\
&= \left\langle\left[\frac{1}{p}f^{pr}\left(|A|^2\right)+\frac{1}{q}g^{qr}\left(|(A^2)^*|\right)\right]x,x\right\rangle.
\end{aligned}$$

It follows from inequality (33) that

$$|\langle Ax,x\rangle|^{2r} \leqslant \left(\frac{1-\alpha}{2}\right)\left\langle\left[\frac{1}{p}f^{pr}\left(|A|^2\right)+\frac{1}{q}g^{qr}\left(|(A^2)^*|\right)\right]x,x\right\rangle + \left(\frac{1+\alpha}{2}\right)\|Ax\|^r\|A^*x\|^r.$$

Taking the supremum over all unit vectors $x\in\mathscr{H}$, we deduce the desired result. □

Inequality (34) induces several numerical radius inequalities as special cases. For example the following result may be stated as well.

COROLLARY 7. *If we take $f(t)=t^{\nu}$, $g(t)=t^{1-\nu}$ and $p=q=2$ in inequality (34), then*

$$w^{2r}(A) \leqslant \frac{1}{2}\left(\frac{1-\alpha}{2}\right)\left\||A|^{4r\nu}+|A^*|^{4r(1-\nu)}\right\| + \left(\frac{1+\alpha}{2}\right)\|A\|^{2r}$$

for any $r\geqslant 1$, $0\leqslant\alpha\leqslant 1$ and $0\leqslant\nu\leqslant 1$.

In addition, by choosing $\alpha=\frac{1}{3}$ and $\nu=\frac{1}{2}$, we have

$$w^{2r}(A) \leqslant \frac{1}{6}\left\||A|^{2r}+|A^*|^{2r}\right\| + \frac{2}{3}\|A\|^{2r}.$$

4. Numerical radius inequalities for product of operators

To prove our main results, we need the following two lemmas.

The first lemma follows from the spectral theorem for positive operators and Jensen's inequality (see [9]).

LEMMA 6. (Hölder Mc-Carty inequality). *Let $T \in \mathscr{B}(\mathscr{H})$, $T \geqslant 0$ and let $x \in \mathscr{H}$ be any unit vector. Then we have*

(i) $\langle Tx, x\rangle^r \leqslant \langle T^r x, x\rangle$ *for* $r \geqslant 1$.

(ii) $\langle T^r x, x\rangle \leqslant \langle Tx, x\rangle^r$ *for* $0 < r \leqslant 1$.

The second lemma concerned with positive real numbers, and it is a consequence of the convexity of the function $f(t) = t^r$, $r \geqslant 1$.

LEMMA 7. *Let $a_i, i = 1, \cdots, n$ be positive real numbers. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leqslant n^{r-1}\sum_{i=1}^{n} a_i^r \quad \text{for } r \geqslant 1. \tag{35}$$

The next result is a generalization of inequality (14) and [21, Theorem 3.3].

THEOREM 6. *Suppose that $A_i, B_i, X_i \in \mathscr{B}(\mathscr{H})$ $(i = 1, \cdots, n)$ such that A_i, B_i $(i = 1, \cdots, n)$ are positive. Then*

$$w^r\left(\sum_{i=1}^{n} A_i^{\alpha} X_i B_i^{\alpha}\right) \leqslant n^{r-1}\|X\|^r \sum_{i=1}^{n}\left\|\frac{1}{p_i}A_i^{2p_i r} + \frac{1}{q_i}B_i^{2q_i r}\right\|^{\alpha} \tag{36}$$

for all $0 \leqslant \alpha \leqslant 1$, $r \geqslant 1$, $p_i, q_i > 1$ with $\frac{1}{p_i} + \frac{1}{q_i} = 1$ $(i = 1, \cdots, n)$ and $p_i r, q_i r \geqslant 2$ and $\|X\| = \max_{1 \leqslant i \leqslant n}\|X_i\|$.

Proof. For any unit vector $x \in \mathscr{H}$ and by the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^{n} A_i^{\alpha} X_i B_i^{\alpha}\right)x, x\right\rangle\right|^r = \left|\sum_{i=1}^{n}\langle A_i^{\alpha} X_i B_i^{\alpha} x, x\rangle\right|^r \\
&= \left|\sum_{i=1}^{n}\langle X_i B_i^{\alpha} x, A_i^{\alpha} x\rangle\right|^r \leqslant \left|\sum_{i=1}^{n}\|X_i\|\,\|A_i^{\alpha}\|\,\|B_i^{\alpha}\|\right|^r \\
&\leqslant n^{r-1}\sum_{i=1}^{n}\|X_i\|^r\|A_i^{\alpha}\|^r\|B_i^{\alpha}\|^r \ \text{(by Lemma 7)} \\
&\leqslant n^{r-1}\|X\|^r\sum_{i=1}^{n}\left\langle A_i^{2\alpha}x, x\right\rangle^{\frac{r}{2}}\left\langle B_i^{2\alpha}x, x\right\rangle^{\frac{r}{2}} \ \left(\|X\| = \max_{1 \leqslant i \leqslant n}\|X_i\|\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant n^{r-1}\|X\|^r\sum_{i=1}^{n}\left[\frac{1}{p_i}\left\langle A_i^{2\alpha}x,x\right\rangle^{\frac{rp_i}{2}}+\frac{1}{q_i}\left\langle B_i^{2\alpha}x,x\right\rangle^{\frac{r}{2}}\right] \text{ (by Lemma 5)}\\
&\leqslant n^{r-1}\|X\|^r\sum_{i=1}^{n}\left[\frac{1}{p_i}\left\langle A_i^{p_ir}x,x\right\rangle^{\alpha}+\frac{1}{q_i}\left\langle B_i^{q_ir}x,x\right\rangle^{\alpha}\right] \text{ (by Lemma 6)}\\
&\leqslant n^{r-1}\|X\|^r\sum_{i=1}^{n}\left[\frac{1}{p_i}\left\langle A_i^{p_ir}x,x\right\rangle+\frac{1}{q_i}\left\langle B_i^{q_ir}x,x\right\rangle\right]^{\alpha}\\
&\qquad \text{(by the convexity of } f(t)=t^{\alpha})\\
&= n^{r-1}\|X\|^r\sum_{i=1}^{n}\left\langle\left(\frac{1}{p_i}A_i^{p_ir}+\frac{1}{q_i}B_i^{q_ir}\right)x,x\right\rangle^{\alpha}.
\end{aligned}
$$

Taking the supremum over all unit vectors $x\in\mathscr{H}$, we deduce the result. □

Our next result is to find an upper bound for power of the numerical radius of $\sum_{i=1}^{n}A_i^{\alpha}X_iB_i^{1-\alpha}$ under assumption $0\leqslant\alpha\leqslant 1$.

THEOREM 7. *Suppose that $A_i,B_i,X_i\in\mathscr{B}(\mathscr{H})$ $(i=1,\cdots,n)$ such that A_i,B_i $(i=1,\cdots,n)$ are positive. Then*

$$
w^r\left(\sum_{i=1}^{n}A_i^{\alpha}X_iB_i^{1-\alpha}\right)\leqslant\|X\|^r\sum_{i=1}^{n}\|\alpha A_i^r+(1-\alpha)B_i^r\| \tag{37}
$$

for all $r\geqslant 2$ and $0\leqslant\alpha\leqslant 1$ and $\|X\|=\max\limits_{1\leqslant i\leqslant n}\|X_i\|$.

Proof. For any unit vector $x\in\mathscr{H}$ and by the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^{n}A_i^{\alpha}X_iB_i^{1-\alpha}\right)x,x\right\rangle\right|^r=\left|\sum_{i=1}^{n}\left\langle A_i^{\alpha}X_iB_i^{1-\alpha}x,x\right\rangle\right|^r\\
&=\left|\sum_{i=1}^{n}\left\langle X_iB_i^{1-\alpha}x,A_i^{\alpha}x\right\rangle\right|^r\leqslant\left|\sum_{i=1}^{n}\|X_i\|\,\|A_i^{\alpha}\|\,\left\|B_i^{1-\alpha}\right\|\right|^r\\
&\leqslant n^{r-1}\sum_{i=1}^{n}\|X_i\|^r\,\|A_i^{\alpha}\|^r\,\left\|B_i^{1-\alpha}\right\|^r \text{ (by Lemma 7)}\\
&\leqslant n^{r-1}\|X\|^r\sum_{i=1}^{n}\left\langle A_i^{2\alpha}x,x\right\rangle^{\frac{r}{2}}\left\langle B_i^{2(1-\alpha)}x,x\right\rangle^{\frac{r}{2}} \ (\|X\|=\max_{1\leqslant i\leqslant n}\|X_i\|)\\
&\leqslant n^{r-1}\|X\|^r\sum_{i=1}^{n}\langle A_i^rx,x\rangle^{\alpha}\langle B_i^rx,x\rangle^{1-\alpha} \text{ (by Lemma 6)}\\
&\leqslant n^{r-1}\|X\|^r\sum_{i=1}^{n}\left(\alpha\langle A_i^rx,x\rangle+(1-\alpha)\langle B_i^rx,x\rangle\right) \text{ (by Lemma 5)}\\
&\leqslant n^{r-1}\|X\|^r\sum_{i=1}^{n}\langle[\alpha A_i^r+(1-\alpha)B_i^r]x,x\rangle.
\end{aligned}
$$

Taking the supremum over all unit vectors $x \in \mathscr{H}$, we deduce the result. □

The next lemma is a direct consequence of [1, Theorem 2.3].

LEMMA 8. *Let f be a non-negative non-decreasing convex function on $[0,\infty)$ and let $A,B \in \mathscr{B}(\mathscr{H})$ be positive operators. Then for any $0<\mu<1$,*

$$\|f(\mu A+(1-\mu)B)\| \leqslant \|\mu f(A)+(1-\mu)f(B)\|. \tag{38}$$

THEOREM 8. *Let $T,S \in \mathscr{B}(\mathscr{H})$, $r \geqslant 1$ and $\alpha,\beta \in [0,1]$. Then*

$$\begin{aligned} w^{2r}(S^*T) &\leqslant \beta \left\|\alpha|A|^{\frac{4r}{\alpha}}+(1-\alpha)|B|^{\frac{4r}{1-\alpha}}\right\|^{\frac{1}{2}} \\ &\qquad +(1-\beta)w^r(S^*T)\left\|\alpha|A|^{\frac{2r}{\alpha}}+(1-\alpha)|B|^{\frac{2r}{1-\alpha}}\right\|^{\frac{1}{2}} \\ &\leqslant \left\|\alpha|A|^{\frac{4r}{\alpha}}+(1-\alpha)|B|^{\frac{4r}{1-\alpha}}\right\|^{\frac{1}{2}} \end{aligned} \tag{39}$$

Proof. For all $\alpha,\beta \in [0,1]$, we have

$$\begin{aligned} w^{2r}(S^*T) &= \beta w^{2r}(S^*T)+(1-\beta)w^{2r}(S^*T) \leqslant \beta \left\|\alpha|A|^{\frac{2r}{\alpha}}+(1-\alpha)|B|^{\frac{2r}{1-\alpha}}\right\| \\ &\qquad +(1-\beta)w^r(S^*T)\left\|\alpha|A|^{\frac{2r}{\alpha}}+(1-\alpha)|B|^{\frac{2r}{1-\alpha}}\right\|^{\frac{1}{2}} \quad \text{(by inequality (12))} \\ &\leqslant \beta \left\|\left(\alpha|A|^{\frac{2r}{\alpha}}+(1-\alpha)|B|^{\frac{2r}{1-\alpha}}\right)^2\right\|^{\frac{1}{2}} \\ &\qquad +(1-\beta)w^r(S^*T)\left\|\alpha|A|^{\frac{2r}{\alpha}}+(1-\alpha)|B|^{\frac{2r}{1-\alpha}}\right\|^{\frac{1}{2}} \\ &\leqslant \beta \left\|\alpha|A|^{\frac{4r}{\alpha}}+(1-\alpha)|B|^{\frac{4r}{1-\alpha}}\right\|^{\frac{1}{2}} \\ &\qquad +(1-\beta)w^r(S^*T)\left\|\alpha|A|^{\frac{2r}{\alpha}}+(1-\alpha)|B|^{\frac{2r}{1-\alpha}}\right\|^{\frac{1}{2}} \quad \text{(by Lemma 8)} \\ &\leqslant \left\|\alpha|A|^{\frac{4r}{\alpha}}+(1-\alpha)|B|^{\frac{4r}{1-\alpha}}\right\|^{\frac{1}{2}}. \quad \square \end{aligned}$$

Note that our inequality in the previous theorem is a generalization of [4, Theorem 2]. Letting $r=1$, $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{3}$, we have

COROLLARY 8. *Let $T,S \in \mathscr{B}(\mathscr{H})$. Then*

$$w^2(S^*T) \leqslant \frac{1}{6}\left\||T|^8+|S|^8\right\|^{\frac{1}{2}}+\frac{1}{3}w(S^*T)\left\||T|^4+|S|^4\right\|^{\frac{1}{2}} \leqslant \frac{1}{2}\left\||A|^8+|B|^8\right\|^{\frac{1}{2}}.$$

Manasrah and Kittaneh [15] obtained the following result which is a refinement of the scalar Young inequality.

LEMMA 9. *Let $a,b>0$, and $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then for $m=1,2,\cdots$, we have*

$$(a^{\frac{1}{p}}b^{\frac{1}{q}})^m + r_0^m(a^{\frac{m}{2}}-b^{\frac{m}{2}})^2 \leqslant \left(\frac{a^r}{p}+\frac{b^r}{q}\right)^{\frac{m}{r}}, \quad r\geqslant 1 \tag{40}$$

where $r_0=\min\{\frac{1}{p},\frac{1}{q}\}$.

The next result is an extension and refinement of the inequality (8).

THEOREM 9. *Let $A,B\in\mathscr{B}(\mathscr{H})$ such that $|A|B=B^*|A|$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If f and g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$ $(t\geqslant 0)$. Then for all $k\in\mathbb{N}$ and $s\geqslant 1$*

$$w^{2s}(AB) \leqslant r^{2s}(B)\left\|\frac{1}{p}f^{2pk}(|A|)+\frac{1}{q}g^{2qk}(|A^*|)\right\|^{\frac{s}{k}} - r^{2s}(B)r_0^k \inf_{\|x\|=1}\phi(x), \tag{41}$$

where $\phi(x):=\left[\left\langle f^{2p}(|A|)x,x\right\rangle^{\frac{k}{2}}-\left\langle g^{2q}(|A^*|)x,x\right\rangle^{\frac{k}{2}}\right]^2$ and $r_0=\min\{\frac{1}{p},\frac{1}{q}\}$.

Proof. Letting $x=y$ in (29), we get

$$\begin{aligned}
|\langle ABx,x\rangle|^{2s} &\leqslant r^{2s}(B)\,\|f(|A|)x\|^{2s}\,\|g(|A^*|)x\|^{2s}\\
&\leqslant r^{2s}(B)\left\langle f^2(|A|)x,x\right\rangle^s\left\langle g^2(|A^*|)x,x\right\rangle^s\\
&\leqslant r^{2s}(B)\left\langle f^{p\cdot\frac{2}{p}}(|A|)x,x\right\rangle^s\left\langle g^{q\cdot\frac{2}{q}}(|A^*|)x,x\right\rangle^s\\
&\leqslant r^{2s}(B)\left(\left\langle f^{2p}(|A|)x,x\right\rangle^{\frac{1}{p}}\left\langle g^{2q}(|A^*|)x,x\right\rangle^{\frac{1}{q}}\right)^s \quad \text{(by Lemma 6)}
\end{aligned}$$

By Lemma 9, we have

$$\begin{aligned}
|\langle ABx,x\rangle|^{2s} &\leqslant r^{2s}(B)\left[\frac{1}{p}\left\langle f^{2p}(|A|)x,x\right\rangle^k+\frac{1}{q}\left\langle g^{2q}(|A^*|)x,x\right\rangle^k\right]^{\frac{s}{k}}\\
&\quad -r_0^k r^{2s}(B)\left[\left\langle f^{2p}(|A|)x,x\right\rangle^{\frac{k}{2}}-\left\langle g^{2q}(|A^*|)x,x\right\rangle^{\frac{k}{2}}\right]^2\\
&\leqslant r^{2s}(B)\left[\frac{1}{p}\left\langle f^{2pk}(|A|)x,x\right\rangle+\frac{1}{q}\left\langle g^{2qk}(|A^*|)x,x\right\rangle\right]^{\frac{s}{k}}\\
&\quad -r_0^k r^{2s}(B)\left[\left\langle f^{2p}(|A|)x,x\right\rangle^{\frac{k}{2}}-\left\langle g^{2q}(|A^*|)x,x\right\rangle^{\frac{k}{2}}\right]^2 \quad \text{(by Lemma 6)}\\
&\leqslant r^{2s}(B)\left\langle\left[\frac{1}{p}f^{2pk}(|A|)+\frac{1}{q}g^{2qk}(|A^*|)\right]x,x\right\rangle^{\frac{s}{k}}\\
&\quad -r_0^k r^{2s}(B)\left[\left\langle f^{2p}(|A|)x,x\right\rangle^{\frac{k}{2}}-\left\langle g^{2q}(|A^*|)x,x\right\rangle^{\frac{k}{2}}\right]^2.
\end{aligned}$$

Taking the supremum all unit vectors $x\in\mathscr{H}$, we get the result. □

Letting $s=k=1$ and $p=q=2$ in Theorem 9, we have

COROLLARY 9. *Let $A,B \in \mathscr{B}(\mathscr{H})$ such that $|A|B = B^*|A|$. If f and g are non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \geqslant 0)$. Then*

$$w^2(AB) \leqslant \frac{1}{2}r^2(B)\left\|f^4(|A|)+g^4(|A^*|)\right\| - \frac{1}{2}r^2(B)\inf_{\|x\|=1}\phi(x), \tag{42}$$

where $\phi(x) := \left[\left\langle f^4(|A|)x,x\right\rangle^{\frac{1}{2}} - \left\langle g^4(|A^*|)x,x\right\rangle^{\frac{1}{2}}\right]^2$.

The following useful estimate of a spectral radius was obtained by Kittaneh in [12].

LEMMA 10. *If $A,B \in \mathscr{B}(\mathscr{H})$, then*

$$r(AB) \leqslant \frac{1}{4}\left(\|AB\|+\|BA\|+\sqrt{(\|AB\|-\|BA\|)^2+4m(A,B)}\right), \tag{43}$$

where $m(A,B) := \min\{\|A\|\,\|BAB\|, \|B\|\,\|ABA\|\}$.

The following fundamental norm estimates is very useful in the proof of our result.

$$\|A+B\| \leqslant \frac{1}{2}\left(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2+4\left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|^2}\right), \tag{44}$$

and

$$\left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\| \leqslant \|AB\|^{\frac{1}{2}}. \tag{45}$$

Both estimates are valid for all positive operators A and B. Also, it should be noted that (44) is sharper than the triangle inequality as pointed out by Kittaneh [13].

THEOREM 10. *Let $A,B \in \mathscr{B}(\mathscr{H})$ such that $|A|B = B^*|A|$. If f and g are non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \geqslant 0)$. Then*

$$\begin{aligned} w^2(AB) \leqslant \frac{1}{8}\left(\|B\|+\left\|B^2\right\|^{\frac{1}{2}}\right)^2 &\Big(\left\|f^4(|A|)\right\|+\left\|g^4(|A^*|)\right\| \\ &+\sqrt{(\|f^4(|A|)\|-\|g^4(|A^*|)\|)^2+4\left\|f^2(|A|)g^2(|A^*|)\right\|^2}\Big) \\ &-\frac{1}{8}\left(\|B\|+\left\|B^2\right\|^{\frac{1}{2}}\right)^2\inf_{\|x\|=1}\eta(x), \end{aligned} \tag{46}$$

where $\eta(x) := \left[\left\langle f^4(|A|)x,x\right\rangle^{\frac{1}{2}} - \left\langle g^4(|A^*|)x,x\right\rangle^{\frac{1}{2}}\right]^2$.

Proof. By using inequality (43) and $A = I$, we have

$$r(B) \leqslant \frac{1}{4}\left(\|B\|+\left\|B^2\right\|^{\frac{1}{2}}\right). \tag{47}$$

Now by using inequality (44), we have

$$\left\|f^4(|A|)\right\|+\left\|g^4(|A^*|)\right\| \leqslant \frac{1}{2}\Big(\left\|f^4(|A|)\right\|+\left\|g^4(|A^*|)\right\| + \sqrt{(\|f^4(|A|)\|-\|g^4(|A^*|)\|)^2+4\left\|f^2(|A|)g^4(|A^*|)\right\|^2}\Big) \tag{48}$$

By substituting (47) and (48) into (42), we deduce the desired result. □

Letting $f(t)=t^{\alpha}, g(t)=t^{1-\alpha}$ in Theorem 10, we have

COROLLARY 10. *Let $A,B\in\mathscr{B}(\mathscr{H})$ such that $|A|B=B^*|A|$. If f and g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$ $(t\geqslant 0)$. Then for all $0\leqslant\alpha\leqslant 1$*

$$\begin{aligned} w^2(AB) \leqslant \frac{1}{8}\left(\|B\|+\left\|B^2\right\|^{\frac{1}{2}}\right)^2 &\left(\left\||A|^{4\alpha}\right\|+\left\||A^*|^{4(1-\alpha)}\right\|\right. \\ &\left.+\sqrt{\left(\left\||A|^{4\alpha}\right\|-\left\||A^*|^{4(1-\alpha)}\right\|\right)^2+4\left\||A|^{2\alpha}|A^*|^{2(1-\alpha)}\right\|^2}\right) \\ &-\frac{1}{8}\left(\|B\|+\left\|B^2\right\|^{\frac{1}{2}}\right)^2 \inf_{\|x\|=1}\psi(x), \end{aligned} \tag{49}$$

where $\psi(x):=\left[\left\langle |A|^{4\alpha}x,x\right\rangle^{\frac{1}{2}}-\left\langle |A^*|^{4(1-\alpha)}x,x\right\rangle^{\frac{1}{2}}\right]^2$.

As an application of Corollary 10, if we take $\alpha=\frac{1}{2}$, we have

COROLLARY 11. *Let $A,B\in\mathscr{B}(\mathscr{H})$ such that $|A|B=B^*|A|$. If f and g are non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$ $(t\geqslant 0)$. Then*

$$\begin{aligned} w^2(AB) \leqslant \frac{1}{8}\left(\|B\|+\left\|B^2\right\|^{\frac{1}{2}}\right)^2 &\left(\left\||A|^2\right\|+\left\||A^*|^2\right\|\right. \\ &\left.+\sqrt{\left(\left\||A|^2-\||A^*|^2\right\|^2+4\left\||A|\,|A^*|\right\|^2\right\|\right)}\right) \\ &-\frac{1}{8}\left(\|B\|+\left\|B^2\right\|^{\frac{1}{2}}\right)^2 \inf_{\|x\|=1}\psi(x), \end{aligned} \tag{50}$$

where $\psi(x):=\left[\left\langle |A|^2x,x\right\rangle^{\frac{1}{2}}-\left\langle |A^*|^2x,x\right\rangle^{\frac{1}{2}}\right]^2$.

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