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NEW RETARDED DYNAMIC INEQUALITIES ON TIME SCALES WITH APPLICATIONS

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Abstract. In this article, we prove new retarded dynamic inequalities on time scales that contain some integral and discrete inequalities reported in the literature. These inequalities can be used as handy tools for the study of qualitative properties of solutions of dynamic equations on time scales. Some examples are included to demonstrate the applications of our results.

1. Introduction

In 2006, Pachpatte [14] established the inequality:

$$w(t) \leqslant c(t) + \int_a^t f(s)w(s)ds + \int_a^b g(s)w(s)ds, \tag{1.1}$$

for all $t \in [a,b] \subseteq \mathbb{R}$. After that, in 2014, Kender et al. [8] established the following further generalizations of inequality (1.1) where the linear term of the unknown function $\omega(t)$ has been replaced by nonlinear term $\omega^p(t)$ in both sides of the inequality as follows:

$$w^p(t) \leqslant c(t) + \int_a^t f(s)w(s)ds + \int_a^b g(s)w^p(s)ds, \tag{1.2}$$

for all $t \in [a,b] \subseteq \mathbb{R}$. Recently, in 2017, El-Deeb and Ahmed [5] studied the retarded version of the inequality (1.2), where they replaced the non-retarded case t with the retarded case $\alpha(t)$ as follows:

$$w^p(t) \leqslant c(t) + \int_a^{\alpha(t)} f(s)w(s)ds + \int_a^b g(s)w^p(s)ds, \tag{1.3}$$

for all $t \in [a,b] \subseteq \mathbb{R}$, where $\alpha(t) \leqslant t$ and $\alpha(a) = a$.

The main aim of this article is to extend the inequalities obtained in [5, 8, 14] to contain corresponding integral inequalities and discrete inequalities as special cases. Our main results will be proved by employing some useful inequalities which will be presented in Section 2 and Theorem 3.1.

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The study of dynamic equations on time scales which goes back to Stefan Hilger [6] and is an area of mathematics has recently received a lot of attention. The general idea is to prove a result for a dynamic equation or a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$; see [2, 3]. One of the purposes of the theory of time scales is to unify continuous and discrete analysis. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see [7]), i.e., when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=\overline{q^{\mathbb{Z}}}=\{q^k : k\in\mathbb{Z}, q>1\}\cup\{0\}$ where $q>1$. The book on the subject of time scales by Bohner and Peterson [4] summarizes and organizes much of time scale calculus. During the past decade a number of dynamic inequalities have been established by some authors which are motivated by some applications, for example, we refer the reader to [1, 4, 10, 11, 13] for contributions, and the references cited therein.

2. Some preliminaries and lemmas

For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The time scales calculus was initiated by Hilger in his Ph.D. thesis in order to unify discrete and continuous analysis [6]. The cases when $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$ represent the classical theories of differential and difference calculus. First we define the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ by

$$\sigma(t) := \inf\{s\in\mathbb{T} : s>t\}, \tag{2.1}$$

and second, the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ by

$$\rho(t) := \sup\{s\in\mathbb{T} : s<t\}. \tag{2.2}$$

In this definition, we put $\inf\emptyset=\sup\mathbb{T}$ and $\sup\emptyset=\inf\mathbb{T}$, where $\emptyset$ is the empty set. A point $t\in\mathbb{T}$ with $\inf\mathbb{T}<t<\sup\mathbb{T}$, is said to be left-dense if $\rho(t)=t$ and is right-dense if $\sigma(t)=t$, points that are simultaneously right-dense and left-dense are said to be dense, is left-scattered if $\rho(t)<t$ and right-scattered if $\sigma(t)>t$, points that are simultaneously right-scattered and left-scattered are said to be isolated. A function $g:\mathbb{T}\to\mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided g is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-sided limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. A function $f:\mathbb{T}\to\mathbb{R}$ is said to be left-dense continuous (ld-continuous) provided f is continuous at left-dense points and at right-dense points in $\mathbb{T}$, right-sided limits exist and are finite. The set of all such ld-continuous functions is denoted by $C_{ld}(\mathbb{T})$.

The forward and backward graininess functions μ and ν for a time scale $\mathbb{T}$ are defined by $\mu(t):=\sigma(t)-t$, and $\nu(t):=t-\rho(t)$, respectively.

Given a time scale $\mathbb{T}$, we introduce the sets $\mathbb{T}^{\kappa}$, $\mathbb{T}_{\kappa}$, and $\mathbb{T}^{\kappa}_{\kappa}$ as follows. If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^{\kappa}=\mathbb{T}-\{t_1\}$, otherwise $\mathbb{T}^{\kappa}=\mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}^{\kappa}=\mathbb{T}-\{t_2\}$, otherwise $\mathbb{T}^{\kappa}=\mathbb{T}$. Finally, $\mathbb{T}^{k}_{\kappa}=\mathbb{T}^{\kappa}\cap\mathbb{T}_{\kappa}$.

Let $f : \mathbb{T} \to \mathbb{R}$ be a real-valued function on a time scale $\mathbb{T}$. Then for all $t \in \mathbb{T}\kappa$, we define $f^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\varepsilon > 0$ there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leqslant \varepsilon|\sigma(t) - s|, \qquad \forall s \in U.$$

For $f : \mathbb{T} \to \mathbb{R}$, we define the function $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ by $f^{\sigma}(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, that is, $f^{\sigma} = f \circ \sigma$. Similarly, we define the function $f^{\rho} : \mathbb{T} \to \mathbb{R}$ by $f^{\rho}(t) = f(\rho(t))$ for all $t \in \mathbb{T}$, that is, $f^{\rho} = f \circ \rho$. A time scale $\mathbb{T}$ is said to be regular if the following two conditions are satisfied simultaneously: (1) $\sigma(\rho(t)) = t$ and (2) $\rho(\sigma(t)) = t$, $\forall t \in \mathbb{T}$. The product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$, here $g^{\sigma} = g \circ \sigma$) of two differentiable functions f and g, are given as the following:

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$$

and

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided that $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{\kappa}$, and the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

We will frequently use the following useful relations between calculus on time scales $\mathbb{T}$ and differential calculus on $\mathbb{R}$, difference calculus on $\mathbb{Z}$, and quantum calculus on $\overline{q^{\mathbb{Z}}}$. Note that if

(i) $\mathbb{T} = \mathbb{R}$, then

$$\sigma(t) = t, \quad \mu(t) = 0, \quad f^{\Delta}(t) = f'(t), \quad \int_a^b f(t)\Delta t = \int_a^b f(t)dt; \tag{2.3}$$

(ii) if $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(t) = t+1, \quad \mu(t) = 1, \quad f^{\Delta}(t) = \Delta f(t), \quad \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t); \tag{2.4}$$

(iii) and if $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, $q > 1$, then

$$\sigma(t) = qt, \quad \mu(t) = (q-1)t, \quad \int_a^b f(t)\Delta t = (q-1) \sum_{k=\log_q(a)}^{\log_q(b)-1} q^k f(q^k), \quad \forall a,b \in q^{\mathbb{N}_0}. \tag{2.5}$$

It can be shown (see [4]) that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^{t} g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. An infinite integral is defined as

$$\int_a^{\infty} f(t)\Delta(t) = \lim_{b \to \infty} \int_a^b f(t)\Delta t.$$

Now, we will give the definition of the generalized exponential function and its derivatives. We say that $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$, we define the set $\Re$ of all regressive and rd-continuous functions. We define the set $\Re^+$ of all positively regressive elements of $\Re$ by $\Re^+ = \{p \in \Re : 1 + \mu(t)p(t) > 0, \forall t \in \mathbb{T}\}$. The set of all regressive functions on a time scale $\mathbb{T}$ forms an Abelian group under the addition $\oplus$ defined by $p \oplus q = p + q + \mu pq$. If $p \in \Re$, then we can define the exponential function by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right), \quad s,t \in \mathbb{T},$$

where $\xi_h(z)$ is the cylinder transformation, which is defined by

$$\xi_h(z) = \begin{cases} \frac{Log(1+hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

If $p \in \Re$, then $e_p(t,s)$ is real-valued and nonzero on $\mathbb{T}$. If $p \in \Re^+$, then $e_p(t,t_0)$ is always positive.

Note that

- if $\mathbb{T} = \mathbb{R}$, then
$$e_a(t,t_0) = \exp\left(\int_{t_0}^t a(s)ds \right); \tag{2.6}$$
- if $\mathbb{T} = \mathbb{Z}$, then
$$e_a(t,t_0) = \prod_{s=t_0}^{t-1} \left(1 + a(s)\right); \tag{2.7}$$
- if $\mathbb{T} = q^{\mathbb{N}_0}$, then
$$e_a(t,t_0) = \prod_{s=t_0}^{t-1} \left(1 + (q-1)sa(s)\right). \tag{2.8}$$

In the following, we present the basic lemmas that will be needed in the proof of our main results.

LEMMA 2.1. ([9]) *If $p,q \in \Re$ and $a,\ b,\ c\ \in \mathbb{T}$, then*

1. $e_p(t,t) = 1$ *and* $e_0(t,s) = 1$*;*
2. $e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s)$*;*

3. *if* $p \in \Re^+$, *then* $e_p(t,t_0) > 0$, $\forall t \in \mathbb{T}$;

4. $\int_a^b p(t)e_p(c,\sigma(t))\Delta t = -\int_a^b [e_p(c,\cdot)]^\Delta(t)\Delta t = e_p(c,a) - e_p(c,b)$.

LEMMA 2.2. (See [9]) *If* $p \in \Re$ *and fix* $t \in \mathbb{T}$, *then the exponential function* $e_p(t,t_0)$ *is the unique solution of the following initial value problem:*

$$\begin{cases} y^\Delta(t) = p(t)y(t), \\ y(t_0) = 1. \end{cases} \tag{2.9}$$

LEMMA 2.3. (See [9]) *Let* $t_0 \in \mathbb{T}^\kappa$ *and* $k : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ *be continuous at* (t,t), *where* $t > t_0$ *and* $t \in \mathbb{T}^\kappa$. *Assume that* $k^\Delta(t,\cdot)$ *is rd-continuous on* $[t_0,\sigma(t)]$. *If for any* $\varepsilon > 0$, *there exists a neighborhood* U *of* t, *independent of* $\tau \in [t_0,\sigma(t)]$, *such that*

$$|[k(\sigma(t),\tau) - k(s,\tau)] - k^\Delta(t,\tau)[\sigma(t) - s]| \leqslant \varepsilon|\sigma(t) - s|, \qquad \forall s \in U.$$

If k^Δ *denotes the derivative of* k *with respect to the first variable, then*

$$f(t) = \int_{t_0}^t k(t,\tau)\Delta\tau$$

yields

$$f^\Delta(t) = \int_{t_0}^t k^\Delta(t,\tau)\Delta\tau + k(\sigma(t),t).$$

LEMMA 2.4. ([9]) *Suppose* u, $b \in C_{rd}$ *and* $a \in \Re^+$. *Then*

$$u^\Delta(t) \leqslant a(t)u(t) + b(t), \quad t \geqslant t_0, t \in \mathbb{T}^\kappa$$

yields

$$u(t) \leqslant u(t_0)e_a(t,t_0) + \int_{t_0}^t e_a(t,\sigma(\tau))b(\tau)\Delta\tau, \quad t \geqslant t_0, t \in \mathbb{T}^\kappa.$$

LEMMA 2.5. ([12]) *If* $x \geqslant 0$ *and* $p \geqslant 1$, *then*

$$x^{\frac{1}{p}} \leqslant m_1 x + m_2, \tag{2.10}$$

where $m_1 = \frac{1}{p}K^{\frac{1-p}{p}}$, $m_2 = \frac{p-1}{p}K^{\frac{1}{p}}$ *and* $K > 0$.

Now we are ready to state and prove our main results.

3. Main results

In this section, we will state and prove the main results and investigate some dynamic Gronwall-Bellman inequalities on time scales.

First, we prove the basic theorem that will be needed in the proofs of the main results and can be considered as the extension of [3, Theorem 5.37, page 139] on time scales.

THEOREM 3.1. *In the following by $f^{\Delta}(t,s)$ we mean the delta derivative of $f(t,s)$ with respect to t. Similarly, $f^{\nabla}(t,s)$ is understood. If f, f^{Δ} and f^{∇} are continuous, and $u,h:\mathbb{T}\to\mathbb{T}$, then the following formulas holds $\forall t\in\mathbb{T}^{\kappa}$*

(i) $\left[\int_{u(t)}^{h(t)} f(t,s)\Delta s\right]^{\Delta} = \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + h^{\Delta}(t)f(\sigma(t),h(t)) - u^{\Delta}(t)f(\sigma(t),u(t))$;

(ii) $\left[\int_{u(t)}^{h(t)} f(t,s)\Delta s\right]^{\nabla} = \int_{u(t)}^{h(t)} f^{\nabla}(t,s)\Delta s + h^{\nabla}(t)f(\rho(t),h(t)) - u^{\nabla}(t)f(\rho(t),u(t))$;

(iii) $\left[\int_{u(t)}^{h(t)} f(t,s)\nabla s\right]^{\Delta} = \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\nabla s + h^{\Delta}(t)f(\sigma(t),h(t)) - u^{\Delta}(t)f(\sigma(t),u(t))$;

(iv) $\left[\int_{u(t)}^{h(t)} f(t,s)\nabla s\right]^{\nabla} = \int_{u(t)}^{h(t)} f^{\nabla}(t,s)\nabla s + h^{\nabla}(t)f(\rho(t),h(t)) - u^{\nabla}(t)f(\rho(t),u(t))$.

Now we are ready to state and prove our main results, which give us the time scales version of the inequalities proved in [5], [8] and [14].

THEOREM 3.2. *Let $a,b\in\mathbb{T}_k^k$ with $a<b$ and ω, g, f, c, $\alpha_i \in C_{rd}([a,b]_{\mathbb{T}^k},\mathbb{R}^+)$ and c, α_i be delta-differentiable on $\mathbb{T}$ with $c^{\Delta}(t)\geqslant 0$, $\alpha_i^{\Delta}(t)\geqslant 0$, $\alpha_i(t)\leqslant t$, $i=1,2$, $\alpha_1(a)=a$, $\alpha_2(a)=b$ and $p\geqslant 1$ be a constant. If*

$$\omega^p(t)\leqslant c(t)+\int_a^{\alpha_1(t)} g(s)\omega(s)\Delta s+\int_a^{\alpha_2(t)} f(s)\omega^p(s)\Delta s, \tag{3.1}$$

for all $t\in[a,b]_{\mathbb{T}^k}$, then

$$\omega(t)\leqslant\left\{\Lambda_1 e_{\eta_1}(t,a)+\int_a^t e_{\eta_1}(t,\sigma(s))\Xi_1(s)\Delta s\right\}^{\frac{1}{p}}, \tag{3.2}$$

where Λ_1 is the best possible constant i.e., it cannot be replaced with a smaller number such that (3.2) remains true for all relevant functions and given by the following equation:

$$\Lambda_1=\frac{c(a)+\int_a^b f(s)\left(\int_a^s e_{\eta_1}(s,\sigma(\lambda))\Xi_1(\lambda)\Delta\lambda\right)\Delta s}{1-\int_a^b f(s)e_{\eta_1}(s,a)\Delta s}, \tag{3.3}$$

such that

$$\int_a^b f(s)e_{\eta_1}(s,a)\Delta s<1, \tag{3.4}$$

and

$$\Xi_1(t)=c^{\Delta}(t)+m_2[\alpha_1^{\Delta}(t)g(\alpha_1(t))+\alpha_2^{\Delta}(t)f(\alpha_2(t))], \tag{3.5}$$

$$\eta_1(t)=m_1[\alpha_1^{\Delta}(t)g(\alpha_1(t))+\alpha_2^{\Delta}(t)f(\alpha_2(t))], \tag{3.6}$$

where m_1, m_2 are defined as in Lemma 2.5, and $e_{\eta_1}(t,a)$ is a solution of the initial value problem (2.9) in Lemma 2.2 when $p(t)$ replaced by η_1.

Proof. Define a function $z_1(t)$ by

$$z_1(t) = c(t) + \int_a^{\alpha_1(t)} g(s)\omega(s)\Delta s + \int_a^{\alpha_2(t)} f(s)\omega^p(s)\Delta s. \tag{3.7}$$

We observe that $z_1(t) \leqslant 0$ nondecreasing on $[a,b]_{\mathbb{T}^k}$. From (3.7), we get

$$z_1(a) = c(a) + \int_a^b f(s)\omega^p(s)\Delta s. \tag{3.8}$$

Then from (3.1), (3.7) and by using the monotonicity of $z_1(t)$, we get

$$\omega(t) \leqslant z_1^{\frac{1}{p}}(t), \omega(\alpha_i(t)) \leqslant z_1^{\frac{1}{p}}(\alpha_i(t)) \leqslant z_1^{\frac{1}{p}}(t), \quad i = 1,2. \tag{3.9}$$

From (3.7), (3.9), and using Theorem 3.1, we have:

$$\begin{aligned} z_1^{\Delta}(t) &= c^{\Delta}(t) + [\alpha_1^{\Delta}(t)g(\alpha_1(t)) + \alpha_2^{\Delta}(t)f(\alpha_2(t))]\omega(\alpha(t)) \\ &\leqslant c^{\Delta}(t) + [\alpha_1^{\Delta}(t)g(\alpha_1(t)) + \alpha_2^{\Delta}(t)f(\alpha_2(t))]z_1^{\frac{1}{p}}(t). \end{aligned} \tag{3.10}$$

Therefore, using Lemma 2.5. From (3.10), we get that

$$\begin{aligned} z_1^{\Delta}(t) &\leqslant m_1[\alpha_1^{\Delta}(t)g(\alpha_1(t)) + \alpha_2^{\Delta}(t)f(\alpha_2(t))]z_1(t) \\ &\quad + (c^{\Delta}(t) + m_2[\alpha_1^{\Delta}(t)g(\alpha_1(t)) + \alpha_2^{\Delta}(t)f(\alpha_2(t))]). \\ &= \eta_1 z_1(t) + \Xi_1(t), \end{aligned} \tag{3.11}$$

where $\Xi_1(t)$ and $\eta_1(t)$ are defined as in (3.5) and (3.6) respectively. Now an application of Lemma 2.3 to (3.11), yields

$$z_1(t) \leqslant z_1(a)e_{\eta_1}(t,a) + \int_a^t e_{\eta_1}(t,\sigma(s))\Xi_1(s)\Delta s. \tag{3.12}$$

From (3.9) and (3.12), we get that

$$\omega^p(t) \leqslant z_1(a)e_{\eta_1}(t,a) + \int_a^t e_{\eta_1}(t,\sigma(s))\Xi_1(s)\Delta s. \tag{3.13}$$

From (3.8) and (3.13), we have

$$\begin{aligned} z_1(a) &= c(a) + \int_a^b f(s)\omega^p(s)\Delta s \\ &\leqslant c(a) + \int_a^b f(s)\left[z_1(a)e_{\eta_1}(s,a) + \int_a^s e_{\eta_1}(s,\sigma(\lambda))\Xi_1(\lambda)\Delta\lambda\right]\Delta s \\ &\leqslant c(a) + z_1(a)\int_a^b f(s)e_{\eta_1}(s,a)\Delta s \\ &\quad + \int_a^b f(s)\left(\int_a^s e_{\eta_1}(s,\sigma(\lambda))\Xi_1(\lambda)\Delta\lambda\right)\Delta s. \end{aligned} \tag{3.14}$$

Thus from (3.14), we obtain

$$z_1(a) \leqslant \Lambda_1, \tag{3.15}$$

where Λ_1 is defined as in (3.3). Then we get the required inequality (3.2) from (3.13) and (3.15). The proof is complete. $\square$

REMARK 3.1. By taking $\mathbb{T}=\mathbb{R}$, $\alpha_1(t)=t$, $\alpha_2(t)=b$, $\alpha_1^\Delta(t)=1$, $\alpha_2^\Delta(t)=0$, $p=1$ and using the relation (2.3), then Theorem 3.2 reduces to [14, Theorem 1.5.1]. If we put $\mathbb{T}=\mathbb{R}$, $\alpha_1(t)=t$, $\alpha_2(t)=b$, $\alpha_1^\Delta(t)=1$, $\alpha_2^\Delta(t)=0$, and using the relation (2.3), then Theorem 3.2 reduces to [8, Theorem 2.1]. If we put $\mathbb{T}=\mathbb{R}$, $\alpha_2(t)=b$, $\alpha_2^\Delta(t)=0$, and using the relation (2.3), then Theorem 3.2, reduces to [5, Theorem 2.1].

THEOREM 3.3. *Let $a,b\in\mathbb{T}_k^k$ with $a<b$ and ω, g, f, c, $\alpha_i\in C_{rd}([a,b]_{\mathbb{T}^k},\mathbb{R}^+)$ and c, α be delta-differentiable on $\mathbb{T}$ with $c^\Delta(t)\geqslant 0$, $\alpha_i^\Delta(t)\geqslant 0$, $\alpha_i(t)\leqslant t$, $i=1,2$, $\alpha_1(a)=a$, $\alpha_2(a)=b$ and $k(t,s)$, $k^\Delta(t,s)\in C_{rd}([a,b]_{\mathbb{T}^k}\times[a,b]_{\mathbb{T}^k},\mathbb{R}^+)$ for $a\leqslant s\leqslant t\leqslant b$ and $p\geqslant 1$ be a constant. If*

$$\omega^p(t)\leqslant c(t)+\int_a^{\alpha_1(t)}k(t,s)\omega(s)\Delta s+\int_a^{\alpha_2(t)}g(s)\omega^p(s)\Delta s, \tag{3.16}$$

for all $t\in[a,b]_{\mathbb{T}^k}$, then

$$\omega(t)\leqslant\left\{\Lambda_2 e_{\eta_2}(t,a)+\int_a^t\Xi_2(s)e_{\eta_2}(t,\sigma(s))\Delta s\right\}^{\frac{1}{p}}, \tag{3.17}$$

where Λ_2 is the best possible constant i.e., it cannot be replaced with a smaller number such that (3.17) remains true for all relevant functions and given by the following equation:

$$\Lambda_2=\frac{c(a)+\int_a^b g(s)\left(\int_a^s e_{\eta_2}(s,\sigma(\tau))\Xi_2(\tau)\Delta\tau\right)\Delta s}{1-\int_a^b g(s)e_{\eta_2}(s,a)\Delta s}, \tag{3.18}$$

such that

$$\int_a^b g(s)e_{\eta_2}(s,a)\Delta s<1, \tag{3.19}$$

and

$$\Xi_2(t)=c^\Delta(t)+m_2\left[\alpha^\Delta(t)k(\sigma(t),\alpha(t))+\int_a^{\alpha(t)}k^\Delta(t,\tau)\Delta\tau\right], \tag{3.20}$$

$$\eta_2(t)=m_1\left[\alpha^\Delta(t)k(\sigma(t),\alpha(t))+\int_a^{\alpha(t)}k^\Delta(t,\tau)\Delta\tau+\alpha_2^\Delta(t)g(\alpha_2(t))\right], \tag{3.21}$$

where m_1,m_2 are defined as in Lemma 2.5, and $e_{\eta_2}(t,a)$ is the solution of the initial value problem (2.9) in Lemma 2.2 when $p(t)$ replaced by $\eta_2(t)$.

Proof. Define a function $z_2(t)$ by

$$z_2(t)=c(t)+\int_a^{\alpha_1(t)}k(t,s)\omega(s)\Delta s+\int_a^{\alpha_2(t)}g(s)\omega^p(s)\Delta s. \tag{3.22}$$

We observe that $z_2(t)\leqslant 0$ is nondecreasing on $[a,b]_{\mathbb{T}^k}$. From (3.22), we get

$$z_2(a)=c(a)+\int_a^b f(s)\omega^p(s)\Delta s. \tag{3.23}$$

Then from (3.16), (3.22) and by using the monotonicity of $z_1(t)$, we obtain

$$\omega(t) \leqslant z_2^{\frac{1}{p}}(t), \omega(\alpha_i(t)) \leqslant z_2^{\frac{1}{p}}(\alpha_i(t)) \leqslant z_2^{\frac{1}{p}}(t), \quad i = 1,2. \tag{3.24}$$

Using Lemma 2.3 and Theorem 3.1 in (3.22). From (3.24), we have

$$\begin{aligned} z_2^{\Delta}(t) &= c^{\Delta}(t) + \alpha_1^{\Delta}(t)k(\sigma(t),\alpha_1(t))\omega(\alpha_1(t)) + \int_a^{\alpha_1(t)} k^{\Delta}(t,\tau)\omega(\tau)\Delta\tau \\ &\quad + \alpha_2^{\Delta}(t)g(\alpha_2(t))w^p(\alpha_2(t)) \\ &\leqslant c^{\Delta}(t) + \alpha_1^{\Delta}(t)k(\sigma(t),\alpha_1(t))z_2^{\frac{1}{p}}(t) + \int_a^{\alpha_1(t)} k^{\Delta}(t,\tau)z_2^{\frac{1}{p}}(\tau)\Delta\tau \\ &\quad + \alpha_2^{\Delta}(t)g(\alpha_2(t))z_2(t) \\ &\leqslant c^{\Delta}(t) + \left[\alpha_1^{\Delta}(t)k(\sigma(t),\alpha_1(t)) + \int_a^{\alpha_1(t)} k^{\Delta}(t,\tau)\Delta_1\tau\right] z_2^{\frac{1}{p}}(t) \\ &\quad + \alpha_2^{\Delta}(t)g(\alpha_2(t))z_2(t). \end{aligned} \tag{3.25}$$

Using Lemma 2.5 in (3.25), the inequality (3.25) can be written as,

$$\begin{aligned} z_2^{\Delta}(t) &\leqslant c^{\Delta}(t) + m_1\left[\alpha_1^{\Delta}(t)k(\sigma(t),\alpha_1(t)) + \int_a^{\alpha_1(t)} k^{\Delta}(t,\tau)\Delta\tau \right. \\ &\qquad \left. + \alpha_2^{\Delta}(t)g(\alpha_2(t))\right] z_2(t) \\ &\qquad + m_2\left[\alpha_1^{\Delta}(t)k(\sigma(t),\alpha_1(t)) + \int_a^{\alpha_1(t)} k^{\Delta}(t,\tau)\Delta\tau\right] \\ &= \eta_2(t)z_2(t) + \Xi_2(t), \end{aligned} \tag{3.26}$$

where $\eta_2(t)$ and $\Xi_2(t)$ are defined as in (3.21) and (3.20) respectively. Now, using Lemma 2.4 in (3.26), yields that

$$z_2(t) \leqslant z_2(a)e_{\eta_2}(t,a) + \int_a^t e_{\eta_2}(t,\sigma(\tau))\Xi_2(s)\Delta\tau. \tag{3.27}$$

From (3.24) and (3.27), we get that

$$\omega^p(t) \leqslant z_2(a)e_{\eta_2}(t,a) + \int_a^t e_{\eta_2}(t,\sigma(\tau))\Xi_2(\tau)\Delta\tau. \tag{3.28}$$

From (3.23) and (3.28), we have

$$\begin{aligned} z_2(a) &= c(a) + \int_a^b g(s)\omega^p(s)\Delta s \\ &\leqslant c(a) + \int_a^b g(s)\left[z_2(a)e_{\eta_2}(s,a) + \int_a^s e_{\eta_2}(s,\sigma(\tau))\Xi_2(\tau)\Delta\tau\right]\Delta s \\ &\leqslant c(a) + z_2(a)\int_a^b g(s)e_{\eta_2}(s,a)\Delta s \\ &\quad + \int_a^b g(s)\left(\int_a^s e_{\eta_2}(s,\sigma(\tau))\Xi_2(\tau)\Delta\tau\right)\Delta s. \end{aligned} \tag{3.29}$$

Thus from (3.29), we obtain

$$z_2(a) \leqslant \frac{c(a)+\int_a^b g(s)\left(\int_a^s e_{\eta_2}(s,\sigma(\tau))\Xi_2(\tau)\Delta\tau\right)\Delta s}{1-\int_a^b g(s)e_{\eta_2}(s,a)\Delta s} = \Lambda_2, \tag{3.30}$$

for all $t \in [a,b]_{\mathbb{T}^k}$, we get the required inequality (3.17) from (3.28) and (3.30). The proof is complete. □

REMARK 3.2. By taking $\mathbb{T}=\mathbb{R}$, $\alpha_1(t)=t$, $\alpha_1^\Delta=1$, $\alpha_2(t)=b$, $\alpha_2^\Delta(t)=0$, $p=1$ and using the relation (2.3), then Theorem 3.3 reduces to [14, Theorem 1.5.2 (b_1)]. If we put $\mathbb{T}=\mathbb{R}$ and $\alpha_1(t)=t$, $\alpha_1^\Delta=1$, $\alpha_2(t)=b$, $\alpha_2^\Delta(t)=0$, and using the relation (2.3), then Theorem 3.3 reduces to [8, Theorem 2.2]. If we put $\mathbb{T}=\mathbb{R}$ and using the relation (2.3), then Theorem 3.4, reduces to [5, Theorem 2.2].

THEOREM 3.4. *Let ω, α_i and c be defined as in Theorem 3.3, $k_1(t,s)$, $k_2(t,s)$, $k_1^\Delta(t,s)$ and $k_2^\Delta(t,s)$ $\in C_{rd}([a,b]_{\mathbb{T}^k}\times[a,b]_{\mathbb{T}^k},\mathbb{R}^+)$ for $a\leqslant s\leqslant t\leqslant b$ and $p\geqslant 1$ be a constant. Assume that $a,b\in\mathbb{T}_k^k$ with $a<b$. If*

$$\omega^p(t) \leqslant c(t)+\int_a^{\alpha_1(t)} k_1(t,s)\omega(s)\Delta s+\int_a^{\alpha_2(t)} k_2(t,s)\omega^p(s)\Delta s, \tag{3.31}$$

for all $t\in[a,b]_{\mathbb{T}^k}$, then

$$\omega(t) \leqslant \left\{\Lambda_3 e_{\eta_3}(t,a)+\int_a^t \Xi_3(s)e_{\eta_3}(t,\sigma(s))\Delta s\right\}^{\frac{1}{p}}, \tag{3.32}$$

where Λ_3 is the best possible constant i.e., it cannot be replaced with a smaller number such that (3.32) remains true for all relevant functions and given by the following equation:

$$\Lambda_3 = \frac{c(a)+\int_a^b k_2(a,s)\left(\int_a^s \Xi_3(\lambda)e_{\eta_3}(s,\sigma(\lambda))\Delta\lambda\right)\Delta s}{1-\int_a^b k_2(a,s)e_{\eta_3}(s,a)\Delta s}, \tag{3.33}$$

such that

$$\int_a^b k_2(s,a)e_{\eta_3}(s,a)\Delta s<1, \tag{3.34}$$

and

$$\begin{aligned}\eta_3(t) = m_1\left[\alpha_1^\Delta(t)k_1(\sigma(t),\alpha_1(t))+\int_a^{\alpha_1(t)} k_1^\Delta(t,s)\Delta s\right]+\alpha_2^\Delta(t)k_2(\sigma(t),\alpha_2(t))\\ +\int_a^{\alpha_2(t)} k_2^\Delta(t,s)\Delta s,\end{aligned} \tag{3.35}$$

$$\Xi_3(t) = c^\Delta(t)+m_2\left[\alpha_1^\Delta(t)k_1(\sigma(t),\alpha_1(t))+\int_a^{\alpha_1(t)} k_1^\Delta(t,s)\Delta s\right], \tag{3.36}$$

where m_1, m_2 are defined as in Lemma 2.5, and $e_{\eta_3}(t,a)$ is a solution of the initial value problem (2.9) in Lemma 2.2, when $p(t)$ replaced by $\eta_3(t)$.

Proof. Define a function $z_3(t)$ by

$$z_3(t) = c(t) + \int_a^{\alpha_1(t)} k_1(t,s)\omega(s)\Delta s + \int_a^{\alpha_2(t)} k_2(t,s)\omega^p(s)\Delta s. \tag{3.37}$$

We observe that $z_3(t) \leqslant 0$ nondecreasing on $[a,b]_{\mathbb{T}^k}$. From (3.37), we get

$$z_3(a) = c(a) + \int_a^b f(s)\omega^p(s)\Delta s. \tag{3.38}$$

Then from (3.31), (3.37) and by using the monotonicity of $z_1(t)$, we obtain

$$\omega(t) \leqslant z_3^{\frac{1}{p}}(t), \omega(\alpha(t)) \leqslant z_3^{\frac{1}{p}}(\alpha(t)) \leqslant z_3^{\frac{1}{p}}(t). \tag{3.39}$$

From (3.37), (3.39) and by using Theorem 3.1, we have

$$\begin{aligned} z_3^{\Delta}(t) &= c^{\Delta}(t) + \alpha_1^{\Delta}(t)k_1(\sigma(t),\alpha_1(t))\omega(\alpha_1(t)) + \int_a^{\alpha_1(t)} k_1^{\Delta}(t,s)\omega(s)\Delta s \\ &\quad + \alpha_2^{\Delta}(t)k_2(\sigma(t),\alpha_2(t))\omega^p(\alpha_2(t)) + \int_a^{\alpha_2(t)} k_2^{\Delta}(t,s)\omega^p(s)\Delta s \\ &\leqslant c^{\Delta}(t) + \alpha_1^{\Delta}(t)k_1(\sigma(t),\alpha_1(t))z_3^{\frac{1}{p}}(t) + \int_a^{\alpha_1(t)} k_1^{\Delta}(t,s)z_3^{\frac{1}{p}}(s)\Delta s \\ &\quad + \alpha_2^{\Delta}(t)k_2(\sigma(t),\alpha_2(t))z_3(t) + \int_a^{\alpha_2(t)} k_2^{\Delta}(t,s)z_3(s)\Delta s \\ &\leqslant c^{\Delta}(t) + \left[\alpha_1^{\Delta}(t)k_1(\sigma(t),\alpha_1(t)) + \int_a^{\alpha_1(t)} k_1^{\Delta}(t,s)\Delta s\right] z_3^{\frac{1}{p}}(t) \\ &\quad + \left[\alpha_2^{\Delta}(t)k_2(\sigma(t),\alpha_2(t)) + \int_a^{\alpha_2(t)} k_2^{\Delta}(t,s)\Delta s\right] z_3(t). \end{aligned} \tag{3.40}$$

By applying Lemma 2.5 on $z_3^{\frac{1}{p}}(t)$ in (3.40), we have

$$\begin{aligned} z_3^{\Delta}(t) &\leqslant c^{\Delta}(t) + m_1\left[\alpha_1^{\Delta}(t)k_1(\sigma(t),\alpha_1(t)) + \int_a^{\alpha_1(t)} k_1^{\Delta}(t,s)\Delta s\right] z_3(t) \\ &\quad + m_2\left[\alpha_1^{\Delta}(t)k_1(\sigma(t),\alpha_1(t)) + \int_a^{\alpha_1(t)} k_1^{\Delta}(t,s)\Delta s\right] \\ &\quad + \left[\alpha_2^{\Delta}(t)k_2(\sigma(t),\alpha_2(t)) + \int_a^{\alpha_2(t)} k_2^{\Delta}(t,s)\Delta s\right] z_3(t) \\ &\leqslant \left\{m_1\left[\alpha_1^{\Delta}(t)k_1(\sigma(t),\alpha_1(t)) + \int_a^{\alpha_1(t)} k_1^{\Delta}(t,s)\Delta s\right]\right. \\ &\quad \left. + \alpha_2^{\Delta}(t)k_2(\sigma(t),\alpha_2(t)) + \int_a^{\alpha_2(t)} k_2^{\Delta}(t,s)\Delta s\right\} z_3(t) \\ &\quad + c^{\Delta}(t) + m_2\left[\alpha_1^{\Delta}(t)k_1(\sigma(t),\alpha_1(t)) + \int_a^{\alpha_1(t)} k_1^{\Delta}(t,s)\Delta s\right] \\ &= \eta_3(t)z_3(t) + \Xi_3(t). \end{aligned} \tag{3.41}$$

Therefore, using Lemma (2.4) in (3.41), we get that

$$z_3(t) \leqslant z_3(a)e_{\eta_3}(t,a) + \int_a^t \Xi_3(s)e_{\eta_3}(t,\sigma(s))\Delta s. \tag{3.42}$$

From (3.39) and (3.42), we get that

$$\omega^p(t) \leqslant z_3(a)e_{\eta_3}(t,a) + \int_a^t \Xi_3(s)e_{\eta_3}(t,\sigma(s))\Delta s. \tag{3.43}$$

From (3.38) and (3.43), we have

$$\begin{aligned} z_3(a) \leqslant c(a) &+ \int_a^b k_2(a,s)\Big[z_3(a)e_{\eta_3}(s,a) \\ &+ \int_a^s \Xi_3(\lambda)e_{\eta_3}(s,\sigma(\lambda))\Delta\lambda\Big]\Delta s \\ &\leqslant c(a) + z_3(a)\int_a^b k_2(a,s)e_{\eta_3}(s,a)\Delta s \\ &+ \int_a^b k_2(a,s)\left(\int_a^s \Xi_3(\lambda)e_{\eta_3}(s,\sigma(\lambda))\Delta\lambda\right)\Delta s. \end{aligned} \tag{3.44}$$

Thus from (3.44), we obtain

$$z_3(a) \leqslant \Lambda_3, \tag{3.45}$$

where Λ_1 is defined as in (3.33). Then we get the required inequality (3.32) from (3.43) and (3.45). The proof is complete. □

REMARK 3.3. By taking $\mathbb{T} = \mathbb{R}$, $\alpha_1(t) = t$, $\alpha_1^\Delta(t) = 1$, $\alpha_2(t) = b$, $\alpha_2^\Delta(t) = 0$, $p = 1$, $c(t)$ = any constant and using the relation (2.3), then Theorem 3.3 reduces to [14, Theorem 1.5.2 (b_2)]. If we put $\mathbb{T} = \mathbb{R}$, $\alpha_1(t) = t$, $\alpha_1^\Delta(t) = 1$, $\alpha_2(t) = b$, $\alpha_2^\Delta(t) = 0$, and using the relation (2.3), then Theorem 3.3 reduces to [8, Theorem 2.3]. If we put $\mathbb{T} = \mathbb{R}$ and using the relation (2.3), then Theorem 3.3, reduces to [5, Theorem 2.3].

4. Applications

In this section, we present some applications of Theorem 3.4 to obtain the explicit estimates on the solutions of certain dynamic equations, and also prove the uniqueness and global existence of solutions for a class of nonlinear dynamic integral equations.

EXAMPLE 4.1. Consider the following nonlinear dynamic integral equation with several delay in time scales:

$$\begin{aligned} &\omega^p(t) = h(t) + \Omega(t, \int_a^{\alpha_1(t)} \Psi_1(s,\omega(s),k_1)\Delta s, \int_a^{\alpha_2(t)} \Psi_2(s,\omega^p(s),k_2)\Delta s), \\ &\omega^p(a) = \tilde{r} \end{aligned} \tag{4.1}$$

if,

$$\begin{aligned}|h(t)| &\leqslant c(t),\\ |\Omega(t,u,\tilde{v})| &\leqslant |u|+|\tilde{v}|,\\ |\Psi_1| &\leqslant k_1(t,s)\omega(s),\\ |\Omega_2| &\leqslant k_2(t,s)\omega(s),\end{aligned} \tag{4.2}$$

where ω, α_i, $c \in C_{rd}([a,b]_{\mathbb{T}^k},\mathbb{R}^+)$ and α_i, c is delta-differentiable on $\mathbb{T}^k$ with $\alpha_i(t) \leqslant t$, $\alpha_i^\Delta(t) \geqslant 0$ $c^\Delta(t) \geqslant 0$, $\alpha_1(a)=a$, $\alpha_2(a)=b$ and $k_1(t,s)$, $k_1^\Delta(t,s)$, $k_2(t,s)$, $k_2^\Delta(t,s)$ $\in C_{rd}([a,b]_{\mathbb{T}^k}\times[a,b]_{\mathbb{T}^k},\mathbb{R}^+)$ for $a \leqslant s \leqslant t \leqslant b$ and $p \geqslant 1$ be constants, then we have the explicit bound estimation of the solution ω of (4.1) as the following:

$$\omega(t) \leqslant \left\{\Lambda_3 e_{\eta_3}(t,a)+\int_a^t \Xi_3(s)e_{\eta_3}(t,\sigma(s))\Delta s\right\}^{\frac{1}{p}}, \tag{4.3}$$

for all $t \in [a,b]_{\mathbb{T}^k}$, where Λ_3, Ξ_3 and $\eta_3(t)$ are defined as in Theorem 3.4.

Proof. From (4.1), (4.2), we have

$$|\omega(t)|^p \leqslant c(t)+\int_a^{\alpha_1(t)} k_1(t,s)|\omega(s)|\Delta s+\int_a^{\alpha_2(t)} k_2(t,s)|\omega(s)|^p\Delta s. \tag{4.4}$$

Now applying Theorem 3.4, to (4.4), we get

$$\omega(t) \leqslant \left\{\Lambda_3 e_{\eta_3}(t,a)+\int_a^t \Xi_3(s)e_{\eta_3}(t,\sigma(s))\Delta s\right\}^{\frac{1}{p}}.$$

This is the desired estimation in (4.3). The proof is complete. □

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