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ON NEW SHARP BOUNDS FOR THE TOADER–QI MEAN INVOLVED IN THE MODIFIED BESSEL FUNCTIONS OF THE FIRST KIND

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Abstract. Let $A(a,b)$, $G(a,b)$, $L(a,b)$ and $TQ(a,b)$ be the arithmetic, geometric, logarithmic and Toader-Qi means of $a,b>0$ with $a\neq b$, respectively. Let $I_v(x)$ be the modified Bessel functions of the first kind of order v. We prove the double inequality

$$\sqrt{\frac{\sinh t}{t}U_q(t)} < I_0(t) < \sqrt{\frac{\sinh t}{t}U_p(t)}$$

holds for $t>0$, or equivalently,

$$\sqrt{L(a,b)\,\mathscr{U}_q(a,b)} < TQ(a,b) < \sqrt{L(a,b)\,\mathscr{U}_p(a,b)},$$

holds for $a,b>0$ with $a\neq b$, if and only if $p\geqslant 11/15$ and $0<q\leqslant 2/\pi$, where

$$U_p(t) = p\cosh t - 4\left(p-\frac{2}{3}\right)\cosh\frac{t}{2} + 3p - \frac{5}{3},$$
$$\mathscr{U}_p = pA - 4\left(p-\frac{2}{3}\right)\sqrt{\frac{A+G}{2}G} + \left(3p-\frac{5}{3}\right)G.$$

These improve some known results, in which $\sqrt{L\mathscr{U}_{2/\pi}}$ is the sharpest lower mean bound for TQ.

1. Introduction

Throughout the paper, we assume that $a,b>0$ with $a\neq b$. The arithmetic mean, geometric mean, logarithmic mean, exponential mean and p-order power mean of a and b are defined by

$$A\equiv A(a,b)=\frac{a+b}{2}, \qquad G\equiv G(a,b)=\sqrt{ab},$$
$$L\equiv L(a,b)=\frac{a-b}{\ln a-\ln b}, \qquad I\equiv I(a,b)=e^{-1}\left(\frac{b^b}{a^a}\right)^{1/(b-a)},$$

$$A_p\equiv A_p(a,b)=\left(\frac{a^p+b^p}{2}\right)^{1/p} \text{ if } p\neq 0 \text{ and } A_0\equiv A_0(a,b)=\sqrt{ab},$$

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respectively. Clearly, $A(a,b) = A_1(a,b)$ and $G(a,b) = A_0(a,b)$. It is a well-known fact that $p \mapsto A_p(a,b)$ is increasing on $\mathbb{R}$. More generally, for a bivariate mean $M : (0,\infty)^2 \to (0,\infty)$ and $p \in \mathbb{R}$, the "p-order M mean" is defined by

$$M_p \equiv M_p(a,b) = M^{1/p}(a^p, b^p) \text{ if } p \neq 0 \text{ and } M_0 \equiv M_0(a,b) = \lim_{p\to 0} M_p(a,b)$$

(see [1]), which satisfies

$$M^{\lambda}_{\lambda p}(a,b) = M^{1/p}\left(a^{\lambda p}, b^{\lambda p}\right) = M_p\left(a^{\lambda}, b^{\lambda}\right)$$

for $\lambda \in \mathbb{R}$. For instance,

$$L_{3/2} \equiv L_{3/2}(a,b) = L^{2/3}\left(a^{3/2}, b^{3/2}\right), \tag{1.1}$$

$$I_{3/4} \equiv I_{3/4}(a,b) = I^{4/3}\left(a^{3/4}, b^{3/4}\right) \tag{1.2}$$

are $(3/2)$-order logarithmic mean and $(3/4)$-order exponential mean, respectively.

There are many inequalities among these classical means. We would like to mention the following inequalities:

$$G < L < A_{1/3} < A_{2/3} < I < A_{\ln 2}, \tag{1.3}$$

$$\sqrt{AG} < \sqrt{LI} < \frac{L+I}{2} < \frac{A+G}{2}, \tag{1.4}$$

$$I > \left(\frac{L^{8/5} + A^{8/5}}{2}\right)^{5/8} > \frac{L+A}{2}, \tag{1.5}$$

where the inequality $L < A_{1/3}$ was prove in [2], the inequality $A_{2/3} < I$ was shown in [3], the inequality $I < A_{\ln 2}$ appeared in [4], the inequalities (1.4) were established in [5], while (1.5) is due to Yang [6] and Sándor [7]. More inequalities involving these elementary means can be seen in [8], [9], [10].

In 1988, Toader [11] introduced a family of means defined, for the strictly monotonic function $f : (0,\infty) \to \mathbb{R}$ and $n \in \mathbb{R}$, by

$$M_{f,n}(a,b) = f^{-1}\left(\frac{2}{\pi}\int_0^{\pi/2} f(r_n(\theta))\, d\theta\right), \tag{1.6}$$

where

$$r_n(\theta) = \begin{cases} \left(a^n \cos^2\theta + b^n \sin^2\theta\right)^{1/n} & \text{if } n \neq 0, \\ a^{\cos^2\theta} b^{\sin^2\theta} & \text{if } n = 0 \end{cases}$$

and f^{-1} is the inverse of the function f. This family of means includes two well-known members, that are,

$$M_{1/x,2}(a,b) = \frac{\pi/2}{\int_0^{\pi/2}\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{-1/2} d\theta} = AGM(a,b), \tag{1.7}$$

$$M_{x,2}(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{1/2} d\theta = \mathscr{T}(a,b), \tag{1.8}$$

where $AGM(a,b)$ is the classical Gaussian arithmetic-geometric mean related to the complete integrals of the first kind (see [12], [13], [14]), while $\mathscr{T}(a,b)$ is the Toader mean involved the complete integrals of the second kind, see [15]. There are a number of papers on the bounds for the two means in terms of other elementary means, for example,

$$L < AGM < L^{3/4}A^{1/4} < L_{3/2}, \tag{1.9}$$

$$A_{3/2} < \mathscr{T} < A_{(\ln 2)/(\ln\pi - \ln 2)}, \tag{1.10}$$

where the inequalities (1.9) appeared in [16], [17], [18], while the double inequality (1.10) was proven in [19], [17]. The latest results on this topic can be found in [20], [21], [22], [23], [24].

Toader's family of means (1.6) also includes a somewhat strange member, that is,

$$M_{x^q,0}(a,b) = \left(\frac{2}{\pi}\int_0^{\pi/2} a^{q\cos^2\theta} b^{q\sin^2\theta} d\theta\right)^{1/q} \quad \text{for } q \neq 0.$$

Qi in [25, Lemma 2.1] and [26] revealed the surprising relationship between this mean and modified Bessel functions of the first kind as follows:

$$M_{x,0}(a,b) = \frac{2}{\pi}\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta} d\theta = \sqrt{ab}\, I_0\left(\ln\sqrt{\frac{b}{a}}\right) \tag{1.11}$$

and

$$M_{x^q,0}(a,b) = \left(\frac{2}{\pi}\int_0^{\pi/2} a^{q\cos^2\theta} b^{q\sin^2\theta} d\theta\right)^{1/q} = \sqrt{ab}\, I_0^{1/q}\left(q\ln\sqrt{\frac{b}{a}}\right), \tag{1.12}$$

where

$$I_v(z) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(v+n+1)}\left(\frac{z}{2}\right)^{2n+v} \quad \text{for } z\in\mathbb{C} \text{ and } v\in\mathbb{R}\setminus\{-1,-2,\ldots\}$$

denotes the modified Bessel functions of the first kind (see [27]) and $\Gamma(z)$ is the classical gamma function. So the mean $M_{x,0}(a,b)$ given by (1.11) is called Toader-Qi mean and denoted by

$$TQ \equiv TQ(a,b) = \frac{2}{\pi}\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta} d\theta = \sqrt{ab}\, I_0\left(\ln\sqrt{\frac{b}{a}}\right) \tag{1.13}$$

(see [29, Theorem 3.3]). Subsequently, the Toader-Qi mean immediately attracted attention of some scholars. Qi et. al. [26] established the chain of inequalities for this mean:

$$L < TQ < \frac{A+G}{2} < \frac{2A+G}{3} < I. \tag{1.14}$$

Yang and Chu [29, Theorem 3.3] presented a series of sharp inequalities for the Toader-Qi mean $TQ(a,b)$ and $I_0(t)$, for example, the inequalities

$$\sqrt{\left[\frac{2}{\pi}A + \left(1-\frac{2}{\pi}\right)G\right]L} < TQ < \sqrt{[\lambda_0 A + (1-\lambda_0)G]L}, \tag{1.15}$$

$$L^{3/4}A^{1/4} < TQ < \frac{3}{4}L + \frac{1}{4}A \tag{1.16}$$

hold with $\lambda_0 = 0.6766\ldots$. In [30], [31], Yang et. al. proved further the double inequality

$$\sqrt{\frac{e}{\pi}}\sqrt{LI} < TQ < \sqrt{LI} \tag{1.17}$$

holds with the best contant $\sqrt{e/\pi} = 0.930\ldots$, and

$$L_{3/2} < TQ. \tag{1.18}$$

Recently, Yang, Tian and Zhu [32] gave an improvement of (1.17). More precisely, they proved

$$\frac{2^{4/3}}{\sqrt{\pi}}\sqrt{LA_{2/3}} < TQ < \sqrt{LA_{3/2}} \tag{1.19}$$

and

$$\sqrt{LA_{(\ln 2)/\ln \pi}} < TQ < \sqrt{LA_{3/2}}. \tag{1.20}$$

Thus, the chain of inequalities given in [29, (3.84)] can be extended to

$$\begin{aligned} &L < AGM < L^{3/4}A^{1/3} < L_{3/2} < TQ < \sqrt{LA_{3/2}} \\ &< \sqrt{LI} < \frac{L+I}{2} < A_{1/2} < \mathscr{T}_{1/3} < I_{3/4}, \end{aligned} \tag{1.21}$$

where $L_{3/2}$ and $I_{3/4}$ are given by (1.1) and (1.2), respectively, whiles $\mathscr{T}_{1/3} = \mathscr{T}(a^{1/3}, b^{1/3})^3$. More inequalities for the Toader-Qi mean can be found in [29, Theorem 3.3] and [33].

The aim of this paper is to find the best constants $p, q > 0$ such that the double inequality

$$\sqrt{L(a,b)\mathscr{U}_q(a,b)} < TQ(a,b) < \sqrt{L(a,b)\mathscr{U}_p(a,b)}, \tag{1.22}$$

where

$$\mathscr{U}_p = pA - 4\left(p - \frac{2}{3}\right)\sqrt{\frac{A+G}{2}G} + \left(3p - \frac{5}{3}\right)G. \tag{1.23}$$

More precisely, we have the following main result.

THEOREM 1.1. *Let* $p, q > 0$. *(i) If* $p \geqslant 11/15$ *then double inequality*

$$\sqrt{\frac{2}{p\pi}L(a,b)\mathscr{U}_p(a,b)} < TQ(a,b) < \sqrt{L(a,b)\mathscr{U}_p(a,b)} \tag{1.24}$$

holds. The lower and upper bounds are sharp.

(ii) The double inequality (1.22) holds if and only if $p \geqslant 11/15$ *and* $q \leqslant 2/\pi$.

Let $b > a > 0$ and $t = \ln\sqrt{a/b}$. Then those means above-mentioned can be represented in terms of hyperbolic functions:

$$\frac{L(a,b)}{\sqrt{ab}} = \frac{\sinh t}{t}, \quad \frac{I(a,b)}{\sqrt{ab}} = \exp\left(\frac{t}{\tanh t} - 1\right), \quad \frac{TQ(a,b)}{\sqrt{ab}} = I_0(t), \tag{1.25}$$

$$\frac{\mathscr{U}_p(a,b)}{\sqrt{ab}} = p\cosh t - 4\left(p - \frac{2}{3}\right)\cosh\frac{t}{2} + 3p - \frac{5}{3} := U_p(t). \tag{1.26}$$

Correspondingly, Theorem 1.1 can be equivalently stated as follows.

THEOREM 1.1′. *Let $p, q > 0$ and $U_p(t)$ be defined by* (1.26).
(i) If $p \geqslant 11/15$ then the double inequality

$$\sqrt{\frac{2}{p\pi}\frac{\sinh t}{t}U_p(t)} < I_0(t) < \sqrt{\frac{\sinh t}{t}U_p(t)} \tag{1.27}$$

holds for $t > 0$. The lower and upper bounds are sharp.
(ii) The double inequality

$$\sqrt{\frac{\sinh t}{t}U_q(t)} < I_0(t) < \sqrt{\frac{\sinh t}{t}U_p(t)} \tag{1.28}$$

holds for $t > 0$ if and only if $p \geqslant 11/15$ and $0 < q \leqslant 2/\pi$.

The rest of this paper is organized as follows. In Section 2, we list two tools and prove several lemmas. In Section 3, we will prove Theorem 1.1′ instead of Theorem 1.1. In Section 4, we will compare the bounds for $TQ(a,b)$ given in Theorem 1.1 with certain known ones to show that our new bounds are better than these ones.

2. Tools and lemmas

To prove our results, we need two tools. The first tool was due to Biernacki and Krzyz [34], which plays an important role in dealing with the monotonicity for the ratio of power series.

LEMMA 2.1. ([34]) *Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=0}^{\infty} b_k t^k$ be two real power series converging on $(-r, r)$ $(r > 0)$ with $b_k > 0$ for all k. If the sequence $\{a_k/b_k\}$ is increasing (decreasing) for all k, then the function $t \mapsto A(t)/B(t)$ is also increasing (decreasing) on $(0, r)$.*

The second tool is another monotonicity rule for the case when the sequence $\{a_k/b_k\}_{k \geqslant 0}$ is piecewise monotonic presented recently in [35, Theorem 1]. The following lemma is a corollary of [35, Theorem 1], which was slightly modified in [36], [37].

LEMMA 2.2. ([35, Corollary 2.3]) *Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=0}^{\infty} b_k t^k$ be two real power series converging on $\mathbb{R}$ with $b_k > 0$ for all k. If for certain $m \in \mathbb{N}$,* *the sequences* $\{a_k/b_k\}_{0\leqslant k\leqslant m}$ *and* $\{a_k/b_k\}_{k\geqslant m}$ *both are non-constant, and they are increasing (decreasing) and decreasing (increasing), respectively,, then there is a unique* $t_0\in(0,\infty)$ *such that the function* A/B *is increasing (decreasing) on* $(0,t_0)$ *and decreasing (increasing) on* (t_0,∞)*.*

The following three lemmas will be used to prove Theorem 1.1$'$.

LEMMA 2.3. ([29, Lemma 2.8]) *There holds*

$$I_0(t)^2=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^4}t^{2n}. \tag{2.29}$$

LEMMA 2.4. ([38, Problems 85, 94]) *If the two given sequences* $\{a_n\}_{n\geqslant0}$ *and* $\{b_n\}_{n\geqslant0}$ *satisfy the conditions*

$$b_n>0;\ \sum_{n=0}^{\infty}b_nt^n \text{ converges for all values of } t;\ \lim_{n\to\infty}\frac{a_n}{b_n}=s.$$

Then, the series $\sum_{n=0}^{\infty}a_nt^n$ *converges too for all values of* t*, and in addition*

$$\lim_{t\to\infty}\frac{\sum_{n=0}^{\infty}a_nt^n}{\sum_{n=0}^{\infty}b_nt^n}=s.$$

LEMMA 2.5. *For* $p>0$*, let*

$$u_n=\frac{p2^{4n}-(3p-2)3^{2n}+(3p-5/3)2^{2n}-(p-2/3)}{2^{2n}(2n+1)!}\ \text{ and }\ v_n=\frac{(2n)!}{2^{2n}n!^4}.$$

Then

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\frac{\pi}{2}p.$$

Proof. Since

$$u_n=\frac{p2^{4n}-(3p-2)3^{2n}+(3p-5/3)2^{2n}-(p-2/3)}{2^{2n}(2n+1)!}\sim p\frac{2^{4n}}{2^{2n}(2n+1)!}$$

as $n\to\infty$, we have

$$\frac{u_n}{v_n}\sim p\frac{2^{4n}n!^4}{(2n)!\,(2n+1)!}:=ps_n,\ \text{as}\ n\to\infty.$$

It thus suffices to show $\lim_{n\to\infty}s_n=\pi/2$. Using the duplication formula for the gamma

$$\Gamma(2z)=(2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma(z+1/2)$$

(see [39, (6.1.18)]), s_n can be simplified to

$$\begin{aligned}s_n&=\frac{2^{4n}\Gamma(n+1)^4}{\Gamma(2n+1)\Gamma(2n+2)}=\frac{(2\pi)2^{4n}\Gamma(n+1)^4}{2n2^{4n+1}\Gamma(n)\Gamma(n+1/2)\Gamma(n+1)\Gamma(n+3/2)}\\&=\frac{\pi}{2}\frac{\Gamma(n+1)^2}{(n+1/2)\Gamma(n+1/2)^2},\end{aligned}$$

which, by the asymptotic formula listed in [39, (6.1.47)]:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}, \text{ as } z \to \infty,$$

s_n tends to $\pi/2$ as $n \to \infty$. This completes the proof. □

LEMMA 2.6. *Let*

$$\alpha_n = 6\left(7n^2+14n+3\right)3^{2n} - 20\left(3n^2+6n+2\right)2^{2n} + 2\left(15n^2+30n+11\right),$$
$$\beta_n = 12 \times 2^{4n} + \left(63n^2+126n+27\right)3^{2n} - 36\left(3n^2+6n+2\right)2^{2n} + 3\left(15n^2+30n+11\right).$$

Then for $n \in \mathbb{N}$ we have that $\alpha_n, \beta_n > 0$ and α_n/β_n is decreasing with $\alpha_1/\beta_1 = 11/15$, $\lim_{n\to\infty}(\alpha_n/\beta_n) = 0$

Proof. (i) Since the third term of the expression of α_n is positive for $n \geqslant 1$, to prove $\alpha_n > 0$ for $n \geqslant 1$, it suffices to show the sum of the first and second term is also positive, that is,

$$\alpha_n' := 6\left(7n^2+14n+3\right)3^{2n} - 20\left(3n^2+6n+2\right)2^{2n} > 0$$

for $n \geqslant 1$. Using the binomial theorem we have

$$\left(\frac{3}{2}\right)^{2n} = \left(1+\frac{5}{4}\right)^n > 1 + n\frac{5}{4} + \frac{n(n-1)}{2}\left(\frac{5}{4}\right)^2 > 1 + \frac{5n}{4} \tag{2.30}$$

for $n \geqslant 1$. It is deduced that

$$\begin{aligned} 2^{2n}\alpha_n' &= 6\left(7n^2+14n+3\right)\left(\frac{3}{2}\right)^{2n} - 20\left(3n^2+6n+2\right) \\ &> 6\left(7n^2+14n+3\right)\left(1+\frac{5}{4}n\right) - 20\left(3n^2+6n+2\right) \\ &= \frac{1}{2}\left(105n^3+174n^2-27n-44\right), \end{aligned}$$

which is clearly positive for $n \geqslant 1$.

Similarly, since the first and fourth terms of the expression of β_n are obviously positive for $n \geqslant 1$, it suffices to show the sum of the second and third terms is also positive, that is,

$$\beta_n' := \left(63n^2+126n+27\right)3^{2n} - 36\left(3n^2+6n+2\right)2^{2n} > 0$$

for $n \geqslant 1$. Applying the inequalities (2.30) we derive

$$\begin{aligned} 2^{-2n}\beta_n' &= \left(63n^2+126n+27\right)\left(\frac{3}{2}\right)^{2n} - 36\left(3n^2+6n+2\right) \\ &> \left(63n^2+126n+27\right)\left(1+\frac{5}{4}n\right) - 36\left(3n^2+6n+2\right) \\ &= \frac{45}{4}\left(7n^3+10n^2-5n-4\right) > 0 \text{ for } n \geqslant 1, \end{aligned}$$

which proves $\beta_n > 0$ for $n \geqslant 1$.

(ii) Due to $\beta_n > 0$ for $n \geqslant 1$, to get that the sequence α_n/β_n is decreasing, it suffices to prove

$$\gamma_n := \frac{\alpha_n\beta_{n+1} - \alpha_{n+1}\beta_n}{2^{4n}} \leqslant 0 \text{ for } n \geqslant 1.$$

Direct computation yields

$$\begin{aligned}\gamma_n &= 504\left(7n^2-4n-24\right)3^{2n} - 2880\left(3n^2+4n-1\right)2^{2n} \\ &\quad +180\left(21n^4+126n^3+254n^2+208n+60\right)\left(\frac{3}{2}\right)^{2n} \\ &\quad -36\left(45n^4+270n^3+558n^2+464n+124\right)2^{-2n}+360\left(15n^2+28n+8\right),\end{aligned}$$

with $\gamma_1 = 111780$, $\gamma_2 = 1576260$, $\gamma_3 = 78160545/4$. Since the sum of the third, fourth and fifth terms is clearly positive for $n \geqslant 1$, to prove $\gamma_n > 0$ for $n \geqslant 1$, we only need to prove the sum of the first and second terms is also positive, that is,

$$\gamma_n' := 504\left(7n^2-4n-24\right)3^{2n} - 2880\left(3n^2+4n-1\right)2^{2n} > 0$$

for $n \geqslant 3$. Using the first inequality of (2.30) we have

$$\begin{aligned}2^{-2n}\gamma_n' &= 504\left(7n^2-4n-24\right)\left(\frac{3}{2}\right)^{2n} - 2880\left(3n^2+4n-1\right) \\ &> 504\left(7n^2-4n-24\right)\left[1+\frac{5}{4}n+\frac{n(n-1)}{2}\frac{25}{16}\right] - 2880\left(3n^2+4n-1\right) \\ &= \frac{9}{4}P_4(n),\end{aligned}$$

where

$$P_4(n) = 1225n^4+35n^3-6892n^2-8536n-4096,$$

which is actually positive for $n \geqslant 3$ due to

$$P_4(n) = 1225(n-3)^4+14735(n-3)^3+59573(n-3)^2+83357(n-3)+8438 > 0,$$

so is γ_n'. Thus the proof is finished. □

3. Proofs of Theorem 1.1′

We are now in a position to prove Theorem 1.1′.

Proof. We prove the desired results by considering the monotonicity of the ratio

$$F_p(t) = \frac{\left(t^{-1}\sinh t\right)U_p(t)}{I_0(t)^2}$$

on $(0,\infty)$. Using "product into sum" formulas and expanding in power series give

$$\begin{aligned}\frac{\sinh t}{t}U_p(t) &= p\frac{\sinh(2t)}{2t}-(3p-2)\frac{\sinh(3t/2)}{3t/2}\\&\quad+\left(3p-\frac{5}{3}\right)\frac{\sinh t}{t}-\left(p-\frac{2}{3}\right)\frac{\sinh(t/2)}{t/2}\\&= p\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n+1)!}t^{2n}-(3p-2)\sum_{n=0}^{\infty}\frac{(3/2)^{2n}}{(2n+1)!}t^{2n}\\&\quad+\left(3p-\frac{5}{3}\right)\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n+1)!}-\left(p-\frac{2}{3}\right)\sum_{n=0}^{\infty}\frac{2^{-2n}}{(2n+1)!}t^{2n}\\&= \sum_{n=0}^{\infty}\frac{p2^{2n}-(3p-2)(3/2)^{2n}+(3p-5/3)-(p-2/3)2^{-2n}}{(2n+1)!}t^{2n}\\&:= \sum_{n=0}^{\infty}u_nt^{2n},\end{aligned}$$

where

$$u_n=\frac{p2^{4n}-(3p-2)3^{2n}+(3p-5/3)2^{2n}-(p-2/3)}{2^{2n}(2n+1)!}.$$

By Lemma 2.3 we see that $I_0(t)^2=\sum_{n=0}^{\infty}v_nt^{2n}$, where

$$v_n=\frac{(2n)!}{2^{2n}n!^4}.$$

Then

$$\lim_{t\to 0}F_p(t)=\lim_{t\to 0}\frac{\left(t^{-1}\sinh t\right)U_p(t)}{I_0(t)^2}=\frac{u_0}{v_0}=1,$$

and by Lemmas 2.4 and 2.5,

$$\lim_{t\to\infty}F_p(t)=\lim_{t\to\infty}\frac{\left(t^{-1}\sinh t\right)U_p(t)}{I_0(t)^2}=\lim_{n\to\infty}\frac{u_n}{v_n}=\frac{p\pi}{2}\text{ if }p>0.$$

By Lemmas 2.1 and 2.2, to confirm the monotonicity of $F_p(t)$, we have to observe the monotonicity of the sequence $\{u_n/v_n\}_{n\geqslant 0}$, which depends on the sign of $w_n=u_{n+1}-(v_{n+1}/v_n)u_n$ due to $v_n>0$. An easy check shows $w_0=0$ and

$$\frac{v_{n+1}}{v_n}=\frac{1}{2}\frac{2n+1}{(n+1)^3}.$$

Therefore, for $n\geqslant 1$ we have

$$\begin{aligned}2^{2n+2}(2n+3)!w_n &= p2^{4n+4}-(3p-2)3^{2n+2}+(3p-5/3)2^{2n+2}-(p-2/3)\\&\quad-2^2(2n+3)(2n+2)\times\frac{1}{2}\frac{2n+1}{(n+1)^3}\\&\quad\times\left(p2^{4n}-(3p-2)3^{2n}+(3p-5/3)2^{2n}-(p-2/3)\right)\\&=\frac{\beta_n}{3(n+1)^2}p-\frac{\alpha_n}{3(n+1)^2}=\frac{\beta_n}{3(n+1)^2}\left(p-\frac{\alpha_n}{\beta_n}\right),\end{aligned}$$

where α_n and β_n are defined in Lemma 2.6. Since $\alpha_n, \beta_n > 0$ and α_n/β_n is decreasing for $n \geqslant 1$ shown as in Lemma 2.6, we will distinguish two cases to discuss the monotonicity of $F_p(t)$ as follows.

Case 1. While $p \geqslant \max_{n\geqslant 1}\{\alpha_n/\beta_n\} = 11/15$. Clearly, $w_n \geqslant 0$ for $n \geqslant 1$. This together with $w_0 = 0$ implies that the sequence $\{u_n/v_n\}_{n\geqslant 0}$ is increasing, so is $F_p(t)$ on $(0,\infty)$ by Lemma 2.1. Then we have

$$1 = \lim_{t\to 0} F_p(t) < F_p(t) < \lim_{t\to\infty} F_p(t) = \frac{p\pi}{2}$$

for $t > 0$, which proves the double inequality (1.27).

Case 2. While $0 < p < 11/15$. Since $\theta_n := p - \alpha_n/\beta_n$ is increasing for $n \geqslant 1$ and $\theta_1 = p - \alpha_1/\beta_1 < 0$, $\theta_\infty = \lim_{n\to\infty}\theta_n = p > 0$, we see that there is an integer $n_0 > 1$ such that $\theta_n \leqslant 0$ for $1 \leqslant n \leqslant n_0$ and $\theta_n > 0$ for $n > n_0$. In view of $\operatorname{sgn} w_n = \operatorname{sgn}\theta_n$ and $w_0 = 0$, we find that the sequence $\{u_n/v_n\}$ is decreasing for $0 \leqslant n \leqslant n_0$ and increasing for $n > n_0$. By Lemma 2.2, there is a $t_0 > 0$ such that the ratio $F_p(t)$ is decreasing on $(0,t_0)$ and increasing on (t_0,∞). This yields

$$F_p(t) < \max\left\{\lim_{t\to 0} F_p(t), \lim_{t\to\infty} F_p(t)\right\} = \max\left\{1, \frac{p\pi}{2}\right\}$$

for $t > 0$. In particular, if $\min\{1, 2/(p\pi)\} = 1$, that is, $0 < p \leqslant 2/\pi$, then $F_p(t) < 1$, which proves the right hand side inequality of (1.28).

It remains to prove the necessity for the double inequality (1.28) to hold. In fact, the necessary condition for the left hand side inequality of (1.28) to hold follows from the limit relation

$$\lim_{t\to 0} \frac{I_0(t)^2 - \left(t^{-1}\sinh t\right) U_p(t)}{t^4} = v_2 - u_2 = -\frac{1}{480}(15p - 11) \leqslant 0.$$

While the necessary condition for the left hand side inequality of (1.28) easily is obtained by the following limit relation

$$\lim_{t\to\infty} \frac{\left(t^{-1}\sinh t\right) U_q(t)}{I_0(t)^2} = \frac{q\pi}{2} \leqslant 1,$$

which completes the proof. □

4. Comparisons of certain bounds for $TQ(a,b)$

In this section, we are ready to compare our new bounds given in Theorem 1.1 for the Toader-Qi mean with certain known bounds.

4.1. Comparing with the lower bounds given in (1.15), (1.18) and (1.20)

LEMMA 4.1. *There hold*

$$\sqrt{LA_{(\ln 2)/\ln\pi}} < \sqrt{L\left[\frac{2}{\pi}A + \left(1 - \frac{2}{\pi}\right)G\right]} < \sqrt{L\mathscr{U}_{2/\pi}} < TQ. \tag{4.31}$$

Proof. The first inequality of (4.31) follows from [32, Eq. (18)]. To prove the second, we note that

$$\mathscr{U}_{2/\pi} = \frac{2}{\pi}A - 4\left(\frac{2}{\pi} - \frac{2}{3}\right)\sqrt{\frac{A+G}{2}G} + \left(3\frac{2}{\pi} - \frac{5}{3}\right)G.$$

Then

$$\mathscr{U}_{2/\pi} - \left[\frac{2}{\pi}A + \left(1 - \frac{2}{\pi}\right)G\right] = \frac{8(\pi-3)}{3\pi}\left(\sqrt{\frac{A+G}{2}G} - G\right) > 0.$$

REMARK 4.2. Lemma 4.1 tells us that our new lower bound $\sqrt{L\mathscr{U}_{2/\pi}}$ for TQ is stronger than $\sqrt{LA_{(\ln 2)/\ln\pi}}$ and $\sqrt{L[2A/\pi + (1-2/\pi)G]}$.

To compare $L_{3/2}$ with $\sqrt{L\mathscr{U}_{2/\pi}}$, we need the following lemma.

LEMMA 4.3. *Let $q > 0$ and $U_q(t)$ be defined by (1.26). The inequality*

$$\left[\frac{\sinh(3t/2)}{3t/2}\right]^{2/3} < \sqrt{\frac{\sinh t}{t}U_q(t)} \tag{4.32}$$

holds for $t > 0$ if and only if $q \geqslant 8/15 = 0.533\ldots$.

Proof. Let

$$H_q(t) = \ln\left[\frac{\sinh(3t/2)}{3t/2}\right]^{4/3} - \ln\left[\frac{\sinh t}{t}U_q(t)\right].$$

Differentiation yields

$$H'_{8/15}(2s) = \frac{h(s)}{6s(8\cosh(2s) + 8\cosh s - 1)\sinh(3s)\sinh(2s)},$$

where

$$\begin{aligned} h(s) = {} & 48s\cosh(4s)\sinh(3s) + 4\sinh(4s)\sinh(3s) - \sinh(3s)\sinh(2s) \\ & -48s\sinh(4s)\cosh(3s) - 6s\sinh(3s)\cosh(2s) + 12s\cosh(3s)\sinh(2s) \\ & +8\sinh(3s)\sinh(2s)\cosh s + 48s\sinh(3s)\cosh(2s)\cosh s \\ & -96s\cosh(3s)\sinh(2s)\cosh s + 24s\sinh(3s)\sinh(2s)\sinh s. \end{aligned}$$

If we prove $h(s) > 0$ for $s > 0$, then $H'_{8/15}(2s) > 0$.

This implies $H_{8/15}(t) < \lim_{t\to 0} H_{8/15}(t) = 0$ for $t > 0$. By the monotonicity of $q \mapsto U_q(t)$ it follows that

$$\left[\frac{\sinh(3t/2)}{3t/2}\right]^{4/3} < \frac{\sinh t}{t}U_{8/15}(t) \leqslant \frac{\sinh t}{t}U_q(t)$$

for $t > 0$ and $q \geqslant 8/15$.

Using "product into sum" formulas for the hyperbolic functions and expanding in power series give

$$\begin{aligned}
h(s) = &\ 2\cosh(7s) - 6s\sinh(6s) + 2\cosh(6s) + 3s\sinh(5s) - \frac{1}{2}\cosh(5s) \\
&-18s\sinh(4s) + 2\cosh(4s) + 30s\sinh(2s) - 2\cosh(2s) \\
&-\frac{3}{2}\cosh s - 57s\sinh s - 2 \\
= &\ 2\sum_{n=0}^{\infty}\frac{7^{2n}}{(2n)!}s^{2n} - 6\sum_{n=1}^{\infty}\frac{(2n)6^{2n-1}}{(2n)!}s^{2n} + 2\sum_{n=0}^{\infty}\frac{6^{2n}}{(2n)!}s^{2n} + 3\sum_{n=1}^{\infty}\frac{(2n)5^{2n-1}}{(2n)!}s^{2n} \\
&-\frac{1}{2}\sum_{n=0}^{\infty}\frac{5^{2n}}{(2n)!}s^{2n} - 18\sum_{n=1}^{\infty}\frac{(2n)4^{2n-1}}{(2n)!}s^{2n} + 2\sum_{n=0}^{\infty}\frac{4^{2n}}{(2n)!}s^{2n} + 30\sum_{n=1}^{\infty}\frac{(2n)2^{2n-1}}{(2n)!}s^{2n} \\
&-2\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}s^{2n} - \frac{3}{2}\sum_{n=0}^{\infty}\frac{1}{(2n)!}s^{2n} - 57\sum_{n=1}^{\infty}\frac{2n}{(2n)!}s^{2n} - 2 \\
= &\sum_{n=1}^{\infty}\frac{d_n}{(2n)!}s^{2n},
\end{aligned}$$

where

$$\begin{aligned}
d_n = &\ 2\times 7^{2n} - 2(n-1)6^{2n} + \left(6n - \frac{5}{2}\right)5^{2n-1} - (9n-2)4^{2n} \\
&+ (15n-1)2^{2n+1} - \frac{3}{2}(76n+1).
\end{aligned}$$

An easy verification yields $d_n = 0$ for $n = 1,2,3$ and $d_4 = 933\,120 > 0$. To show $d_n > 0$ for $n \geqslant 5$, it suffices to show that, for $n \geqslant 5$,

$$\begin{aligned}
d_n' &= 2\times 7^{2n} - 2(n-1)6^{2n} > 0, \\
d_n'' &= \left(6n - \frac{5}{2}\right)5^{2n-1} - (9n-2)4^{2n} > 0, \\
d_n''' &= (15n-1)2^{2n+1} - \frac{3}{2}(76n+1) > 0.
\end{aligned}$$

It is easy to check that d_n' satisfies

$$d_{n+1}' - 49d_n' = 2(13n-49)6^{2n} > 0,$$

which together with $d_5' = 81\,221\,090 > 0$ yields $d_n' > 0$ for $n \geqslant 5$. Similarly, d_n'' satisfies

$$\frac{d_{n+1}''}{9n+7} - 16\frac{d_n''}{9n-2} = \frac{3}{10}\frac{324n^2+117n+70}{81n^2+45n-14}5^{2n} > 0,$$

which together with $d_5'' = 17\,244\,339/2 > 0$ yields $d_n'' > 0$ for $n \geqslant 5$. Finally, d_n''' satisfies

$$d_{n+1}''' - d_n''' = (90n+114)2^{2n} - 114 > 0,$$

which together with $d_5''' = 301\,961/2 > 0$ yields $d_n''' > 0$ for $n \geqslant 5$. This proves $h(s) > 0$ for $s > 0$, and this proof is done. $\square$

REMARK 4.4. By Lemma 4.3 and the monotonicity of $q \mapsto U_q(t)$, we immediately get

$$\left[\frac{\sinh(3t/2)}{3t/2}\right]^{2/3} < \sqrt{\frac{\sinh t}{t} U_{8/15}(t)} < \sqrt{\frac{\sinh t}{t} U_{2/\pi}(t)} < I_0(t)$$

for $t > 0$. Equivalently, the inequalities

$$L_{3/2}(a,b) < \sqrt{L(a,b)\mathscr{U}_{8/15}(a,b)} < \sqrt{L(a,b)\mathscr{U}_{2/\pi}(a,b)} < TQ(a,b)$$

hold, which indicate that lower bound $\sqrt{L\mathscr{U}_{2/\pi}}$ for TQ in (1.22) is superior to $L_{3/2}$.

REMARK 4.5. From Remarks 4.2 and 4.4 we find that $\sqrt{L\mathscr{U}_{2/\pi}}$ for TQ is the sharpest among exsisting lower mean bounds.

4.2. Comparing with the upper bound given in (1.16)

REMARK 4.6. We have

$$TQ(a,b) < \sqrt{L(a,b)\mathscr{U}_{11/15}(a,b)} < \frac{3}{4}L(a,b) + \frac{1}{4}A(a,b).$$

It suffices to show the inequality

$$D(t) = \frac{\sinh t}{t} U_{11/15}(t) - \left(\frac{3}{4}\frac{\sinh t}{t} + \frac{1}{4}\cosh t\right)^2 < 0$$

for $t > 0$. Expanding it in power series gives

$$\begin{aligned} -\left(480t^2\right)D(t) &= 15t^2\cosh 2t + 135\cosh 2t + 90t\sinh 2t \\ &\quad -352t^2\cosh t + 128t^2\cosh\frac{t}{2} - 241t^2 - 135 \\ &= 15\sum_{n=1}^{\infty}\frac{2^{2n-2}}{(2n-2)!}t^{2n} + 135\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n} + 90\sum_{n=1}^{\infty}\frac{2^{2n-1}}{(2n-1)!}t^{2n}t \\ &\quad -352\sum_{n=1}^{\infty}\frac{1}{(2n-2)!}t^{2n} + 128\sum_{n=1}^{\infty}\frac{(1/2)^{2n-2}}{(2n-2)!}t^{2n} - 241t^2 \\ &:= \sum_{n=2}^{\infty}\frac{\delta_n}{(2n-2)!}t^{2n}, \end{aligned}$$

where

$$\delta_n = 15\frac{2n^2 + 11n + 18}{2n(2n-1)}2^{2n-1} - 352 + \frac{128}{2^{2n-2}}.$$

It is not difficult to check that $\delta_n > 0$ for $n \geqslant 2$, then $D(t) < 0$ for $t > 0$.

4.3. Comparing with the upper bounds in (1.17)

Before comparing the double inequality (1.24) with (1.17), we first give the following lemma.

LEMMA 4.7. *Let $p>0$ and $U_p(t)$ be defined by (1.26). (i) If $p \geqslant 38/45$, then the double inequality*

$$\frac{2}{pe}U_p(t) < \exp\left(\frac{t}{\tanh t}-1\right) < U_p(t) \tag{4.33}$$

holds for $t>0$. (ii) If $0<p\leqslant 2/3$, then the double inequality (4.33) is reversed. (iii) If $2/3<p<38/45$, then the inequality

$$\min\left\{\frac{2}{pe},1\right\}U_p(t) < \exp\left(\frac{t}{\tanh t}-1\right)$$

holds for $t>0$. In particular, if $0<p\leqslant 2/e$, then the inequality

$$U_p(t) < \exp\left(\frac{t}{\tanh t}-1\right) \tag{4.34}$$

holds for $t>0$.

Proof. Let

$$G_p(t) = \frac{t}{\tanh t} - 1 - \ln\left[p\cosh t - 4\left(p-\frac{2}{3}\right)\cosh\frac{t}{2} + 3p - \frac{5}{3}\right].$$

Differentiation yields

$$G_p'(2s) = 2\frac{g_2(s) - p\times g_1(s)}{[3p\cosh(2s) - 4(3p-2)\cosh s + 9p - 5]\sinh^2(2s)}, \tag{4.35}$$

where

$$\begin{aligned} g_1(s) = {} & 6\sinh(4s)\cosh s - 6\sinh^2(2s)\sinh s - \frac{9}{2}\sinh(4s) \\ & -3\sinh(2s) + 6s\cosh(2s) - 24s\cosh s + 18s, \end{aligned}$$

$$g_2(s) = 4\sinh(4s)\cosh s - 4\sinh^2(2s)\sinh s - \frac{5}{2}\sinh(4s) - 16s\cosh s + 10s.$$

Using "product into sum" formulas for hyperbolic functions and expanding in power series give

$$\begin{aligned} g_1(s) = {} & \frac{3}{2}\sinh(5s) - \frac{9}{2}\sinh(4s) + \frac{9}{2}\sinh(3s) \\ & +6s\cosh(2s) - 3\sinh(2s) + 3\sinh s - 24s\cosh s + 18s \end{aligned}$$

$$= \frac{3}{2}\sum_{n=0}^{\infty}\frac{5^{2n+1}}{(2n+1)!}s^{2n+1} - \frac{9}{2}\sum_{n=0}^{\infty}\frac{4^{2n+1}}{(2n+1)!}s^{2n+1} + \frac{9}{2}\sum_{n=0}^{\infty}\frac{3^{2n+1}}{(2n+1)!}s^{2n+1}$$
$$+6\sum_{n=0}^{\infty}\frac{(2n+1)2^{2n}}{(2n+1)!}s^{2n+1} - 3\sum_{n=0}^{\infty}\frac{2^{2n+1}}{(2n+1)!}s^{2n+1} + 3\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}s^{2n+1}$$
$$-24\sum_{n=0}^{\infty}\frac{(2n+1)}{(2n+1)!}s^{2n+1} + 18s$$
$$= \sum_{n=1}^{\infty} a_n s^{2n+1},$$

where

$$a_n = \frac{3}{2}\frac{5^{2n+1} - 3\times 4^{2n+1} + 3^{2n+2} + n\times 2^{2n+3} - 2(16n+7)}{(2n+1)!};$$

$$g_2(s) = \sinh(5s) - \frac{5}{2}\sinh(4s) + 3\sinh(3s) + 2\sinh s - 16s\cosh s + 10s$$
$$= \sum_{n=0}^{\infty}\frac{5^{2n+1}}{(2n+1)!}s^{2n+1} - \frac{5}{2}\sum_{n=0}^{\infty}\frac{4^{2n+1}}{(2n+1)!}s^{2n+1} + 3\sum_{n=0}^{\infty}\frac{3^{2n+1}}{(2n+1)!}s^{2n+1}$$
$$+2\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}s^{2n+1} - 16\sum_{n=0}^{\infty}\frac{(2n+1)}{(2n+1)!}s^{2n+1} + 10s$$
$$:= \sum_{n=1}^{\infty} b_n s^{2n+1},$$

where

$$b_n = \frac{5^{2n+1} - (5/2)\times 4^{2n+1} + 3^{2n+2} - 2(16n+7)}{(2n+1)!}.$$

We now prove the ratio $g_2(s)/g_1(s)$ is decreasing from $(0,\infty)$ onto $(2/3, 38/45)$. To this end, we note first that $a_1 = b_1 = 0$ and $a_n, b_n > 0$ for $n \geqslant 2$. Second, if we prove the sequence $\{b_n/a_n\}_{n\geqslant 2}$ is decreasing, then so is $g_2(s)/g_1(s)$ on $(0,\infty)$ by Lemma 2.1. A direct computation yields

$$c_n := \frac{3}{2^{2n+1}}a_n a_{n+1}\left(\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}\right)$$
$$= 9\times 10^{2n+1} - 21\times 6^{2n+1} - 8(21n-4)\times 5^{2n+1} + 20(3n-1)\times 4^{2n+2}$$
$$-8(5n-4)\times 3^{2n+2} + (240n+89)2^{2n+2} - 16\left(48n^2+69n+28\right)$$

and $c_2 = 192\,000 > 0$. To show $c_n > 0$ for $n \geqslant 2$, it suffices to show that, for $n \geqslant 3$,

$$c_n' = 9\times 10^{2n+1} - 21\times 6^{2n+1} - 8(21n-4)\times 5^{2n+1} > 0,$$
$$c_n'' = 20(3n-1)\times 4^{2n+2} - 8(5n-4)\times 3^{2n+2} > 0,$$
$$c_n''' = (240n+89)2^{2n+2} - 16\left(48n^2+69n+28\right) > 0.$$

It is easy to check that c_n' satifies

$$c_{n+1}' - 100c_n' = 224\times 6^{2n+2} + 24(21n-11)\times 5^{2n+3} > 0,$$

which in combination with $c_3' = 47\,246\,344 > 0$ yields $c_n' > 0$ for $n \geqslant 3$. The fact that $c_n'' > 0$ for $n \geqslant 3$ is evident. For c_n''', we have

$$c_{n+1}''' - c_n''' = (2880n + 4908) \times 2^{2n} - (1536n + 1872) > 0,$$

which together with $c_3''' = 196\,432 > 0$ gives $c_n''' > 0$ for $n \geqslant 3$. It is readily verified that

$$\lim_{s\to 0} \frac{g_2(s)}{g_1(s)} = \frac{b_2}{a_2} = \frac{38}{45} \quad \text{and} \quad \lim_{s\to\infty} \frac{g_2(s)}{g_1(s)} = \lim_{n\to\infty} \frac{b_n}{a_n} = \frac{2}{3},$$

where the second limit holds due to Lemma 2.4.

Now, let us return to (4.35). We have

$$\operatorname{sgn}\left[G_p'(2s)\right] = \operatorname{sgn}\left[\frac{g_2(s)}{g_1(s)} - p\right].$$

Due to the monotoncity of the ratio $g_2(s)/g_1(s)$, it is deduced that G_p is decreasing (or increasing) on $(0,\infty)$ if and only if

$$p \geqslant \sup_{s>0}\left\{\frac{g_2(s)}{g_1(s)}\right\} = \frac{38}{45} \quad \text{or} \quad p \leqslant \inf_{s>0}\left\{\frac{g_2(s)}{g_1(s)}\right\} = \frac{2}{3}.$$

While $2/3 < p < 38/45$, there is a $t_0 > 0$ such that G_p is increasing on $(0,t_0)$ and decreasing on (t_0,∞). Consequently, the following inequalities

$$\ln\frac{2}{pe} = \lim_{t\to\infty} G_p(t) < G_p(t) < \lim_{t\to 0} G_p(t) = 0 \text{ if } p \geqslant \frac{38}{45},$$
$$0 = \lim_{t\to 0} G_p(t) < G_p(t) < \lim_{t\to\infty} G_p(t) = \ln\frac{2}{pe} \text{ if } 0 \leqslant p \leqslant \frac{2}{3},$$

and

$$\min\left\{\ln\frac{2}{pe}, 0\right\} < G_p(t) \leqslant G_p(t_0) \text{ if } \frac{2}{3} < p < \frac{38}{45}$$

hold for $t > 0$, which prove the desired inequalities. The proof is completed. □

REMARK 4.8. The inequalities (1.27) for $p = 11/15$ and (4.34) for $p = 2/e$, as well as the monotonicity of $q \mapsto U_q(t)$ with $11/15 < 2/e$ lead to

$$I_0(t) < \sqrt{\frac{\sinh t}{t} U_{11/15}(t)} < \sqrt{\frac{\sinh t}{t} U_{2/e}(t)} < \sqrt{\frac{\sinh t}{t} \exp\left(\frac{t}{\tanh t} - 1\right)}$$

for $t > 0$, or equivalently, the inequalities

$$TQ(a,b) < \sqrt{L(a,b)\mathscr{U}_{11/15}(a,b)} < \sqrt{L(a,b)\mathscr{U}_{2/e}(a,b)} < \sqrt{L(a,b)I(a,b)}$$

hold. This shows that our upper bound $\sqrt{L\mathscr{U}_{11/15}}$ for the Toader-Qi mean TQ is better than $\sqrt{LI}$.

4.4. Comparing with the bounds given in (1.20)

REMARK 4.9. Our new upper bound $\sqrt{L\mathscr{U}_{11/15}}$ for TQ is not comparable with than $\sqrt{LA_{2/3}}$. Let

$$J_p(t) = \frac{3}{2}\ln\cosh\left(\frac{2t}{3}\right) - \ln U_p(t).$$

We easily get

$$\lim_{t\to 0}\frac{J_{11/15}(t)}{t^4} = \frac{13}{12960} > 0,$$
$$\lim_{t\to\infty} J_{11/15}(t) = -\frac{3}{2}\ln 2 - \ln\frac{11}{30} < 0.$$

These show that there are $t_1, t_2 > 0$ such that $J_{11/15}(t) > 0$ for $t \in (0, t_1)$ and $J_{11/15}(t) < 0$ for $t \in (t_2, \infty)$. Therefore, $\sqrt{L\mathscr{U}_{11/15}}$ is not comparable with than $\sqrt{LA_{2/3}}$.

5. Conclusions

In this paper, we obtained the best constants p and q such that the double inequality (1.28) holds for $t > 0$, or equivalently, (1.22) holds for $a, b > 0$ with $a \neq b$. Via comparisons, we found that the chain of inequalities

$$\max\left\{\left[\frac{\sinh(3t/2)}{3t/2}\right]^{2/3}, \sqrt{\frac{\sinh t}{t}\left(\frac{2}{\pi}\cosh t + 1 - \frac{2}{\pi}\right)}\right\}$$
$$< \sqrt{\frac{\sinh t}{t}U_{2/\pi}(t)} < I_0(t) < \min\left\{\sqrt{\frac{\sinh t}{t}U_{11/15}(t)}, \sqrt{\frac{\sinh t}{t}\cosh^{3/2}\frac{2t}{3}}\right\}$$
$$< \min\left\{\frac{3}{4}\frac{\sinh t}{t} + \frac{1}{4}\cosh t, \sqrt{\frac{\sinh t}{t}\exp\left(\frac{t}{\tanh t} - 1\right)}\right\}$$

hold for $t > 0$. Or equivalently, the inequalities for means

$$\begin{aligned} \max\left\{L_{3/2}, \sqrt{L\left(\frac{2}{\pi}A + \left(1 - \frac{2}{\pi}\right)G\right)}\right\} &< \sqrt{L\mathscr{U}_{2/\pi}} < TQ \\ < \min\left\{\sqrt{L\mathscr{U}_{11/15}}, \sqrt{LA_{2/3}}\right\} &< \min\left\{\frac{3}{4}L + \frac{1}{4}A, \sqrt{LI}\right\} \end{aligned} \tag{5.36}$$

hold for $a, b > 0$ with $a \neq b$. It thus can be seen that our new bounds for $TQ(a,b)$ and $I_0(t)$ are better than those known ones. Moreover, taking into account (1.21) and (5.36) we can derive a more refine chain of inequalities for means:

$$L < AGM < L^{3/4}A^{1/3} < L_{3/2} < \sqrt{L\mathscr{U}_{2/\pi}} < TQ < \min\left\{\sqrt{L\mathscr{U}_{11/15}}, \sqrt{LA_{3/2}}\right\}$$
$$< \min\left\{\frac{3}{4}L + \frac{1}{4}A, \sqrt{LI}\right\} < \frac{L+I}{2} < \frac{A+G}{2} < \mathscr{T}_{1/3} < I_{3/4},$$

where the inequality

$$\frac{3}{4}L+\frac{1}{4}A<\frac{L+I}{2}$$

holds due to (1.5).

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REFERENCES

[1] Z.-H. YANG AND Y.-M. CHU, *An optimal inequalities chain for bivarite means*, J. Math. Inequal. **9** (2015), no. 2, 331–343, doi:10.7153/jmi-09-27.
[2] T.-P. LIN, *The power mean and the logarithmic mean*, Amer. Math. Monthly **81** (1974), 879–883.
[3] K. B. STOLARSKY, *The power and generalized logarithmic means*, Amer. Math. Monthly **87** (1980), no. 7, 545–548.
[4] A. O. PITTENGER, *Inequalities between arithmetic and logarithmic means*, Univ. Beograd. Publ. Elektrotehn. Fak., Ser. Mat. Fiz. **678–715** (1980), 15–18.
[5] H. ALZER, *Ungleichungen für Mittelwerte*, Arch. Math. **47** (1986), no. 5, 422–426, https://link.springer.com/article/10.1007/BF01189983.
[6] Z.-H. YANG, *New sharp bounds for identric mean in terms of logarithmic mean and arithmetic mean*, J. Math. Inequal. **6** (2012), no. 4, 533–543, doi:10.7153/jmi-06-51.
[7] J. SÁNDOR, *On the identric and logarithmic means*, Aequat. Math. **40** (1990), 261–270, https://link.springer.com/article/10.1007/BF02112299.
[8] ZS. PÁLES, *Inequalities for differences of powers*, J. Math. Anal. Appl. **131** (1988), 271–281, https://core.ac.uk/download/pdf/8257720.
[9] Z.-H. YANG, *On the log-convexity of two-parameter homogeneous functions*, Math. Inequal. Appl. **10** (2007), no. 3, 499–516, doi:10.7153/mia-10-47.
[10] Z.-H. YANG, *On the monotonicity and log-convexity of a four-parameter homogeneous mean*, J. Inequal. Appl. **2008** (2008), Art. ID 149286, 12 pages, doi:10.1155/2008/149286.
[11] GH. TOADER, *Some mean values related to the arithmetic-geometric mean*, J. Math. Anal. Appl. **218** (1998), no. 2, 358–368, doi:10.1006/jmaa.1997.5766.
[12] J. M. BORWEIN AND P. B. BORWEIN, *Pi and the AGM*, John Wiley and Sons, New York, 1987.
[13] M. K. VAMANAMURTHY AND M. VUORINEN, *Inequalities for means*, J. Math. Anal. Appl. **183** (1994), no. 1, 155–166, doi:10.1155/S0161171295000792.
[14] Z.-H. YANG, *A new proof of inequalities for Gauss compound mean*, Int. Journal of Math. Analysis **4** (2010), no. 21, 1013–1018, doi:ijma-21-24-2010/yangzhIJMA21-24-2010.
[15] Y.-M. CHU AND M.-K. WANG, *Optimal Lehmer mean bounds for the Toader mean*, Results Math. **61** (2012), no. 3–4, 223–229, doi:10.1007/s00025-010-0090-9.
[16] B. C. CARLSON AND M. VUORINEN, *An inequality of the AGM and the logarithmic mean*, SIAM Rev. **33** (1991), 655, Problem 91–117, doi:10.1137/1033141.

[17] H. ALZER, S.-L. QIU, *Monotonicity theorems and inequalities for the complete elliptic integrals*, J. Comput. Appl. Math. **172** (2004), 289–312, doi:10.1016/j.cam.2004.02.009.

[18] J. M. BORWEIN AND P. B. BORWEIN, *Inequalities for compound mean iterations with logarithmic asymptotes*, J. Math. Anal. Appl. **177** (1993), no. 2, 572–582, doi:10.1006/jmaa.1993.1278.

[19] R. W. BARNARD, K. PEARCE AND K. C. RICHARDS, *An inequality involving the generalized hypergeometric function and the arclength of an ellipse*, SIAM J. Math. Anal. **31** (2000), no. 3, 693–699, doi:10.1137/S0036141098341575.

[20] Z.-H. YANG, W.-M. QIAN, Y.-M. CHU AND W. ZHANG, *On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind*, J. Math. Anal. Appl. **462** (2018), no. 2, 1714–1726, doi:10.1016/j.jmaa.2018.03.005.

[21] Z.-H. YANG, *Sharp approximations for the complete elliptic integrals of the second kind by one-parameter means*, J. Math. Anal. Appl. **467** (2018), 446–461, doi:10.1016/j.jmaa.2018.07.020.

[22] J.-L. WANG, W.-M. QIAN, Z.-Y. HE AND Y.-M. CHU, *On approximating the Toader mean by other bivariate means*, J. Funct. Spaces **2019** (2019), Art. ID 6082413, 7 pages, doi:10.1155/2019/6082413.

[23] Z.-H. YANG, Y.-M. CHU, W. ZHANG, *High accuracy asymptotic bounds for the complete elliptic integral of the second kind*, Appl. Math. Comput. **348** (2019), 552–564, doi:10.1016/j.amc.2018.12.025.

[24] Z.-H. YANG, W.-M. QIAN, W. ZHANG AND Y.-M. CHU, *Notes on the complete elliptic integral of the first kind*, Math. Inequal. Appl. **23** (2020), no. 1, 77–93, doi:10.7153/mia-2020-23-07.

[25] F. QI, X.-T. SHI, F.-F. LIU, AND Z.-H. YANG, *A double inequality for an integral mean in terms of the exponential and logarithmic means*, ResearchGate Research, doi:10.13140/RG.2.1.2353.6800.

[26] F. QI, X.-T. SHI, F.-F. LIU, AND Z.-H. YANG, *A double inequality for an integral mean in terms of the exponential and logarithmic means*, Period Math. Hung. **75** (2017), 180–189, doi:10.1007/s10998-016-0181-9.

[27] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1922.

[28] Z.-H. YANG AND J.-F. TIAN, *Asymptotic expansions for the gamma function in terms of hyperbolic functions*, J. Math. Anal. Appl. **478** (2019), 133–155, doi:10.1016/j.jmaa.2019.05.022.

[29] Z.-H. YANG AND Y.-M. CHU, *On approximating the modified Bessel function of the first kind and Toader-Qi mean*, J. Inequal. Appl. **2016** (2016): 40, doi:10.1186/s13660-016-0988-1.

[30] Z.-H. YANG, Y.-M. CHU AND Y.-Q. SONG, *Sharp bounds for Toader-Qi mean in terms of logarithmic and identric means*, Math. Inequal. Appl. **19** (2016), no. 2, 721–730, doi:10.7153/mia-19-52.

[31] Z.-H. YANG AND Y.-M. CHU, *A sharp lower bound for Toader-Qi mean with applications*, J. Func. Spaces **2016** (2016), Art. ID 4165601, 5 pages, available online at doi:10.1155/2016/4165601.

[32] Z.-H. YANG, J.-F. TIAN, AND Y.-R. ZHU, *New sharp bounds for the modified Bessel function of the first kind and Toader-Qi mean*, Mathematics 2020, 8, 901, doi:10.3390/math8060901.

[33] W.-M. QIAN, X.-H. ZHANG AND Y.-M. CHU, *Sharp bounds for the Toader-Qi mean in terms of harmonic and geometric means*, J. Math. Inequal. 11 (2017), no. 1, 121–127, doi:10.7153/jmi-11-11.

[34] M. BIERNACKI AND J. KRZYZ, *On the monotonicity of certain functionals in the theory of analytic functions*, Annales Universitatis Mariae Curie-Sklodowska **9** (1955), 135–147.

[35] Z.-H. YANG, Y.-M. CHU, AND M.-K. WANG, *Monotonicity criterion for the quotient of power series with applications*, J. Math. Anal. Appl. **428** (2015), 587–604, doi:10.1016/j.jmaa.2015.03.043.

[36] Z.-H. YANG AND S.-Z. ZHENG, *Sharp Bounds for the Ratio of Modified Bessel Functions*, Mediterr. J. Math. **14** (2017): 169, doi:10.1007/s00009-017-0971-1.

[37] Z.-H. YANG AND J. TIAN, *Sharp inequalities for the generalized elliptic integrals of the first kind*, Ramanujan J. **48** (2019), 91–116, doi:10.1007/s11139-018-0061-4.

[38] G. PÓLYA AND G. SZEGÖ, *Problems and Theorems in Analysis I: Series. Integral Calculus, Theory of Functions*, Reprint of the 1978 edition, Classics in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, 1998.

[39] M. ABRAMOWITZ AND I. A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series 55, 10th printing, Dover Publications, New York and Washington, 1972.

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