Journal of
Mathematical
Inequalities
Volume 16, Number 2 (2022), 649–657 doi:10.7153/jmi-2022-16-46

EIGENVALUES OF THE SOLUTION OF THE LYAPUNOV TENSOR EQUATION

LI LIANG

(*Communicated by M. Krnić*)

Abstract. The paper is concerned with $\mathrm{H(Z)}$-eigenvalues of the solution of the Lyapunov tensor equation. According to the Lyapunov algebraic theorem on tensors, bounds for $\mathrm{H(Z)}$-eigenvalues of the solution are given firstly, then based on the relationship between the Lyapunov tensor equation and the continuous-time linear uncertain system, conditional inequalities for the asymptotic stability of the system are shown by $\mathrm{H(Z)}$-eigenvalues of the solution of the Lyapunov tensor equation.

1. Introduction

In the paper, we consider the Lyapunov tensor equation

$$\mathscr{P}\times_1 A+\mathscr{P}\times_2 A+\cdots+\mathscr{P}\times_{2m} A=-\mathscr{Q} \tag{1}$$

where both the matrix $A\in\mathbb{R}^{n\times n}$ and the 2mth-order tensor $\mathscr{Q}\in\mathbb{R}^{n\times n\times\cdots\times n}$ are known, the other 2mth-order tensor $\mathscr{P}\in\mathbb{R}^{n\times n\times\cdots\times n}$ is unknown and the operator $\times_k$ $(k=1,2,\ldots,m)$ is the k-mode product [5]. Usually, equation (1) comes from finite difference or spectral method discretization of a linear partial differential equation in high dimension, and estimates on the solution of the Lyapunov tensor equation can simplify solving processes of these equations. Hence, similar to estimations on solutions of matrix equations [3, 4, 7, 8], the paper discusses bounds for the solution of the Lyapunov tensor equation firstly.

With the rapid development of tensors, the Lyapunov tensor equation has gradually attracted attentions of scholars. In 2010, Li et al. [9] presented an algorithm for solving the third-order Lyapunov tensor equation when dealing with the three-dimensional radiative transfer equation. In 2013, Wei et al. [14] considered the backward error and disturbance bounds on the solution of the Lyapunov tensor equation. In 2016, Ali Beik et al. [1] proposed a new tensor multiplication and solved the Lyapunov tensor equation based on it by some known iterative methods. In 2019, Zheng et al. [10] discussed the sensitivity of the solution of the Lyapunov tensor equation.

In particular, Zheng et al. [11] established the relationship between the Lyapunov tensor equation and the linear system $\dot{x}(t)=Ax(t)$ by the positive definite tensor [13], and then showed the Lyapunov algebraic theorem on tensors as follows

Mathematics subject classification (2020): 15A18, 15A69, 93D05.
Keywords and phrases: Lyapunov tensor equation, $\mathrm{H(Z)}$-eigenvalue, linear uncertain system, asymptotic stability.

LEMMA 1. [11] *Let* $A \in \mathbb{R}^{n\times n}$ *be a matrix and* $\mathscr{Q} \in \mathbb{R}^{n\times n\times\cdots\times n}$ *be a* $2m$*th-order positive definite tensor. Then* A *is stable if and only if the Lyapunov tensor equation*

$$\mathscr{P}\times_1 A+\mathscr{P}\times_2 A+\cdots+\mathscr{P}\times_{2m} A=-\mathscr{Q}$$

has the unique solution $\mathscr{P}$ *and* $\mathscr{P}\in\mathbb{R}^{n\times n\times\cdots\times n}$ *is positive definite.*

However, based on possible errors in the calculation process and uncertainties in the real world, it is necessary to consider the asymptotic stability of the uncertain linear system

$$\dot{x}(t)=(A+\triangle A)x(t) \tag{2}$$

where $x(t)\in\mathbb{R}^n$ is the system state vector, $A\in\mathbb{R}^{n\times n}$ is known and $\triangle A\in\mathbb{R}^{n\times n}$ is unknown. In 1994, Zelentsovsky [15] used the even-order homogeneous polynomial Lyapunov function

$$V(x)=\sum_{i_1,i_2,\ldots,i_{2m}=1}^{n} p_{i_1i_2\cdots i_{2m}}x_{i_1}x_{i_2}\cdots x_{i_{2m}} \tag{3}$$

where $x=(x_1,x_2,\ldots,x_n)$ and coefficients $p_{i_1i_2\cdots i_{2m}}\in\mathbb{R}$, to obtain conditions for the asymptotic stability of system (2), and then these results became the basis of following numerous studies. Recently, according to the Lyapunov stability theorem and relevant knowledge of tensors [6, 13], we rewrite $V(x)$ and $\dot{V}(x)$ as follows

$$V(x)=\mathscr{P}x^{2m}$$

$$\dot{V}(x)=(\mathscr{P}\times_1(A+\triangle A)^{\mathrm{T}}+\mathscr{P}\times_2(A+\triangle A)^{\mathrm{T}}+\cdots+\mathscr{P}\times_{2m}(A+\triangle A)^{\mathrm{T}})x^{2m}$$

and find that basing on the Lyapunov tensor equation to consider the asymptotic stability of system (2) can reduce the computation complexity compared with [15]. Hence, the paper applies eigenvalues of the solution of the Lyapunov tensor equation to give conditional inequalities for the asymptotic stability of system (2) secondly.

The outline of the paper is as follows. In Section 2, we introduce some preliminary materials. In Section 3, we estimate the solution of the Lyapunov tensor equation and give bounds for eigenvalues of the solution. In Section 4, we apply the eigenvalues of the solution to show conditional inequalities for the asymptotic stability of system (2).

2. Preliminaries

In 2005, Qi [13] defined the eigenvalue of tensors and Lim [12] defined the eigenvalue and the singular value of tensors respectively. On this basis, the paper is concerned with eigenvalues of the solution of the Lyapunov tensor equation (1).

DEFINITION 1. [13] Let $\mathscr{A}\in\mathbb{R}^{n\times n\times\cdots\times n}$ be a $2m$th-order symmetric tensor. If a real number λ and a nonzero real vector x are solutions of the following homogeneous equation:

$$\mathscr{A}x^{m-1}=\lambda x^{[m-1]},$$

then called the solution x H-eigenvector and λ H-eigenvalue of $\mathscr{A}$.

DEFINITION 2. [13] Let $\mathscr{A} \in \mathbb{R}^{n\times n\times\cdots\times n}$ be a $2m$th-order symmetric tensor. If a real number λ and a nonzero real vector x are solutions of the following homogeneous equation:

$$\begin{cases} \mathscr{A}x^{m-1} = \lambda x \\ x^{\mathrm{T}}x = 1 \end{cases},$$

then called the solution x Z-eigenvector and λ Z-eigenvalue of $\mathscr{A}$.

DEFINITION 3. [2, 12] Let $\mathscr{Q} \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_{2m}}$ and $1 < p_1,\ldots,p_{2m} < \infty$, then the $l^{p_1,\ldots,p_{2m}}$-singular value of $\mathscr{Q}$ is defined as the critical value of the function

$$f : (\mathbb{R}^{n_1}\setminus\{o\}) \times\cdots\times (\mathbb{R}^{n_{2m}}\setminus\{o\}) \to \mathbb{R}$$

given by

$$f(x_1,\ldots,x_{2m}) := \frac{|\mathscr{Q}(x_1,\ldots,x_{2m})|}{\|x_1\|_{p_1}\cdot\cdots\cdot\|x_{2m}\|_{p_{2m}}}$$

3. Bounds for eigenvalues

Estimates on the solution of the Lyapunov tensor equation can simplify the solving process of a linear partial differential equation in high dimension, so we discuss bounds for eigenvalues of the solution of the Lyapunov tensor equation (1) in this section.

THEOREM 1. *Let $A \in \mathbb{R}^{n\times n}$ be a stable matrix and $\mathscr{Q} \in \mathbb{R}^{n\times n\times\cdots\times n}$ be a $2m$th-order positive definite tensor. Then when $\lambda_{\max}^{\mathrm{H(Z)}}(\mathscr{I}^{\mathrm{H(Z)}}\times_1 A+\cdots+\mathscr{I}^{\mathrm{H(Z)}}\times_{2m} A) < 0$, H(Z)-eigenvalues of the solution $\mathscr{P}$ of $\mathscr{P}\times_1 A+\mathscr{P}\times_2 A+\cdots+\mathscr{P}\times_{2m} A = -\mathscr{Q}$ satisfy*

$$\begin{aligned} &\frac{\lambda_{\min}^{\mathrm{H(Z)}}(\mathscr{Q})}{|\lambda_{\min}^{\mathrm{H(Z)}}(\mathscr{I}^{\mathrm{H(Z)}}\times_1 A+\cdots+\mathscr{I}^{\mathrm{H(Z)}}\times_{2m} A)|} \\ &\leqslant \lambda^{\mathrm{H(Z)}}(\mathscr{P}) \\ &\leqslant \frac{\lambda_{\max}^{\mathrm{H(Z)}}(\mathscr{Q})}{|\lambda_{\max}^{\mathrm{H(Z)}}(\mathscr{I}^{\mathrm{H(Z)}}\times_1 A+\cdots+\mathscr{I}^{\mathrm{H(Z)}}\times_{2m} A)|} \end{aligned}$$

where $\mathscr{I}^{\mathrm{H}} = \mathscr{I}$, $\mathscr{I}$ is a $2m$th-order identity tensor, $\mathscr{I}^{\mathrm{Z}} = \mathrm{sym}(I_2^m)$ and

$$\mathrm{sym}(I_2^m)_{i_1,i_2,\ldots,i_{2m-1},i_{2m}} = \begin{cases} 1, & i_1 = i_2,\ldots,i_{2m-1} = i_{2m} \\ 0, & others \end{cases}.$$

Proof. (i) Choose a vector $y \in \mathbb{R}^n$ satisfying

$$\mathscr{P}y^{2m-1} = \lambda^{\mathrm{H}}(\mathscr{P})y^{[2m-1]}$$

then multiplying y^{2m} on both sides of equation (1) yields

$$\lambda^{\mathrm{H}}(\mathscr{P})(\mathscr{I}^{\mathrm{H}}\times_1 A+\mathscr{I}^{\mathrm{H}}\times_2 A+\cdots+\mathscr{I}^{\mathrm{H}}\times_{2m} A)y^{2m} = -\mathscr{Q}y^{2m}$$

for $y^{[2m-1]} = Iy^{2m-1} = \mathscr{I}^{\mathrm{H}}y^{2m-1}$. Therefore, based on the positive definite tensor $\mathscr{Q}$ and the stable matrix A,

$$\begin{aligned} &\frac{2m\lambda_{\min}^{\mathrm{H}}(\mathscr{Q})}{|\lambda_{\min}^{\mathrm{H}}(\mathscr{I}^{\mathrm{H}}\times_1 A+\cdots+\mathscr{I}^{\mathrm{H}}\times_{2m} A)|} \\ &\leqslant \lambda^{\mathrm{H}}(\mathscr{P}) = \frac{\mathscr{Q}y^{2m}}{-(\mathscr{I}^{\mathrm{H}}\times_1 A+\mathscr{I}^{\mathrm{H}}\times_2 A+\cdots+\mathscr{I}^{\mathrm{H}}\times_{2m} A)y^{2m}} \\ &\leqslant \frac{2m\lambda_{\max}^{\mathrm{H}}(\mathscr{Q})}{|\lambda_{\max}^{\mathrm{H}}(\mathscr{I}^{\mathrm{H}}\times_1 A+\cdots+\mathscr{I}^{\mathrm{H}}\times_{2m} A)|} \end{aligned}$$

valid when $\lambda_{\max}^{\mathrm{H}}(\mathscr{I}^{\mathrm{H}}\times_1 A+\cdots+\mathscr{I}^{\mathrm{H}}\times_{2m} A)<0$.

(ii) Choose a vector $y\in\mathbb{R}^n$ satisfying

$$\begin{cases} \mathscr{P}y^{2m-1} = \lambda^{\mathrm{Z}}(\mathscr{P})y \\ y^2 = 1 \end{cases}$$

then according to the property $y = \mathrm{sym}(I_2^m)y^{2m-1} = \mathscr{I}^{\mathrm{Z}}y^{2m-1}$,

$$\begin{aligned} &\frac{2m\lambda_{\min}^{\mathrm{Z}}(\mathscr{Q})}{|\lambda_{\min}^{\mathrm{Z}}(\mathscr{I}^{\mathrm{Z}}\times_1 A+\cdots+\mathscr{I}^{\mathrm{Z}}\times_{2m} A)|} \\ &\leqslant \lambda^{\mathrm{Z}}(\mathscr{P}) = \frac{\mathscr{Q}y^{2m}}{-(\mathscr{I}^{\mathrm{Z}}\times_1 A+\mathscr{I}^{\mathrm{Z}}\times_2 A+\cdots+\mathscr{I}^{\mathrm{Z}}\times_{2m} A)y^{2m}} \\ &\leqslant \frac{2m\lambda_{\max}^{\mathrm{Z}}(\mathscr{Q})}{|\lambda_{\max}^{\mathrm{Z}}(\mathscr{I}^{\mathrm{Z}}\times_1 A+\cdots+\mathscr{I}^{\mathrm{Z}}\times_{2m} A)|} \end{aligned}$$

valid when $\lambda_{\max}^{\mathrm{Z}}(\mathscr{I}^{\mathrm{Z}}\times_1 A+\cdots+\mathscr{I}^{\mathrm{Z}}\times_{2m} A)<0$. □

Particularly, based on Theorem 1, H(Z)-eigenvalues of the solution of equation (1) with the symmetric coefficient matrix A possess better properties.

PROPOSITION 2. *Let $A\in\mathbb{R}^{n\times n}$ be a stable symmetric matrix and $\mathscr{Q}\in\mathbb{R}^{n\times n\times\cdots\times n}$ be a $2m$th-order positive definite tensor. Then when $\lambda_{\max}^{\mathrm{H(Z)}}(\mathscr{I}^{\mathrm{H(Z)}}\times_1 A+\cdots+\mathscr{I}^{\mathrm{H(Z)}}\times_{2m} A)<0$, the minimal and maximal* H(Z)*-eigenvalue of the solution $\mathscr{P}$ of*

$$\mathscr{P}\times_1 A+\mathscr{P}\times_2 A+\cdots+\mathscr{P}\times_{2m} A = -\mathscr{Q}$$

satisfy

$$\frac{\lambda_{\min}^{\mathrm{H(Z)}}(\mathscr{Q})}{|\lambda_{\min}^{\mathrm{H(Z)}}(\mathscr{I}^{\mathrm{H(Z)}}\times_1 A+\cdots+\mathscr{I}^{\mathrm{H(Z)}}\times_{2m} A)|} \leqslant \lambda_{\min}^{\mathrm{H(Z)}}(\mathscr{P}) \leqslant \frac{\lambda_{\max}^{\mathrm{H(Z)}}(\mathscr{Q})}{2m|\lambda_{\min}(A)|}$$

$$\frac{\lambda_{\min}^{\mathrm{H(Z)}}(\mathscr{Q})}{2m|\lambda_{\max}(A)|} \leqslant \lambda_{\max}^{\mathrm{H(Z)}}(\mathscr{P}) \leqslant \frac{\lambda_{\max}^{\mathrm{H(Z)}}(\mathscr{Q})}{|\lambda_{\max}^{\mathrm{H(Z)}}(\mathscr{I}^{\mathrm{H(Z)}}\times_1 A+\cdots+\mathscr{I}^{\mathrm{H(Z)}}\times_{2m} A)|}$$

where $\mathscr{I}^{\mathrm{H}} = \mathscr{I}$, $\mathscr{I}$ is a $2m$th-order identity tensor and $\mathscr{I}^{\mathrm{Z}} = \mathrm{sym}(I_2^m)$.

Proof. Choose a vector $y \in \mathbb{R}^n$ satisfying $Ay = \lambda y$, then multiplying y^{2m} on both sides of equation (1) yields

$$2m\lambda \mathscr{P} y^{2m} = -\mathscr{Q} y^{2m}$$

(i) Considering $\|y\|_{2m}$ implies

$$2m|\lambda|\lambda_{\min}^{\mathrm{H}}(\mathscr{P}) \leqslant 2m|\lambda| \frac{\mathscr{P} y^{2m}}{\|y\|_{2m}^{2m}} = \frac{\mathscr{Q} y^{2m}}{\|y\|_{2m}^{2m}} \leqslant \lambda_{\max}^{\mathrm{H}}(\mathscr{Q})$$

$$2m|\lambda|\lambda_{\max}^{\mathrm{H}}(\mathscr{P}) \geqslant 2m|\lambda| \frac{\mathscr{P} y^{2m}}{\|y\|_{2m}^{2m}} = \frac{\mathscr{Q} y^{2m}}{\|y\|_{2m}^{2m}} \geqslant \lambda_{\min}^{\mathrm{H}}(\mathscr{Q})$$

then

$$\lambda_{\min}^{\mathrm{H}}(\mathscr{P}) \leqslant \frac{\lambda_{\max}^{\mathrm{H}}(\mathscr{Q})}{2m|\lambda_{\min}|}$$

with the choice of $\lambda = \lambda_{\min}$,

$$\lambda_{\max}^{\mathrm{H}}(\mathscr{P}) \geqslant \frac{\lambda_{\min}^{\mathrm{H}}(\mathscr{Q})}{2m|\lambda_{\max}|}$$

with the choice of $\lambda = \lambda_{\max}$. Therefore, based on Theorem 1,

$$\frac{\lambda_{\min}^{\mathrm{H}}(\mathscr{Q})}{|\lambda_{\min}^{\mathrm{H}}(\mathscr{I}^{\mathrm{H}} \times_1 A + \cdots + \mathscr{I}^{\mathrm{H}} \times_{2m} A)|} \leqslant \lambda_{\min}^{\mathrm{H}}(\mathscr{P}) \leqslant \frac{\lambda_{\max}^{\mathrm{H}}(\mathscr{Q})}{2m|\lambda_{\min}(A)|}$$

$$\frac{\lambda_{\min}^{\mathrm{H}}(\mathscr{Q})}{2m|\lambda_{\max}(A)|} \leqslant \lambda_{\max}^{\mathrm{H}}(\mathscr{P}) \leqslant \frac{\lambda_{\max}^{\mathrm{H}}(\mathscr{Q})}{|\lambda_{\max}^{\mathrm{H}}(\mathscr{I}^{\mathrm{H}} \times_1 A + \cdots + \mathscr{I}^{\mathrm{H}} \times_{2m} A)|}$$

valid when

$$\lambda_{\max}^{\mathrm{H}}(\mathscr{I}^{\mathrm{H}} \times_1 A + \cdots + \mathscr{I}^{\mathrm{H}} \times_{2m} A) < 0$$

(ii) Consider $\|y\|_2$, then based on the choice of λ and Theorem 1,

$$\frac{\lambda_{\min}^{\mathrm{Z}}(\mathscr{Q})}{|\lambda_{\min}^{\mathrm{Z}}(\mathscr{I}^{\mathrm{Z}} \times_1 A + \cdots + \mathscr{I}^{\mathrm{Z}} \times_{2m} A)|} \leqslant \lambda_{\min}^{\mathrm{Z}}(\mathscr{P}) \leqslant \frac{\lambda_{\max}^{\mathrm{Z}}(\mathscr{Q})}{2m|\lambda_{\min}(A)|}$$

$$\frac{\lambda_{\min}^{\mathrm{Z}}(\mathscr{Q})}{2m|\lambda_{\max}(A)|} \leqslant \lambda_{\max}^{\mathrm{Z}}(\mathscr{P}) \leqslant \frac{\lambda_{\max}^{\mathrm{Z}}(\mathscr{Q})}{|\lambda_{\max}^{\mathrm{Z}}(\mathscr{I}^{\mathrm{Z}} \times_1 A + \cdots + \mathscr{I}^{\mathrm{Z}} \times_{2m} A)|}$$

valid when

$$\lambda_{\max}^{\mathrm{Z}}(\mathscr{I}^{\mathrm{Z}} \times_1 A + \cdots + \mathscr{I}^{\mathrm{Z}} \times_{2m} A) < 0. \quad \square$$

4. Conditional inequalities for the asymptotic stability

For system (2), choose the homogeneous polynomial function (3) as the Lyapunov function, then according to the Lyapunov stability theorem, we try to use the Lyapunov tensor equation to consider the asymptotic stability of system (2).

THEOREM 3. *Let* $A \in \mathbb{R}^{n\times n}$ *be a stable matrix and* $\mathscr{Q} \in \mathbb{R}^{n\times n\times\cdots\times n}$ *be a* $2m$*th-order positive definite tensor. If*

$$\|\triangle A\|_{2m} < \frac{\lambda^{\mathrm{H}}_{\min}(\mathscr{Q})}{2m\sigma^{\mathrm{H}}_{\max}(\mathscr{P})}$$

then $A+\triangle A$ *is stable, where* $\mathscr{P} \in \mathbb{R}^{n\times n\times\cdots\times n}$ *is the solution of the Lyapunov tensor equation*

$$\mathscr{P}\times_1 A^{\mathrm{T}} + \mathscr{P}\times_2 A^{\mathrm{T}} + \cdots + \mathscr{P}\times_{2m} A^{\mathrm{T}} = -\mathscr{Q} \tag{4}$$

Proof. Let $A \in \mathbb{R}^{n\times n}$ be a stable matrix , then according to Lemma 1, there exist positive definite tensors $\mathscr{P}$ and $\mathscr{Q}$ satisfying the Lyapunov tensor equation (4).

Consider system (2), we can choose the Lyapunov function

$$V(x) = \mathscr{P}x^{2m}$$

implying $V(x) > 0$ for $\forall x \in \mathbb{R}^n$, then according to definitions in Section 2,

$$\begin{aligned}\dot V(x) &= (\mathscr{P}\times_1 \triangle A^{\mathrm{T}} + \mathscr{P}\times_2 \triangle A^{\mathrm{T}} + \cdots + \mathscr{P}\times_{2m} \triangle A^{\mathrm{T}})x^{2m} - 2m\mathscr{Q}x^{2m}\\ &\leqslant (2m\|\triangle A^{\mathrm{T}}\|_{2m}\sigma^{\mathrm{H}}_{\max}(\mathscr{P}) - \lambda^{\mathrm{H}}_{\min}(\mathscr{Q}))\|x\|^{2m}_{2m}\end{aligned}$$

for

$$\frac{(\mathscr{P}\times_1 \triangle A^{\mathrm{T}} + \mathscr{P}\times_2 \triangle A^{\mathrm{T}} + \cdots + \mathscr{P}\times_{2m} \triangle A^{\mathrm{T}})x^{2m}}{\|x\|^{2m}_{2m}}$$

$$\leqslant \left(\sup\frac{|\mathscr{P}\times_1 \triangle Ax\times_2\cdots\times_{2m}x|}{\|x\|^{2m-1}_{2m}\|\triangle Ax\|_{2m}} + \cdots + \sup\frac{|\mathscr{P}\times_1 x\times_2\cdots\times_{2m}\triangle Ax|}{\|x\|^{2m-1}_{2m}\|\triangle Ax\|_{2m}}\right)\frac{\|\triangle Ax\|_{2m}}{\|x\|_{2m}}$$

$$\leqslant 2m\|\triangle A\|_{2m}\sigma^{\mathrm{H}}_{\max}(\mathscr{P})$$

and

$$\frac{\mathscr{Q}x^{2m}}{\|x\|^{2m}_{2m}} \geqslant \inf_{\mathrm{x}\in\mathbb{R}^{\mathrm{n}}}\frac{|\mathscr{Q}x^{2m}|}{\|x\|^{2m}_{2m}} = \lambda^{\mathrm{H}}_{\min}(\mathscr{Q})$$

Obviously, if

$$\|\triangle A\|_{2m} < \frac{\lambda^{\mathrm{H}}_{\min}(\mathscr{Q})}{2m\sigma^{\mathrm{H}}_{\max}(\mathscr{P})}$$

then $\dot V(x) < 0$ for $\forall x \in \mathbb{R}^n$.

Therefore, according to the Lyapunov stability theory, system (2) is asymptotically stable, i.e. $A+\triangle A$ is stable. □

Due to the uncertainty of the $2m$th-order positive definite tensor $\mathscr{Q}$ in equation (4), sometimes it is necessary to find a special $\mathscr{Q}$ making $\frac{\lambda^{\mathrm{H}}_{\min}(\mathscr{Q})}{2m\sigma^{\mathrm{H}}_{\max}(\mathscr{P})}$ maximum.

PROPOSITION 4. *Let $A \in \mathbb{R}^{n\times n}$ be a stable matrix and $\mathscr{I} \in \mathbb{R}^{n\times n\times\cdots\times n}$ be a 2mth-order identity tensor. If*

$$\|\triangle A\|_{2m} < \frac{1}{\sigma_{\max}^{\mathrm{H}}(\bar{\mathscr{P}})}$$

then $A+\triangle A$ is stable, where $\bar{\mathscr{P}} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is the solution of equation

$$\bar{\mathscr{P}} \times_1 A^{\mathrm{T}} + \bar{\mathscr{P}} \times_2 A^{\mathrm{T}} + \cdots + \bar{\mathscr{P}} \times_{2m} A^{\mathrm{T}} = -2m\mathscr{I} \tag{5}$$

Proof. According $\frac{\mathscr{I}x^{2m}}{\|x\|_{2m}^{2m}} = 1$, the proof is obvious. □

LEMMA 2. [10] *Let $A \in \mathbb{R}^{n\times n}$ be a stable matrix and $\mathscr{Q} \in \mathbb{R}^{n\times n\times\cdots\times n}$ be a 2mth-order positive definite tensor, then equation*

$$\mathscr{P} \times_1 A^{\mathrm{T}} + \mathscr{P} \times_2 A^{\mathrm{T}} + \cdots + \mathscr{P} \times_{2m} A^{\mathrm{T}} = -\mathscr{Q}$$

has the unique solution

$$\mathscr{P} = \int_0^{+\infty} \mathscr{Q} \times_1 e^{A^{\mathrm{T}}t} \times_2 e^{A^{\mathrm{T}}t} \times \cdots \times_{2m} e^{A^{\mathrm{T}}t} dt$$

PROPOSITION 5. *Let $A \in \mathbb{R}^{n\times n}$ be a stable matrix, $\mathscr{Q} \in \mathbb{R}^{n\times n\times\cdots\times n}$ be a 2mth-order positive definite tensor and $\mathscr{I} \in \mathbb{R}^{n\times n\times\cdots\times n}$ be a 2mth-order identity tensor. Then*

$$\frac{\lambda_{\min}^{\mathrm{H}}(\mathscr{Q})}{2m\sigma_{\max}^{\mathrm{H}}(\mathscr{P})} \leqslant \frac{1}{\sigma_{\max}^{\mathrm{H}}(\bar{\mathscr{P}})}$$

where $\mathscr{P} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is the solution of the Lyapunov tensor equation

$$\mathscr{P} \times_1 A^{\mathrm{T}} + \mathscr{P} \times_2 A^{\mathrm{T}} + \cdots + \mathscr{P} \times_{2m} A^{\mathrm{T}} = -\mathscr{Q}$$

$\bar{\mathscr{P}} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is the solution of the Lyapunov tensor equation

$$\bar{\mathscr{P}} \times_1 A^{\mathrm{T}} + \bar{\mathscr{P}} \times_2 A^{\mathrm{T}} + \cdots + \bar{\mathscr{P}} \times_{2m} A^{\mathrm{T}} = -2m\mathscr{I}$$

Proof. Consider the Lyapunov tensor equation (4)

$$\mathscr{P} \times_1 A^{\mathrm{T}} + \mathscr{P} \times_2 A^{\mathrm{T}} + \cdots + \mathscr{P} \times_{2m} A^{\mathrm{T}} = -\mathscr{Q}$$

and

$$\hat{\mathscr{P}} \times_1 A^{\mathrm{T}} + \hat{\mathscr{P}} \times_2 A^{\mathrm{T}} + \cdots + \hat{\mathscr{P}} \times_{2m} A^{\mathrm{T}} = -\hat{\mathscr{Q}} \tag{6}$$

i) Suppose $\hat{\mathscr{Q}} = \omega\mathscr{Q}(\omega$ is a positive scalar$)$, the Lyapunov tensor equation (6) can be changed into

$$\left(\frac{1}{\omega}\hat{\mathscr{P}}\right) \times_1 A^{\mathrm{T}} + \left(\frac{1}{\omega}\hat{\mathscr{P}}\right) \times_2 A^{\mathrm{T}} + \cdots + \left(\frac{1}{\omega}\hat{\mathscr{P}}\right) \times_{2m} A^{\mathrm{T}} = -\hat{\mathscr{Q}}$$

then according to Lemma 1 and definitions in Section 2,

$$\frac{\lambda_{\min}^{\mathrm{H}}(\hat{\mathscr{Q}})}{2m\sigma_{\max}^{\mathrm{H}}(\hat{\mathscr{P}})} = \frac{\lambda_{\min}^{\mathrm{H}}(\omega\mathscr{Q})}{2m\sigma_{\max}^{\mathrm{H}}(\omega\mathscr{P})} = \frac{\lambda_{\min}^{\mathrm{H}}(\mathscr{Q})}{2m\sigma_{\max}^{\mathrm{H}}(\mathscr{P})}$$

ii) Choose $\omega = \frac{2m}{\lambda_{\min}^{\mathrm{H}}(\mathscr{Q})}$, then $\lambda_{\min}^{\mathrm{H}}(\hat{\mathscr{Q}}) = \frac{2m}{\lambda_{\min}^{\mathrm{H}}(\mathscr{Q})} \cdot \inf_{\mathrm{x} \in \mathbb{R}^{\mathrm{n}}} \frac{\mathscr{Q}x^{2m}}{\|x\|_{2m}^{2m}} = 2m$.

If $\mathscr{Q} = 2m\mathscr{I}$, equation (4) is just equation (5), then according to Lemma 2,

$$\hat{\mathscr{P}} - \bar{\mathscr{P}} = \int_0^{+\infty} (\hat{\mathscr{Q}} - 2m\mathscr{I}) \times_1 e^{A^{\mathrm{T}}t} \times_2 e^{A^{\mathrm{T}}t} \times \cdots \times_{2m} e^{A^{\mathrm{T}}t} dt$$

Obviously, $\hat{\mathscr{P}} - \bar{\mathscr{P}}$ is a positive definite tensor for

$$\frac{(\hat{\mathscr{Q}} - 2m\mathscr{I})x^{2m}}{\|x\|_{2m}^{2m}} \geqslant \lambda_{\min}^{\mathrm{H}}(\hat{\mathscr{Q}}) - 2m = 0$$

Therefore,

$$\frac{\lambda_{\min}^{\mathrm{H}}(\mathscr{Q})}{2m\sigma_{\max}^{\mathrm{H}}(\mathscr{P})} = \frac{\lambda_{\min}^{\mathrm{H}}(\hat{\mathscr{Q}})}{2m\sigma_{\max}^{\mathrm{H}}(\hat{\mathscr{P}})} = \frac{1}{\sigma_{\max}^{\mathrm{H}}(\hat{\mathscr{P}})} \leqslant \frac{1}{\sigma_{\max}^{\mathrm{H}}(\bar{\mathscr{P}})}. \quad \square$$

Similar to H-eigenvalues and H-singular values, Z-eigenvalues and Z-singular values of the solution of equation (4) can also be applied to discuss the asymptotic stability of system (2).

THEOREM 6. *Let $A \in \mathbb{R}^{n\times n}$ be a stable matrix and $\mathscr{Q} \in \mathbb{R}^{n\times n\times\cdots\times n}$ be a 2mth-order positive definite tensor. If*

$$\|\triangle A\|_2 < \frac{\lambda_{\min}^{\mathrm{Z}}(\mathscr{Q})}{2m\sigma_{\max}^{\mathrm{Z}}(\mathscr{P})}$$

then $A + \triangle A$ is stable, where $\mathscr{P} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is the solution of equation

$$\mathscr{P} \times_1 A^{\mathrm{T}} + \mathscr{P} \times_2 A^{\mathrm{T}} + \cdots + \mathscr{P} \times_{2m} A^{\mathrm{T}} = -\mathscr{Q}$$

PROPOSITION 7. *Let $A \in \mathbb{R}^{n\times n}$ be a stable matrix and $\mathscr{I}^{\mathrm{Z}} = \mathrm{sym}(I_2^m)$ be a 2mth-order tensor. If*

$$\|\triangle A\|_2 < \frac{1}{\sigma_{\max}^{\mathrm{Z}}(\bar{\mathscr{P}})}$$

then $A + \triangle A$ is stable, where $\bar{\mathscr{P}} \in \mathbb{R}^{n\times n\times\cdots\times n}$ is the solution of equation

$$\bar{\mathscr{P}} \times_1 A^{\mathrm{T}} + \bar{\mathscr{P}} \times_2 A^{\mathrm{T}} + \cdots + \bar{\mathscr{P}} \times_{2m} A^{\mathrm{T}} = -2m\mathscr{I}^{\mathrm{Z}}$$

PROPOSITION 8. *Let $A \in \mathbb{R}^{n\times n}$ be a stable matrix, $\mathscr{Q} \in \mathbb{R}^{n\times n\times\cdots\times n}$ be a 2mth-order positive definite tensor and $\mathscr{I}^{\mathrm{Z}} = \mathrm{sym}(I_2^m)$ be a 2mth-order tensor. Then*

$$\frac{\lambda_{\min}^{\mathrm{Z}}(\mathscr{Q})}{2m\sigma_{\max}^{\mathrm{Z}}(\mathscr{P})} \leqslant \frac{1}{\sigma_{\max}^{\mathrm{Z}}(\bar{\mathscr{P}})}$$

where $\mathscr{P} \in \mathbb{R}^{n\times n\times\cdots\times n}$ *is the solution of equation*

$$\mathscr{P} \times_1 A^{\mathrm{T}} + \mathscr{P} \times_2 A^{\mathrm{T}} + \cdots + \mathscr{P} \times_{2m} A^{\mathrm{T}} = -\mathscr{Q}$$

$\bar{\mathscr{P}} \in \mathbb{R}^{n\times n\times\cdots\times n}$ *is the solution of equation*

$$\bar{\mathscr{P}} \times_1 A^{\mathrm{T}} + \bar{\mathscr{P}} \times_2 A^{\mathrm{T}} + \cdots + \bar{\mathscr{P}} \times_{2m} A^{\mathrm{T}} = -2m\mathscr{I}^{Z}$$

Acknowledgement. The research is supported by Scientific Research Project of Hunan Provincial Education Department (Grant no. 20C0073).

REFERENCES

[1] F. P. ALI BEIK, F. S. MOVAHED, A. S. AHMADI, *On the Krylov Subspace Methods Based on Tensor Format for Positive Definite Sylvester Tensor Equations*, Numerical Linear Algebra and Its Applications, (23) 2016: 444–466.

[2] A. GAUTIER, M. HEIN, *Tensor Norm and Maximal Singular Vectors of Nonnegative Tensors, a Perron–Frobenius Theorem, a Collatz–Wielandt Characterization and a Generalized Power Method*, Linear Algebra and its Applications, 505 (2016) 313–343.

[3] R. DAVIES, P. SHI, R. WILTSHIRE, *New Upper Solution Bounds for Perturbed Continuous Algebraic Riccati Equations Applied to Automatic Control*, Chaos, Solitons and Fractals, (32) 2007: 487–495.

[4] R. DAVIES, P. SHI, R. WILTSHIRE, *New Upper Solution Bounds of the Discrete Algebraic Riccati Matrix Equation*, Journal of Computational and Applied Mathematics, (213) 2008: 307–315.

[5] T. G. KOLDA, B. W. BADER, *Tensor decompositions and Applications*, Siam Review. (51) 2009: 455–500.

[6] TAMARA G. KOLDA, BRETT W. BADER, *Tensor Decompositions and Applications*, Siam Review, (51) 2009: 455–500.

[7] C. H. LEE, *Solution Bounds of the Continuous and Discrete Lyapunov Matrix Equations*, Journal of Optimization Theory and Applications, (120) 2004: 559–578.

[8] C. H. LEE, *New Upper Solution Bounds of the Continuous Algebraic Riccati Matrix Equation*, IEEE Transactions Automatic Control, (51) 2006: 330–334.

[9] B. W. LI, S. TIAN, Y. S. SUN, *Schur-decomposition for 3D Matrix Equations and its Application in Solving Radiative Discrete Ordinates Equations Discretized by Chebyshev Collocation Spectral Method*, Journal of Computational Physics, (229) 2010: 1198–1212.

[10] L. LIANG, B. ZHENG, *Sensitivity Analysis for the Lyapunov Tensor Equarion*, Linear and Multilinear Algebra, 2019, 67: 555–572.

[11] L. LIANG, B. ZHENG, Y. TIAN, *Algebraic Lyapunov and Stein Stability Results for Tensors*, Linear and Multilinear Algebra, 2018, 66: 731–741.

[12] L. H. LIM, *Singular Values and Eigenvalues of Tensors: a Variational Approach*, IEEE International Workshop on Computational Advances in Multi-sensor Adaptive Processing, (2005) 2006: 129–132.

[13] L. QI, *Eigenvalues of a Real Supersymmetric Tensor*, Journal of Symbolic Computation, (40) 2005: 1302–1324.

[14] X. SHI, Y. WEI, S. LING, *Backward Error and Perturbation Bounds for High Order Sylvester Tensor Equation*, Linear and Multilinear Algebra, 61 (2013): 1436–1446.

[15] A. L. ZELENTSOVSKY, *Nonquadratic Lyapunov Functions for Robust Stability Analysis of Linear Uncertain Systems*, IEEE Transactions on Automatic Control, (39) 1994: 135–138.

(Received April 12, 2021)

Li Liang
School of Mathematics and Statistics
Changsha University of Science and Technology
Changsha 410114, China
e-mail: `liangli@csust.edu.cn`