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ESTIMATES FOR WEIGHTED HARDY–LITTLEWOOD AVERAGES AND THEIR COMMUTATORS ON MIXED CENTRAL MORREY SPACES

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Abstract. In this paper, we study the boundedness of the weighted Hardy-Littlewood average H_φ and its commutator H_φ^b on mixed central Morrey spaces. More precisely, we first obtain the sufficient and necessary condition for the boundedness of H_φ on the mixed central Morrey space $\dot{M}_{\vec{q}}^p(\mathbb{R}^n)$, and also obtain the sharp constant simultaneously. Then we give a characterization for the boundedness of the commutator formed by H_φ and a central bounded mean oscillation function b on $\dot{M}_{\vec{q}}^p(\mathbb{R}^n)$.

1. Introduction

Let $\varphi : [0,1] \to [0,\infty)$ be a measurable function, and f be a complex-valued measurable function on $\mathbb{R}^n$. The weighted Hardy-Littlewood average $H_\varphi f$ is defined by

$$H_\varphi f(x) := \int_0^1 f(xt)\varphi(t)dt. \tag{1.1}$$

The operator H_φ on $L^p(\mathbb{R}^n)$ was first studied by Carton-Lebrun and Fosset [2] under certain conditions on φ, and then extended by Xiao [30], in which the sufficient and necessary condition on φ was obtained to guarantee the boundedness of H_φ on Lebesgue spaces. Moreover, the sharp constant was also obtained in [30].

The weighted Hardy-Littlewood average H_φ is of great importance since it contains many classical average operators as special cases. For example, when $n = 1$ and $\varphi(x) = 1$, $x \in [0,1]$, H_φ is just the classical Hardy operator $Hf(x) = \frac{1}{x}\int_0^x f(t)dt$. In addition, the operator H_φ is closely related to the Riemann-Liouville fractional integral operator, see [10] for more details.

As is well known, commutators are also important operators and play a key role in harmonic analysis. Recall that for a locally integrable function b and an integral operator T, the commutator formed by b and T is defined by $[b,T] = bT - Tb$.

In 2009, Fu et al. [11] first studied the commutators of weighted Hardy-Littlewood averages, and gave the sufficient and necessary conditions for the boundedness of H_φ^b

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on Lebesgue spaces. The mapping property of H_φ and H_φ^b on Morrey spaces was also considered in [13].

Nowadays, there are amounts of papers concerned with weighted Hardy-Littlewood averages and their commutators. Among them, the boundedness for H_φ and H_φ^b on different function spaces is one of the most active research areas in the study of weighted Hardy-Littlewood averages. For example, the boundedness of weighted Hardy-Littlewood averages and their commutators were considered on Morrey type spaces [8, 10], variable spaces [5, 8] and other non-standard function spaces [7, 26]. For more about weighted Hardy-Littlewood averages and their commutators, we refer readers to [4, 6, 9, 14, 15, 16, 17, 20, 27, 29].

The classical Morrey spaces are natural generalizations of Lebesgue spaces, which were introduced by Morrey [21] to study the regularity of elliptic partial differential equations. Till now, Morrey spaces have become one of the most important function spaces in the theory of function spaces, see the book [25] for a complete theory of Morrey type spaces. In 2019, Nogayama [23] introduced a new Morrey type space, which is called mixed Morrey spaces. Mixed Morrey spaces are mixtures of mixed Lebesgue spaces [1] and classical Morrey spaces. Some important operators in harmonic analysis, such as Hardy-Littlewood maximal operator, fractional integral operator and their commutators, were proved to be bounded on mixed Morrey spaces [22, 23, 24]. A natural question is whether weighted Hardy-Littlewood averages and their commutators perform well in mixed central Morrey spaces, the central version of mixed Morrey spaces. In this paper, we will give an affirmative answer. More precisely, we will investigate the boundedness of H_φ and H_φ^b on mixed central Morrey spaces.

The organization of the remainder of this article is as follows. The definitions and some preliminaries are presented in Sect. 2. The necessary and sufficient condition for the boundedness of H_φ on mixed central Morrey spaces will be given in Sect. 3. The corresponding sharp constant is also obtained in Sect. 3. The necessary and sufficient condition on φ is presented in Sect. 4 to guarantee the boundedness for the commutator of H_φ.

2. Definitions and preliminaries

Throughout the paper, we use the following notations.

For any $r > 0$ and $x \in \mathbb{R}^n$, let $Q(x,r) = \{y : |y_i - x_i| < \frac{r}{2}, i = 1,\cdots,n\}$ be the cube in $\mathbb{R}^n$ centered at x with side length r. Let $\mathbb{Q} = \{Q(x,r) : x \in \mathbb{R}^n, r > 0\}$ be the set of all such cubes. Let $L_{\text{loc}}(\mathbb{R}^n)$ denote the locally integrable functions. The letter $\mathbb{C}$ represents all the complex numbers. We also use χ_E and $|E|$ to denote the characteristic function and the Lebesgue measure of a measurable set E, respectively. Let $\mathscr{M}(\mathbb{R}^n)$ denote the class of all Lebesgue measurable functions.

The letter $\vec{p}$ denotes n-tuples of the numbers in $[0,\infty]$, $(n \geqslant 1)$, $\vec{p} = (p_1,\cdots,p_n)$. By the definition, the inequality, for example, $0 < \vec{p} < \infty$ means $0 < p_i < \infty$ for all i. For $1 \leqslant \vec{p} \leqslant \infty$, we denote $\vec{p}\,' = (p_1',\cdots,p_n')$, where p_i' satisfies $\frac{1}{p_i} + \frac{1}{p_i'} = 1$. By $A \lesssim B$, we mean that $A \leqslant CB$ for some constant $C > 0$, and $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$.

To get a better understanding on mixed Morrey spaces, we first recall definitions of mixed norm spaces [1,3], and classical Morrey spaces [21].

For $\vec{p}=(p_1,\cdots,p_n)\in(0,\infty]^n$, the mixed Lebesgue norm $\|\cdot\|_{\vec{p}}$ or $\|\cdot\|_{(p_1,\cdots,p_n)}$ is defined by

$$\|f\|_{\vec{p}}=\|f\|_{(p_1,\cdots,p_n)}:=\left(\int_{\mathbb{R}}\cdots\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|f(x_1,x_2,\cdots,x_n)|^{p_1}dx_1\right)^{\frac{p_2}{p_1}}dx_2\right)^{\frac{p_3}{p_2}}\cdots dx_n\right)^{\frac{1}{p_n}},$$

where $f:\mathbb{R}^n\to\mathbb{C}$ is a measurable function. If $p_j=\infty$ for some $j=1,\cdots,n$, then we have to make appropriate modifications. We define the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)=L^{(p_1,\cdots,p_n)}(\mathbb{R}^n)$ to be the set of all $f\in\mathscr{M}(\mathbb{R}^n)$ with $\|f\|_{\vec{p}}<\infty$.

The classical Morrey space $M_q^p(\mathbb{R}^n)$ is the set of all measurable functions f for which

$$\|f\|_{M_q^p(\mathbb{R}^n)}:=\sup\left\{|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_Q|f(x)|^q dx\right)^{\frac{1}{q}},Q\in\mathbb{Q}\right\}$$

is finite.

If one replaces the $L^q(\mathbb{R}^n)$ norm by the mixed Lebesgue norm $L^{\vec{q}}(\mathbb{R}^n)$ in the definition of classical Morrey spaces, then we get the mixed Morrey spaces $M_{\vec{q}}^p(\mathbb{R}^n)$ introduced by Nagayama et al. [22,23,24]. Now we give the definition of mixed Morrey spaces.

For $\vec{q}=(q_1,\cdots,q_n)\in(0,\infty]^n$ and $p\in(0,\infty]$ satisfying

$$\sum_{j=1}^n\frac{1}{q_j}\geqslant\frac{n}{p},$$

the mixed Morrey quasi-norm $\|\cdot\|_{M_{\vec{q}}^p(\mathbb{R}^n)}$ is defined by

$$\|f\|_{M_{\vec{q}}^p(\mathbb{R}^n)}:=\sup\left\{|Q|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{i=1}^n\frac{1}{q_j}\right)}\|f\chi_Q\|_{\vec{q}},Q\in\mathbb{Q}\right\},$$

and the mixed Morrey space $M_{\vec{q}}^p(\mathbb{R}^n)$ is the set of all $f\in\mathscr{M}(\mathbb{R}^n)$ for which $\|f\|_{M_{\vec{q}}^p(\mathbb{R}^n)}$ is finite.

In this paper, we mainly consider mixed central Morrey spaces, which are defined as follows.

DEFINITION 2.1. For $\vec{q}=(q_1,\cdots,q_n)\in(0,\infty]^n$ and $p\in(0,\infty]$ satisfies

$$\sum_{j=1}^n\frac{1}{q_j}\geqslant\frac{n}{p},$$

the mixed central Morrey quasi-norm $\|\cdot\|_{\dot{M}_{\vec{q}}^p(\mathbb{R}^n)}$ is defined by

$$\|f\|_{\dot{M}_{\vec{q}}^p(\mathbb{R}^n)}:=\sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n\frac{1}{q_j}\right)}\|f\chi_{Q(0,r)}\|_{\vec{q}},$$

and the mixed central Morrey space $\dot{M}^p_{\vec{q}}(\mathbb{R}^n)$ is the set of all $f \in \mathscr{M}(\mathbb{R}^n)$ for which $\|f\|_{\dot{M}^p_{\vec{q}}(\mathbb{R}^n)}$ is finite.

In the study of the commutator of H_φ on mixed central Morrey spaces, the symbol function b under consideration can be in a more general function space, rather than $\mathrm{BMO}(\mathbb{R}^n)$ or central $\mathrm{BMO}(\mathbb{R}^n)$. We first recall the definition of $\mathrm{BMO}(\mathbb{R}^n)$ and central $\mathrm{BMO}(\mathbb{R}^n)$.

A function $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to the bounded mean oscillation space $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{Q\in\mathbb{Q}} \frac{1}{|Q|}\int_Q |f(x)-f_Q|dx < \infty. \tag{2.2}$$

For $1 \leqslant q < \infty$, a function $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to the central bounded mean oscillation space $\dot{\mathrm{CMO}}^q(\mathbb{R}^n)$ if

$$\|f\|_{\dot{\mathrm{CMO}}^q(\mathbb{R}^n)} := \sup_{r>0}\left(\frac{1}{|Q(0,r)|}\int_{Q(0,r)} |f(x)-f_{Q(0,r)}|^q dx\right)^{\frac{1}{q}} < \infty. \tag{2.3}$$

The John-Nirenberg ineuqalitiy for $\mathrm{BMO}(\mathbb{R}^n)$ spaces yields that for any $1<q<\infty$ and $f \in \mathrm{BMO}(\mathbb{R}^n)$, the $\mathrm{BMO}(\mathbb{R}^n)$ norm of f is equivalent to

$$\|f\|_{\mathrm{BMO}^q(\mathbb{R}^n)} := \sup_{Q\in\mathbb{Q}}\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^q dx\right)^{\frac{1}{q}}.$$

However, this is not the case for $\dot{\mathrm{CMO}}^q(\mathbb{R}^n)$, since the John-Nirenberg ineuqalitiy does not hold for central bounded mean oscillation spaces. In fact, the inclusion $\dot{\mathrm{CMO}}^{q_2}(\mathbb{R}^n) \subseteq \dot{\mathrm{CMO}}^{q_1}(\mathbb{R}^n)$ holds for all $1 \leqslant q_1 < q_2 < \infty$ by Hölder's inequality, and the inclusion is strict [12].

Recall that for any $\vec{q} = (q_1, \cdots, q_n) \in [1,\infty)^n$, a new John-Nirenberg inequality for $\mathrm{BMO}(\mathbb{R}^n)$ [19] showed that the BMO norm of all $f \in \mathrm{BMO}(\mathbb{R}^n)$ is also equivalent to

$$\|f\|_{\mathrm{BMO}^{\vec{q}}(\mathbb{R}^n)} := \sup_{Q\in\mathbb{Q}} \frac{\|(f-f_Q)\chi_Q\|_{\vec{q}}}{\|\chi_Q\|_{\vec{q}}}. \tag{2.4}$$

This fact inspires us to study the central version of the above norm, which is exactly the mixed central bounded mean oscillation space $\dot{\mathrm{CMO}}^{\vec{q}}(\mathbb{R}^n)$ introduced in [28].

DEFINITION 2.2. Let $\vec{q} = (q_1, \cdots, q_n) \in [1,\infty)^n$. Then the mixed central bounded mean oscillation space $\dot{\mathrm{CMO}}^{\vec{q}}(\mathbb{R}^n)$ is defined by

$$\|f\|_{\dot{\mathrm{CMO}}^{\vec{q}}(\mathbb{R}^n)} := \sup_{r>0} \frac{\|(f-f_{Q(0,r)})\chi_{Q(0,r)}\|_{\vec{q}}}{\|\chi_{Q(0,r)}\|_{\vec{q}}} < \infty. \tag{2.5}$$

Similarly, we have $\dot{\mathrm{CMO}}^{\vec{q}}(\mathbb{R}^n) \subseteq \dot{\mathrm{CMO}}^{\vec{r}}(\mathbb{R}^n)$ if $1 \leqslant \vec{r} < \vec{q} < \infty$, and the mixed central bounded mean oscillation space $\dot{\mathrm{CMO}}^{\vec{q}}(\mathbb{R}^n)$ is a Banach space in the sense that two functions differ by a constant are regarded as a function in this space. Obviously, $\mathrm{BMO}(\mathbb{R}^n) \subseteq C\dot{M}O^{\vec{q}}(\mathbb{R}^n)$ for all $\vec{q} = (q_1, \cdots, q_n) \in (1,\infty)^n$, and $C\dot{M}O^{q}(\mathbb{R}^n)$ is a special case of $C\dot{M}O^{\vec{q}}(\mathbb{R}^n)$.

For more properties of $C\dot{M}O^{\vec{q}}(\mathbb{R}^n)$, $\vec{q} = (q_1, \ldots, q_n) \in (1,\infty)^n$, the reader is referred to [28].

3. Sharp constant for H_φ on $\dot{M}^p_{\vec{q}}(\mathbb{R}^n)$

This section is devoted to investigating the boundedness of weighted Hardy-Littlewood averages on mixed central Morrey spaces. Moreover, we will also show that the obtained upper bound is optimal.

THEOREM 3.1. *Let $0 < p < \infty$, $1 < \vec{q} < \infty$ and $\sum_{j=1}^n \frac{1}{q_j} \geqslant \frac{n}{p}$. Then H_φ is bounded on $\dot{M}^p_{\vec{q}}(\mathbb{R}^n)$ if and only if*

$$C_p = \int_0^1 t^{-\frac{n}{p}} \varphi(t)dt < \infty. \tag{3.6}$$

Moreover, $\|H_\varphi\|_{\dot{M}^p_{\vec{q}}(\mathbb{R}^n) \to \dot{M}^p_{\vec{q}}(\mathbb{R}^n)} = C_p$.

Proof. For any cube $Q(0,r)$, $r > 0$, by Minkowski's inequality and the dilation property of mixed Lebesgue spaces, we have

$$\begin{aligned}
&|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)} \|H_\varphi f \chi_{Q(0,r)}\|_{\vec{q}} \\
\leqslant &\int_0^1 \varphi(t) |Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)} \|f(t\cdot)\chi_{Q(0,r)(\cdot)}\|_{\vec{q}} dt \\
\leqslant &\int_0^1 t^{-\frac{n}{p}} \varphi(t) |Q(0,tr)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{q_j}\right)} \|f \chi_{Q(0,tr)}\|_{\vec{q}} dt \\
\leqslant &\|f\|_{\dot{M}^p_{\vec{q}}(\mathbb{R}^n)} \int_0^1 t^{-\frac{n}{p}} \varphi(t) dt.
\end{aligned}$$

Taking the supremum over all such cubes $Q(0,r)$ with $r > 0$, we obtain the estimate

$$\|H_\varphi f\|_{\dot{M}^p_{\vec{q}}(\mathbb{R}^n)} \leqslant C_p \|f\|_{\dot{M}^p_{\vec{q}}(\mathbb{R}^n)}. \tag{3.7}$$

Next we will prove the constant C_p is sharp.

When $\sum_{j=1}^n \frac{1}{q_j} > \frac{n}{p}$, we choose $f_0(x_1, \cdots, x_n) = \prod_{j=1}^n |x_j|^{-\frac{1}{p_j}}$, in which $0 < q_j < p_j$ and $\sum_{j=1}^n \frac{1}{p_j} = \frac{n}{p}$. Note that such f_0 must exist since $\sum_{j=1}^n \frac{1}{q_j} > \frac{n}{p}$.

It is proved in [23] that $f_0 \in M^p_{\vec{q}}(\mathbb{R}^n) \subseteq \dot{M}^p_{\vec{q}}(\mathbb{R}^n)$. By a direct computation, it yields

$$H_\varphi f_0(x_1, \cdots, x_n) = C_p \cdot f_0(x_1, \cdots, x_n). \tag{3.8}$$

As a consequence,

$$\|H_\varphi f_0\|_{\dot{M}^p_{\vec{q}}(\mathbb{R}^n)} = C_p \|f_0\|_{\dot{M}^p_{\vec{q}}(\mathbb{R}^n)}.$$

When $\sum_{j=1}^n \frac{1}{q_j} = \frac{n}{p}$, then $\dot{M}^p_{\vec{q}}(\mathbb{R}^n)$ is just the mixed Lebesgue space $L^{\vec{q}}(\mathbb{R}^n)$.

In this case we take $f_1(x_1,\cdots,x_n) = \prod_{j=1}^n |x_j|^{-\frac{1}{q_j}-\varepsilon} \chi_{\{|x_j|>1\}}$, for some sufficiently small $\varepsilon > 0$.

By a routine calculation, we get $\|f_1\|_{\vec{q}} = \prod_{j=1}^n 2^{\frac{1}{q_j}} (q_i\varepsilon)^{-\frac{1}{q_j}}$.

Inserting f_1 into $H_\varphi f$, we get

$$H_\varphi f_1(x_1,\cdots,x_n) = \prod_{j=1}^n |x_j|^{-\frac{1}{q_j}-\varepsilon} \int_{\max_{i=1}^n \left\{\frac{1}{|x_i|}\right\}}^1 t^{-\sum_{i=1}^n \frac{1}{q_i} - n\varepsilon} \varphi(t)dt.$$

Therefore, for sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned} \|H_\varphi f_1\|_{\vec{q}} &\geqslant \int_\varepsilon^1 t^{-\sum_{i=1}^n \frac{1}{q_i} - n\varepsilon} \varphi(t)dt \cdot \left\| \prod_{j=1}^n |x_j|^{-\frac{1}{q_j}-\varepsilon} \chi_{\{|x_j|>\frac{1}{\varepsilon}\}} \right\|_{\vec{q}} \\ &= \varepsilon^{n\varepsilon} \prod_{j=1}^n 2^{\frac{1}{q_j}} (q_j\varepsilon)^{-\frac{1}{q_j}} \int_\varepsilon^1 t^{-\sum_{i=1}^n \frac{1}{q_i} - n\varepsilon} \varphi(t)dt \\ &= \varepsilon^{n\varepsilon} \|f_1\|_{\vec{q}} \int_\varepsilon^1 t^{-\sum_{i=1}^n \frac{1}{q_i} - n\varepsilon} \varphi(t)dt. \end{aligned}$$

Letting $\varepsilon \to 0^+$, and using the fact $\lim_{\varepsilon\to 0^+} \varepsilon^\varepsilon = 1$, we obtain

$$\|H_\varphi^b\|_{L^{\vec{q}}(\mathbb{R}^n)\to L^{\vec{q}}(\mathbb{R}^n)} \geqslant \int_0^1 t^{-\sum_{i=1}^n \frac{1}{q_i}} \varphi(t)dt = C_p.$$

Combining the above two cases, we complete the proof. □

Since the classical central Morrey spaces are particular cases of mixed central Morrey spaces, Theorem 3.1 extends the results in [10, 13] to mixed central Morrey spaces.

4. Boundedness of H_φ^b from $\dot{M}^p_{\vec{k}}(\mathbb{R}^n)$ to $\dot{M}^p_{\vec{s}}(\mathbb{R}^n)$

Now we study the condition on φ such that H_φ^b is bounded from $\dot{M}^p_{\vec{k}}(\mathbb{R}^n)$ to $\dot{M}^p_{\vec{s}}(\mathbb{R}^n)$. Our result can be read as follows.

THEOREM 4.1. *Let* $0 < p < \infty$, $\vec{s} = (s_1,\cdots,s_n)$, $\vec{k} = (k_1,\cdots,k_n)$, $\vec{q} = (q_1,\cdots,q_n)$ *satistify* $1 < \vec{s} < \vec{k} < \infty$, $1 < \vec{q} < \infty$ *and* $\frac{1}{s_i} = \frac{1}{k_i} + \frac{1}{q_i}$, $i = 1,\cdots,n$. *Assume further that* $\sum_{j=1}^n \frac{1}{k_j} > \frac{n}{p}$ *and* φ *is a non-negative integrable function on* $[0,1]$. *Then* H_φ^b *is bounded form* $\dot{M}^p_{\vec{k}}(\mathbb{R}^n)$ *to* $\dot{M}^p_{\vec{s}}(\mathbb{R}^n)$ *for all* $b \in C\dot{M}O^{\vec{q}}(\mathbb{R}^n)$ *if and ony if*

$$C_p^* = \int_0^1 t^{-\frac{n}{p}} \varphi(t) \log\frac{2}{t} dt < \infty.$$

Proof. We first prove the "if" part.

For any cube $Q(0,r)$, $r>0$, one has

$$
\begin{aligned}
&\|H_\varphi^b f\|_{\dot{M}^p_{\vec{s}}(\mathbb{R}^n)}\\
&= \sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n\frac{1}{s_j}\right)}\left\|H_\varphi^b f\cdot\chi_{Q(0,r)}\right\|_{\vec{s}}\\
&\leqslant \sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n\frac{1}{s_j}\right)}\left\|\int_0^1(b(\cdot)-b_{Q(0,r)})f(t\cdot)\varphi(t)dt\cdot\chi_{Q(0,r)}(\cdot)\right\|_{\vec{s}}\\
&\quad+\sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n\frac{1}{s_j}\right)}\left\|\int_0^1(b_{Q(0,r)}-b_{Q(0,tr)})f(t\cdot)\varphi(t)dt\cdot\chi_{Q(0,r)}(\cdot)\right\|_{\vec{s}}\\
&\quad+\sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n\frac{1}{s_j}\right)}\left\|\int_0^1(b_{Q(0,tr)}-b(t\cdot))f(t\cdot)\varphi(t)dt\cdot\chi_{Q(0,r)}(\cdot)\right\|_{\vec{s}}\\
&:= I+II+III.
\end{aligned}
$$

For the first term I, by Hölder's inequality on mixed Lebesgue spaces [1], we get

$$
\begin{aligned}
I &= \sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n\frac{1}{s_j}\right)}\left\|\int_0^1(b(\cdot)-b_{Q(0,r)})f(t\cdot)\varphi(t)dt\cdot\chi_{Q(0,r)}(\cdot)\right\|_{\vec{s}}\\
&= \sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n\frac{1}{s_j}\right)}\left\|H_\varphi^b f(\cdot)(b(\cdot)-b_{Q(0,r)})\chi_{Q(0,r)}(\cdot)\right\|_{\vec{s}}\\
&\leqslant \sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n\frac{1}{s_j}\right)}\left\|H_\varphi^b f(\cdot)\chi_{Q(0,r)}(\cdot)\right\|_{\vec{k}}\cdot\left\|(b(\cdot)-b_{Q(0,r)})\chi_{Q(0,r)}(\cdot)\right\|_{\vec{q}}\\
&\leqslant \sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n\frac{1}{k_j}\right)}\left\|H_\varphi^b f(\cdot)\chi_{Q(0,r)}(\cdot)\right\|_{\vec{k}}|Q(0,r)|^{-\frac{1}{n}\left(\sum_{i=1}^n\frac{1}{q_j}\right)}\\
&\quad\cdot\left\|(b(\cdot)-b_{Q(0,r)})\chi_{Q(0,r)}(\cdot)\right\|_{\vec{q}}\\
&\leqslant \int_0^1 t^{-\frac{n}{p}}\varphi(t)dt\|b\|_{C\dot{M}O^{\vec{q}}(\mathbb{R}^n)}\|f\|_{\dot{M}^p_{\vec{k}}(\mathbb{R}^n)},
\end{aligned}
$$

where in the last inequality, we have used Theorem 3.1.

For the last term III, by using Minkowski's inequalitiy and Hölder's inequality, we obtain

$$
\begin{aligned}
III &= \sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n\frac{1}{s_j}\right)}\left\|\int_0^1(b_{Q(0,tr)}-b(t\cdot))f(t\cdot)\varphi(t)dt\cdot\chi_{Q(0,r)}(\cdot)\right\|_{\vec{s}}\\
&\leqslant \sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n\frac{1}{s_j}\right)}\int_0^1\varphi(t)\left\|(b_{Q(0,tr)}-b(t\cdot))f(t\cdot)\chi_{Q(0,r)}(\cdot)\right\|_{\vec{s}}dt\\
&\leqslant \sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n\frac{1}{s_j}\right)}\int_0^1\varphi(t)\left\|f(t\cdot)\chi_{Q(0,r)}(\cdot)\right\|_{\vec{k}}\\
&\quad\cdot\left\|(b_{Q(0,tr)}-b(t\cdot))\chi_{Q(0,r)}(\cdot)\right\|_{\vec{q}}dt
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \sup_{r>0}\int_0^1 t^{-\frac{n}{p}}\varphi(t)|Q(0,tr)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{i=1}^n \frac{1}{s_j}\right)}\left\|f(\cdot)\chi_{Q(0,tr)}(\cdot)\right\|_{\vec{k}} \\
&\quad \cdot \left\|(b_{Q(0,tr)}-b(\cdot))\chi_{Q(0,tr)}(\cdot)\right\|_{\vec{q}}dt \\
&\leqslant \int_0^1 t^{-\frac{n}{p}}\varphi(t)dt\|b\|_{C\dot{M}O^{\vec{q}}(\mathbb{R}^n)}\|f\|_{\dot{M}^p_{\vec{k}}(\mathbb{R}^n)}.
\end{aligned}
$$

For the term II, using Minkowski's inequality and Hölder's inequality again, one has

$$
\begin{aligned}
II &= \sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{s_j}\right)}\left\|\int_0^1 (b_{Q(0,r)}-b_{Q(0,tr)})f(t\cdot)\varphi(t)dt\cdot\chi_{Q(0,r)}(\cdot)\right\|_{\vec{s}} \\
&\leqslant \sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{s_j}\right)}\int_0^1 \varphi(t)\left|b_{Q(0,r)}-b_{Q(0,tr)}\right|\|f(t\cdot)\chi_{Q(0,r)}(\cdot)\|_{\vec{s}}dt \\
&\leqslant \sup_{r>0}|Q(0,r)|^{\frac{1}{p}-\frac{1}{n}\left(\sum_{j=1}^n \frac{1}{k_j}\right)}\int_0^1 \varphi(t)\left|b_{Q(0,r)}-b_{Q(0,tr)}\right|\|f(t\cdot)\chi_{Q(0,r)}(\cdot)\|_{\vec{k}}dt \\
&\leqslant \|f\|_{\dot{M}^p_{\vec{k}}(\mathbb{R}^n)}\int_0^1 t^{-\frac{n}{p}}\varphi(t)\left|b_{Q(0,r)}-b_{Q(0,tr)}\right|dt
\end{aligned}
$$

For any $0<t<1$, there exists some $l\in\mathbb{N}\backslash\{0\}$, such that $2^{-l}<t\leqslant 2^{-l+1}$. By changing the variables, and using Hölder's inequality, we get

$$
II \leqslant \|f\|_{\dot{M}^p_{\vec{k}}(\mathbb{R}^n)}\int_0^1 t^{-\frac{n}{p}}\varphi(t)\left(\sum_{i=1}^l\left|b_{Q(0,2^{-i}r)}-b_{Q(0,2^{-i+1}r)}\right|+\left|b_{Q(0,2^{-l}r)}-b_{Q(0,tr)}\right|\right)dt.
$$

For any $i\in\mathbb{N}$, from the definition of $C\dot{M}O^{\vec{q}}(\mathbb{R}^n)$, we have

$$
\begin{aligned}
&\left|b_{Q(0,2^{-i}r)}-b_{Q(0,2^{-i+1}r)}\right| \\
&= \frac{1}{|Q(0,2^{-i}r)|}\left|\int_{Q(0,2^{-i}r)} b(x)-b_{Q(0,2^{-i+1}r)}dx\right| \\
&\lesssim \frac{1}{|Q(0,2^{-i+1}r)|}\int_{Q(0,2^{-i+1}r)}\left|b(x)-b_{Q(0,2^{-i+1}r)}\right|dx \\
&\leqslant \frac{1}{|Q(0,2^{-i+1}r)|^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}}}\left\|\left(b(\cdot)-b_{Q(0,2^{-i+1}r)}\right)\chi_{Q(0,2^{-i+1}r)}(\cdot)\right\|_{\vec{q}} \\
&\leqslant \|b\|_{C\dot{M}O^{\vec{q}}(\mathbb{R}^n)}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
II &\lesssim \|b\|_{C\dot{M}O^{\vec{q}}(\mathbb{R}^n)}\|f\|_{\dot{M}^p_{\vec{k}}(\mathbb{R}^n)}\sum_{l=1}^\infty\int_{2^{-l}}^{2^{-l+1}} t^{-\frac{n}{p}}\varphi(t)\times(l+1)dt \\
&\lesssim \|b\|_{C\dot{M}O^{\vec{q}}(\mathbb{R}^n)}\|f\|_{\dot{M}^p_{\vec{k}}(\mathbb{R}^n)}\sum_{l=1}^\infty\int_{2^{-l}}^{2^{-l+1}} t^{-\frac{n}{p}}\varphi(t)\times(\log 2^l+1)dt
\end{aligned}
$$

$$\begin{aligned}
&\lesssim \|b\|_{C\dot{M}O^{\vec{q}}(\mathbb{R}^n)}\|f\|_{\dot{M}^p_{\vec{k}}(\mathbb{R}^n)}\sum_{l=1}^{\infty}\int_{2^{-l}}^{2^{-l+1}} t^{-\frac{n}{p}}\varphi(t)\times(\log\frac{1}{t}+1)dt\\
&\lesssim \|b\|_{C\dot{M}O^{\vec{q}}(\mathbb{R}^n)}\|f\|_{\dot{M}^p_{\vec{k}}(\mathbb{R}^n)}\int_0^1 t^{-\frac{n}{p}}\varphi(t)\times\Big(\log\frac{1}{t}+1\Big)dt\\
&\lesssim \|b\|_{C\dot{M}O^{\vec{q}}(\mathbb{R}^n)}\|f\|_{\dot{M}^p_{\vec{k}}(\mathbb{R}^n)}\int_0^1 t^{-\frac{n}{p}}\varphi(t)\times\log\frac{2}{t}dt.
\end{aligned}$$

Combining all the estimates of I, II, III, it yields

$$\|H^b_\varphi f\|_{\dot{M}^p_{\vec{s}}(\mathbb{R}^n)} \lesssim C^*_p\|b\|_{C\dot{M}O^{\vec{q}}(\mathbb{R}^n)}\|f\|_{\dot{M}^p_{\vec{k}}(\mathbb{R}^n)}$$

Next we show the "only if" part.

We take $f_0(x_1,\cdots,x_n)=\prod_{j=1}^n|x_j|^{-\frac{1}{p_j}}$, in which $0<k_j<p_j$ and $\sum_{j=1}^n\frac{1}{p_j}=\frac{n}{p}$ as in Theorem 3.1. Note that $f_0\in\dot{M}^p_{\vec{k}}(\mathbb{R}^n)$ has been proved in the proof of Theorem 3.1. By using the same method, it is not hard to verify that $f_0\in\dot{M}^p_{\vec{s}}(\mathbb{R}^n)$ since $1<\vec{s}<\vec{k}<\infty$. Take $b_0(x)=\log|x|$, $x\in\mathbb{R}^n$. Obviously, b_0 is in $\mathrm{BMO}(\mathbb{R}^n)$, see [18]. From (2.4) and (2.5), we have $b_0\in C\dot{M}O^{\vec{q}}(\mathbb{R}^n)$.

From a routine computation, there holds

$$\begin{aligned}
H^{b_0}_\varphi f_0(x_1,\cdots,x_n) &= \prod_{j=1}^n|x_j|^{-\frac{1}{p_j}}\int_0^1 t^{-\frac{n}{p}}\varphi(t)\log\frac{1}{t}dt\\
&= f_0(x_1,\cdots,x_n)\int_0^1 t^{-\frac{n}{p}}\varphi(t)\log\frac{1}{t}dt.
\end{aligned}$$

Using the boundedness of $H^{b_0}_\varphi$ form $\dot{M}^p_{\vec{k}}(\mathbb{R}^n)$ to $\dot{M}^p_{\vec{s}}(\mathbb{R}^n)$, and the fact $f_0\in\dot{M}^p_{\vec{s}}(\mathbb{R}^n)$, we obtain

$$\frac{\|f_0\|_{\dot{M}^p_{\vec{s}}(\mathbb{R}^n)}}{\|f_0\|_{\dot{M}^p_{\vec{k}}(\mathbb{R}^n)}}\times\int_0^1 t^{-\frac{n}{p}}\varphi(t)\log\frac{1}{t}dt<\infty.$$

That is to say,

$$\int_0^1 t^{-\frac{n}{p}}\varphi(t)\log\frac{1}{t}dt<\infty. \tag{4.9}$$

From (4.9), we can also get

$$\int_0^{\frac{1}{2}} t^{-\frac{n}{p}}\varphi(t)dt\lesssim\int_0^{\frac{1}{2}} t^{-\frac{n}{p}}\varphi(t)\log\frac{1}{t}dt<\infty. \tag{4.10}$$

On the other hand, since φ is integrable on $[\frac{1}{2},1]$ and $p>0$, we know that

$$\int_{\frac{1}{2}}^1 t^{-\frac{n}{p}}\varphi(t)dt<\infty. \tag{4.11}$$

In view of (4.10) and (4.11), we obtain

$$\int_0^1 t^{-\frac{n}{p}}\varphi(t)dt<\infty. \tag{4.12}$$

Combining (4.9) and (4.12),we get the desired result $C_p^* < \infty$.

Therefore we complete the proof. □

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