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THRESHOLD DYNAMICS BEHAVIORS OF A STOCHASTIC SIRS EPIDEMIC MODEL WITH A PARAMETER FUNCTIONAL VALUE

JIANGUO SUN* AND MIAOMIAO GAO

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Abstract. This article pays the main attention to the notions of a spreading threshold dynamical model for a stochastic SIRS with environmental noise. A unique positive solution of the stochastic model is proved to be existed in this article. Furthermore, by appropriate Lyapunov functions, the ergodic stationary distribution is introduced. The conditions of extinction or permanence of the SIRS epidemic model are also considered in this arcticle.

1. Introduction

For the reason that the health of human beings are threaten seriously by infectious diseases, how to control of the infectious diseases is one of the most important research topics in the study of the epidemic models in mathematical biology. The main diseases can be modeled as SIR, SIRS, or SIS models [1–8]. Hethcode H.W. [8] considered the deterministic SIRS epidemic model by the following system:

$$\begin{cases} dS(t) = (\mu - \mu S(t) - \beta S(t)I(t) + \gamma R(t))dt, \\ dI(t) = (-(\mu+\lambda)I(t) + \beta S(t)I(t))dt, \\ dR(t) = (-(\mu+\gamma)R(t) + \lambda I(t))dt, \end{cases} \tag{1.1}$$

where $S(t), I(t), R(t)$ stand for the population fractions of susceptible, the infective, and the removed at time t, respectively. The positive constant μ stands for the death rates, β stands for the infection coefficient, λ stands for the recovery rate, and γ stands for the lost immunity rat. The author [8] showed that the system (1.1) has a unique globally asymptotically stable disease-free equilibrium state.

Actually, in real life, the epidemic model has a lot of randomness, which is affected by environmental noise [1, 4, 6, 10–23]. Compared to deterministic models, stochastic models are closer to reality. Lahrouz A. et al. [4] gave the conditions of extinction and persistence of stochastic SIRS epidemic model; Herwaarden et al. [6] put their theory to the test that an endemic equilibrium can disappear by stochastic fluctuations.

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* Corresponding author.

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The stochastic SIRS epidemic model with a double epidemic asymmetry assumption is studied by Chang et al. [17]. About the system (1.1), Tornatore et. al. [5] obtained the stability of the disease-free equilibrium state E_0 of the following improved stochastic SIRS model:

$$\begin{cases} dS(t) = (\mu - \mu S(t) - \beta S(t)I(t) + \gamma R(t))dt - \sigma S(t)I(t)dB(t), \\ dI(t) = (-(\mu+\lambda)I(t) + \beta S(t)I(t))dt + \sigma S(t)I(t)dB(t), \\ dR(t) = (-(\mu+\gamma)R(t) + \lambda I(t))dt. \end{cases} \tag{1.2}$$

Meanwhile, A. Lahrouz etc. [4] considered when the system (1.2) is extinct and persist.

Motivated by the above facts, in this article, we consider the ergodic stationary distribution of the stochastic SIRS epidemic system (1.2), influenced with $dS(t)$, $dI(t)$, and $dR(t)$ as

$$\begin{cases} dS(t) = (\mu - \mu S(t) - \beta S(t)I(t) + \gamma R(t))dt + \sigma_1 S(t)dB_1(t), \\ dI(t) = (-(\mu+\lambda)I(t) + \beta S(t)I(t))dt + \sigma_2 I(t)dB_2(t), \\ dR(t) = (-(\mu+\gamma)R(t) + \lambda I(t))dt + \sigma_3 R(t)dB_3(t), \end{cases} \tag{1.3}$$

where $B_i(t)$ stand for standard Brownian motions and $B_i(0) = 0$, $\sigma_i^2 > 0$ are the environmental noise, $i = 1,2,3$.

We organize the present manuscript as follows. In the second section, we mainly give some basic concepts and conclusions. In the third section, we gave the uniqueness properties of the positive solution in the system (1.3). We demonstrate the extinction and persistence of the system (1.3) in the fourth section. Meanwhile, the existence and uniqueness properties of an ergodic stationary distribution of the system (1.3) are obtained in the fifth section. The main theoretical results are illuminated by an example and many kinds of numerical simulations in the sixth section. Finally, we give the conclusion and the assumption that we can continue the research work in the future in the last section.

2. Preliminaries

We define the general d-dimensional stochastic differential equation

$$dx(t) = f(x(t),t)dt + g(x(t),t)dB(t), \quad \text{for } t \geqslant t_0 \tag{2.1}$$

with initial value $x(t_0) = x_0 \in \mathbb{R}^n$, where $B(t)$ is d-dimensional standard Brownian motion. A differential operator L is defined in the system (2.1) as Mao [9]:

$$L = \frac{\partial}{\partial t} + \Sigma f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\Sigma[g^T(x,t)g(x,t)]_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

If L acts on a function $V \in C^{2,1}(\mathbb{R}^n \times \bar{\mathbb{R}}_+; \bar{\mathbb{R}}_+)$, then

$$LV(x,t) = V_t(x,t) + V_x(x,t) + \frac{1}{2}\operatorname{trac}[g^T(x,t)V_{xx}(x,t)g(x,t)],$$

where $V_t = \frac{\partial V}{\partial t}, V_x = (\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_d})$ and $V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{d\times d}$.

Let $x(t)$ be a homogeneous Markov process in $\mathbb{R}^d$ described as,

$$dV(x(t),t) = LV(x(t),t)dt + V_x(x(t),t)g(x(t),t)dB(t).$$

We can obtain the diffusion matrix as follows

$$A(x) = (a_{ij}(x)), \quad a_{ij} = \sum_{r=1}^{k} g_r^i(x) g_r^j(x).$$

3. Existence and uniqueness of the global positive solution

The solution of the system (1.3) may explode for its not linear increase. We can obtain the following conclusion,

THEOREM 1. (Main) *For any initial value $(S(0),I(0),R(0)) \in \mathbb{R}_+^3$, there is a unique positive solution $(S(t),I(t),R(t)) \in \mathbb{R}_+^3$ of the system (1.3) on $t \geqslant 0$, and the solution will remain in $\mathbb{R}_+^3$ with probability one.*

Proof. There is a explosion time τ_0 with the coefficients locally Lipschitz continuous. Let n_0 be an arbitrarily large positive number lying in $[\frac{1}{n_0}, n_0]$, for any $n \geqslant n_0$, the stopping time is defined by

$$\tau_n = \inf\{t \in [0,\tau_0) : \min\{S(t),I(t),R(t)\} \leqslant \tfrac{1}{n} \text{ or } \max\{S(t),I(t),R(t)\} \geqslant n\}. \quad (3.1)$$

Obviously, τ_n is increasing as $n \to \infty$. $\tau_\infty = \lim_{n\to\infty} \tau_n$, $\tau_\infty \leqslant \tau_0$ a.s. if $\tau_\infty = \infty$ a.s. and $\tau_0 = \infty$. Here, we verify that $\tau_\infty = \infty$ a.s. for all $(S(t),I(t),R(t)) \in \mathbb{R}_+^3$ a.s. $t \geqslant 0$. If this assertion is false, there are two constants $T \geqslant 0$ and $\varepsilon \in (0,1)$, such that

$$P\{\tau_\infty \leqslant T\} \geqslant \varepsilon,$$

and there is an integer $n_1 \geqslant n_0$ such that

$$P\{\tau_n \leqslant T\} \geqslant \varepsilon \text{ for all } n \geqslant n_1.$$

We define a fundamental C^2-function $\tilde{V} : \mathbb{R}_+^3 \to \bar{\mathbb{R}}_+$, which is

$$\tilde{V} = (S(t) - a - a\ln\frac{S(t)}{a}) + (I(t) - 1 - \ln I(t)) + (R(t) - 1 - \ln R(t)), \quad (3.2)$$

where a is a positive constant, which will be determined in the following text. The non-negativity of the function $\tilde{V}(S(t),I(t),R(t))$ can be seen from $x - 1 - \ln x \geqslant 0$ for any $x > 0$.

Applying Itô's formula [9], we obtain

$$d\tilde{V}(S,I,R) = L\tilde{V}dt + \sigma_1 a(S(t) - a)dB_1(t) + \sigma_2(I(t) - 1)dB_2(t) + \sigma_3(R(t) - 1)dB_3(t), \quad (3.3)$$

where

$$\begin{aligned} L\tilde{V}(S,I,R) &= (1-\tfrac{a}{S})dS + (1-\tfrac{1}{I})dI + (1-\tfrac{1}{R})dR + \tfrac{1}{2}(a\sigma_1^2+\sigma_2^2+\sigma_3^2 \\ &= \mu - \mu S - \beta SI + \gamma R - \tfrac{a\mu}{S} + \mu a + a\beta I - \tfrac{a\gamma R}{S} \\ &\quad -(\mu+\lambda)I + \beta SI + (\mu+\lambda) - \beta S \\ &\quad -(\mu+\gamma)R + \lambda I + (\mu+\gamma) - \tfrac{\lambda I}{R} + \tfrac{1}{2}(a\sigma_1^2+\sigma_2^2+\sigma_3^2) \\ &\leqslant 3\mu + a\mu + \lambda + \gamma + (a\beta-\mu)I + \tfrac{1}{2}(a\sigma_1^2+\sigma_2^2+\sigma_3^2). \end{aligned} \tag{3.4}$$

Choosing $a = \frac{\mu}{\beta}$, such that $a\beta - \mu = 0$, then,

$$L\tilde{V}(S,I,R) \leqslant 3\mu + \frac{\mu^2}{\beta} + \lambda + \gamma + \frac{1}{2}((\mu/\beta)\sigma_1^2+\sigma_2^2+\sigma_3^2) := K, \tag{3.5}$$

where K is a positive constant. The remainder of the proof is similar to Theorem 3.1 in Mao. [21]. Hence, we omit it here. □

4. Extinction and persistence of the system (1.3)

Define a parameter constant value

$$\widehat{R}_0^s = \frac{\beta\mu}{(\mu+\sigma_1^2)(\mu+\lambda+\frac{\sigma_2^2}{2})}.$$

According to the results in [21], we can obtain the following lemma.

LEMMA 1. *For any initial value, the solution of stochastic model satisfies*

$$\lim_{t\to\infty}\frac{\ln S(t)}{t} \leqslant 0, \quad \lim_{t\to\infty}\frac{\ln I(t)}{t} \leqslant 0, \quad \lim_{t\to\infty}\frac{\ln R(t)}{t} \leqslant 0 \quad a.s. \tag{4.1}$$

$$\lim_{t\to\infty}\frac{S(t)+I(t)+R(t)}{t} = 0, \quad a.s. \tag{4.2}$$

Moreover, if $\mu > \frac{\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2}{2}$*, we obtain*

$$\begin{gathered} \lim_{t\to 0}\frac{1}{t}\int_0^t S(m)dB_1(m) = 0, \qquad \lim_{t\to 0}\frac{1}{t}\int_0^t I(m)dB_2(m) = 0, \\ \lim_{t\to 0}\frac{1}{t}\int_0^t R(m)dB_3(m) = 0 \quad a.s. \end{gathered} \tag{4.3}$$

THEOREM 2. (Main) *Let* $(S(t),I(t),R(t))$ *be the solution of the system (1.3) with any initial value* $(S(0),I(0),R(0)) \in \mathbb{R}_+^3$.

(1) If $\widehat{R}_0^s < 1$*, then the solution* $(S(t), I(t), R(t))$ *of the system (1.3) satisfies*

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \leqslant \Big(\mu+\lambda+\frac{\sigma_2^2}{2}\Big)(\widehat{R}_0^s - 1) < 0 \ \ a.s.$$

Namely, the disease will be eradicated in a long term.

(2) If $\widehat{R}_0^s > 1$*, then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t I(s)ds \geqslant \frac{(\mu+\lambda+\frac{\sigma_2^2}{2})(\widehat{R}_0^s - 1)}{K_2} > 0 \ \ a.s.,$$

where $K_2 = \frac{\mu\beta^2}{(\mu+\frac{\sigma_1^2}{2})^2} > 0$*, which implies the disease will persist in a long term.*

Proof. (1). Consider the following auxiliary logistic equation with random perturbation

$$d\ln I(t) = \Big(-\Big(\mu+\lambda+\frac{\sigma_1^2}{2}\Big)+\beta S\Big)dt + \sigma_2 dB_2(t).$$

Integrating above formula from 0 to t on both sides, then

$$\ln I(t) - \ln I(0) = \int_0^t \Big[-\Big(\mu+\lambda+\frac{\sigma_1^2}{2}\Big)+\beta S(m)\Big]dm + \sigma_2\int_0^t dB_2(m).$$

According to the strong law of large numbers [18], we have

$$\lim_{t\to 0}\frac{1}{t}\int_0^t dB_2(m) = 0 \quad \text{a.s.},$$

$$\begin{aligned} d(S(t)+I(t)+R(t)) = {} & [\mu - \mu(S(t)+I(t)+R(t))] + \sigma_1 S(t)dB_1(t) \\ & + \sigma_2 I(t)dB_2(t) + \sigma_3 R(t)dB_3(t). \end{aligned} \tag{4.4}$$

On the other hand,

$$\langle f\rangle = \frac{1}{t}\int_0^t f(m)dm,$$

using (1), we can obtain

$$\begin{aligned} & \frac{S(t)-S(0)}{t} + \frac{I(t)-I(0)}{t} + \frac{R(t)-R(0)}{t} \\ & = \mu - \mu(\langle S\rangle + \langle I\rangle + \langle R\rangle) + \frac{\sigma_1\int_0^t S(m)dB_1(m)}{t} \\ & \quad + \frac{\sigma_2\int_0^t I(m)dB_2(m)}{t} + \frac{\sigma_3\int_0^t R(m)dB_3(m)}{t} \\ & = \mu - \mu(\langle S\rangle + \langle I\rangle + \langle R\rangle) \\ & \leqslant \mu - \mu\langle S\rangle. \end{aligned} \tag{4.5}$$

Hence, we have $\lim_{t\to\infty}\langle S\rangle \leqslant 1$, $\lim_{t\to\infty}\langle I\rangle \leqslant 1, \lim_{t\to\infty}\langle R\rangle \leqslant 1$.

Together with the Equation (4.5), for σ_1 is a very small positive number, we know

$$\lim_{t\to\infty}\langle S\rangle \leqslant \frac{\mu}{\mu+\frac{\sigma_1^2}{2}}. \tag{4.6}$$

Taking the superior limit and using stochastic comparison theorem, combining Equations (1) and (4.6), we obtain

$$\begin{aligned}
\limsup_{t\to\infty}\frac{\ln I(t)}{t} &= \limsup_{t\to\infty}\frac{1}{t}\int_0^t[-(\mu+\lambda+\tfrac{\sigma_1^2}{2})+\beta S]dm \\
&= -(\mu+\lambda+\tfrac{\sigma_1^2}{2})+\limsup_{t\to\infty}\frac{1}{t}\int_0^t\beta S dm \\
&\leqslant -(\mu+\lambda+\tfrac{\sigma_1^2}{2})+\beta\frac{\mu}{\mu+\frac{\sigma_1^2}{2}} \\
&= (\mu+\lambda+\tfrac{\sigma_1^2}{2})(\widehat{R}_0^s-1) \\
&< 0 \quad \text{a.s.}
\end{aligned} \tag{4.7}$$

Therefore, it indicates that

$$\lim_{t\to\infty} I(t) = 0 \quad \text{a.s.}$$

Consequently, it means that the disease will be eradicated in a long time.

(2). Define a C^2-function V_1 as

$$V_1(S,I) = -\ln I - c_1 \ln S.$$

Applying Itô's formula [9], we obtain

$$\begin{aligned}
LV_1(S,I) &= (\mu+\lambda)-\beta S+c_1(-\tfrac{\mu}{S}+\mu+\beta I-\gamma R)+\tfrac{1}{2}(c_1\sigma_1^2+\sigma_2^2) \\
&\leqslant -\beta S - c_1\tfrac{\mu}{S}+c_1(\mu+\tfrac{1}{2}\sigma_1^2)+(\mu+\lambda)+\sigma_2^2+c_1\beta I \\
&\leqslant -2\sqrt{c_1\beta\mu}+c_1(\mu+\tfrac{1}{2}\sigma_1^2)+(\mu+\lambda)+\sigma_2^2+c_1\beta I.
\end{aligned} \tag{4.8}$$

Supposing $f(c_1) = -2\sqrt{c_1\beta\mu}+c_1(\mu+\frac{1}{2}\sigma_1^2)$ and $f'(c_1)=0$, choosing $c_1 = \frac{\beta\mu}{(\mu+\frac{\sigma_1^2}{2})^2}$ such that

$$-2\sqrt{c_1\beta\mu}+c_1\left(\mu+\frac{1}{2}\sigma_1^2\right) = -\frac{\beta\mu}{\mu+\frac{\sigma_1^2}{2}}.$$

Then,

$$\begin{aligned}
LV_1(S,I) &\leqslant -\frac{\beta\mu}{\mu+\frac{\sigma_1^2}{2}}+(\mu+\lambda)+\sigma_2^2+c_1\beta I \\
&\leqslant -(\mu+\lambda+\sigma_2^2)\left(\frac{\beta\mu}{(\mu+\frac{1}{2}\sigma_1^2)(\mu+\lambda+\sigma_2^2)}-1\right)+K_2 I,
\end{aligned} \tag{4.9}$$

where $\widehat{R}_0^s = \frac{\beta\mu}{(\mu+\frac{1}{2}\sigma_1^2)(\mu+\lambda+\sigma_2^2)}$.

Let $K_2 = \frac{\mu\beta^2}{(\mu+\frac{\sigma_1^2}{2})^2}$. Consequently,

$$dV_1(S,I) = LV_1 dt - \sigma_2 dB_2(t) - c_1\sigma_1 dB_1(t). \tag{4.10}$$

Integrating both sides of Equation (4.10), we have

$$\begin{aligned}&\frac{V_1(S(t),I(t),R(t)) - V_1(S(0),I(0),R(0))}{t}\\ &\leqslant -(\mu+\lambda+\sigma_2^2)(\widehat{R}_0^s - 1) + K_2\frac{1}{t}\int_0^t I(m)dm\\ &\quad -\frac{1}{t}\int_0^t \sigma_2 dB_2(m) - c_1\frac{1}{t}\int_0^t \sigma_1 dB_1(m).\end{aligned} \tag{4.11}$$

By the strong law of large number for martingales again, we also have

$$\lim_{t\to 0}\frac{1}{t}\int_0^t dB_2(m) = 0 \quad \text{a.s.}$$

In view of Lemma 1, we obtain from (4.11)

$$\begin{aligned}&\frac{1}{K_2}\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(m)dm\\ &\geqslant \frac{1}{K_2}(\mu+\lambda+\sigma_2^2)(\widehat{R}_0^s - 1) + \liminf_{t\to\infty}\frac{V_1(S(t),I(t),R(t)) - V_1(S(0),I(0),R(0))}{t}\\ &\geqslant \frac{1}{K_2}(\mu+\lambda+\sigma_2^2)(\widehat{R}_0^s - 1)\\ &> 0 \quad \text{a.s.}\end{aligned} \tag{4.12}$$

Therefore, it implies that the disease will persist when $\widehat{R}_0^s > 1$. □

5. Ergodic stationary distribution of the system (1.3)

LEMMA 2. [23] *The Markov process $X(t)$ has a unique ergodic stationary distribution $\mu(\cdot)$ if there exists a bounded domain $U \subset E_l$ with regular boundary Γ and*

(A.1) there is a positive number M such that $\sum_{i,j=1}^{l} a_{ij}(x)\xi_i\xi_j \geqslant M|\xi|^2$, $x \in U$, $\xi \in R^l$.

(A.2) there exists a non-negative C^2 function V such that LV is negative for any $E_l \backslash U$. Then

$$P_x\Big\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))dt = \int_{E_l} f(x)\mu(dx)\Big\} = 1,$$

for all $x \in E_l$, where $f(\cdot)$ is a function integrable with respect to the measure μ.

THEOREM 3. (Main) *Assuming that $\widehat{R}_0^s > 1$, for the initial values $(S(0),I(0),R(0)) \in \mathbb{R}_+^3$. A stationary distribution $\mu(\cdot)$ of the system (1.3) and the ergodicity are held.*

Proof. *Step* 1: Verify that $(A.1)$ holds. Apparently, the corresponding diffusion matrix of the system (1.3) is given by

$$A=\begin{pmatrix}\sigma_1^2S^2 & 0 & 0\\ 0 & \sigma_2^2I^2 & 0\\ 0 & 0 & \sigma_3^2R^2\end{pmatrix}.$$

Choosing $M=\min_{(S,I,R)\in D_\varepsilon}\{\sigma_1^2S^2,\sigma_2^2I^2,\sigma_3^2R^2\}>0$, we obtain

$$\sum_{i,j=1}^{3}a_{ij}(S,I,R)\xi_i\xi_j=\sigma_1^2S^2\xi_1^2+\sigma_2^2I^2\xi_2^2+\sigma_3^2R^2\xi_3^2\geqslant M\mid\xi\mid^2,$$

for all $(S,I,R)\in D_\varepsilon,\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}_+^3$, which implies condition $(A.1)$ is satisfied.

Step 2: Now we will construct a C^2-function $V:\mathbb{R}_+^3\to\mathbb{R}$ as follows,

$$V(S,I,R)=MV_1-\ln S-\ln R+\frac{1}{\theta+1}(S+R+I)^{\theta+1},$$

where $\theta\in(0,1)$ is a positive constant satisfying

$$\rho:\ \ \mu-\frac{\theta}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)>0.$$

There exists a positive constant M satisfying the following condition

$$f_1^{\mu}-M(\mu+\lambda+\sigma_2^2)(\widehat{R}_0^s-1)\leqslant-2,$$

where $f_i^{\mu}=\sup_{(S,I,R)\in\mathbb{R}_+^3}(f_i)$. It is easy to check that

$$\lim_{k\to\infty,(S,I,R)\in\mathbb{R}_+^3\setminus U_k}V(S,I,R)=+\infty,$$

where $U_k=(\frac{1}{k},k)\times(\frac{1}{k},k)\times(\frac{1}{k},k)$. Moreover, $V(S,I,R)$ is a continuous function and have a minimum point (S_0,I_0,R_0) in the interior of $\mathbb{R}_+^3$. Then we define a nonnegative C^2-function $V_s:\mathbb{R}_+^3\to\overline{\mathbb{R}}_+$ as follows,

$$V_s(S,I,R)=V(S,I,R)-V(S_0,I_0,R_0).$$

Applying Itô's formula to the function $V(S,I,R)$. Denote

$$V_2=-\ln S(t),\quad V_3=-\ln R(t),\quad V_4=\frac{1}{\theta+1}(S+R+I)^{\theta+1}.$$

We can act the differential operator L on the above functions, respectively

$$\begin{aligned}LV_2&=\mu-\tfrac{\mu}{S}+\beta I-\tfrac{\gamma R}{S}+\tfrac{1}{2}\sigma_1^2\\ &\leqslant\mu-\tfrac{\mu}{S}+\beta I+\tfrac{1}{2}\sigma_1^2;\end{aligned}\tag{5.1}$$

$$LV_3 = (\mu+\gamma) - \frac{\lambda I}{R} + \frac{1}{2}\sigma_3^2; \tag{5.2}$$

$$\begin{aligned} LV_4 &= (S+R+I)^{\theta}\mu[1-(S+R+I)] + \tfrac{\theta}{2}(S+R+I)^{\theta-1}(\sigma_1^2 S^2 + \sigma_2^2 I^2 + \sigma_3^2 R^2) \\ &\leqslant \mu(S+R+I)^{\theta} - \mu(S+R+I)^{\theta+1} + \tfrac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)(S+R+I)^{\theta+1} \\ &= \mu(S+R+I)^{\theta} - [\mu - \tfrac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)](S+R+I)^{\theta+1} \\ &\leqslant A - \tfrac{\rho}{2}(S+I+R)^{\theta+1} \\ &\leqslant A - \tfrac{\rho}{2}(S^{\theta+1} + I^{\theta+1} + R^{\theta+1}), \end{aligned} \tag{5.3}$$

where

$$A = \mu(S+R+I)^{\theta} - \frac{\rho}{2}(S+R+I)^{\theta+1}.$$

From the above analysis, we have

$$\begin{aligned} LV_s(S(t),I(t),R(t)) \leqslant\ & M[-(\mu+\lambda+\sigma_2^2)(\widehat{R}_0^s - 1) + c_1\beta I] \\ & + \mu - \tfrac{\mu}{S} + \beta I + \tfrac{1}{2}\sigma_1^2 \\ & + (\mu+\gamma) - \tfrac{\lambda I}{R} + \tfrac{1}{2}\sigma_3^2 + A - \tfrac{\rho}{2}(S^{\theta+1} + I^{\theta+1} + R^{\theta+1}). \end{aligned} \tag{5.4}$$

Define

$$f_1(S) = 3\mu + \frac{1}{2}\sigma_1^2 + \gamma + \frac{1}{2}\sigma_3^2 + A - \frac{\mu}{S} - \frac{\rho}{2}S^{\theta+1},$$

$$f_2(I) = M[-(\mu+\lambda+\sigma_2^2)(\widehat{R}_0^s - 1) + c_1\beta I] + \beta I - \frac{\rho}{2}I^{\theta+1},$$

$$f_3(R) = -\frac{\lambda I}{R} - \frac{\rho}{2}R^{\theta+1}.$$

Then, a bounded closed set is defined as

$$D_{\varepsilon} = \left\{(S,I,R) \in \mathbb{R}_+^3 : \varepsilon < S < \frac{1}{\varepsilon}, \varepsilon < I < \frac{1}{\varepsilon}, \varepsilon < R < \frac{1}{\varepsilon}\right\},$$

where $\varepsilon > 0$ is a arbitrarily small number. For the set $\mathbb{R}_+^3 \setminus D_{\varepsilon}$,

$$D_1 = \{(S,I,R) \in \mathbb{R}_+^3 : 0 < S < \varepsilon\}; \quad D_2 = \left\{(S,I,R) \in \mathbb{R}_+^3 : S > \frac{1}{\varepsilon}\right\};$$

$$D_3 = \{(S,I,R) \in \mathbb{R}_+^3 : 0 < I < \varepsilon\}; \quad D_4 = \left\{(S,I,R) \in \mathbb{R}_+^3 : I > \frac{1}{\varepsilon}\right\};$$

$$D_5 = \left\{(S,I,R) \in \mathbb{R}_+^3 : 0 < R < \varepsilon^2, I > \varepsilon\right\}; \quad D_6 = \left\{(S,I,R) \in \mathbb{R}_+^3 : R > \frac{1}{\varepsilon^2}\right\}.$$

$LV(S,I,R) \leqslant -1$ on $\mathbb{R}_+^3 \setminus D_{\varepsilon}$ is used to prove the result on the six domains, respectively.

Case 1. If $(S,I,R) \in D_1$ or $(S,I,R) \in D_2$, one can see that

$$f_1(S) + f_2(I) + f_3(R) \leqslant f(S) + f_2^{\mu}(I) \to -\infty;$$

Case 2. If $(S,I,R) \in D_3$, then

$$f_1(S)+f_2(I)+f_3(R) \leqslant f_1^{\mu}+f_2(I) \to f_1^{\mu}+M[-(\mu+\lambda+\sigma_2^2)(\widehat{R}_0^s-1)] \leqslant -2;$$

Case 3. If $(S,I,R) \in D_4$, then

$$f_1(S)+f_2(I)+f_3(R) \leqslant f_1^{\mu}+f_2(I) \to -\infty;$$

Case 4. If $(S,I,R) \in D_5$ or $(S,I,R) \in D_6$, then

$$f_1(S)+f_2(I)+f_3(R) \leqslant f_1^{\mu}+f_2^{\mu}+f_3(R) \to -\infty;$$

Therefore, for all $(S,I,R) \in \mathbb{R}_+^3 \setminus D_\varepsilon$, $V(S,I,R) \leqslant -1$, which indicates assumption $(A.2)$ holds. $\square$

6. Concluding remarks and future directions

Given the suitable stochastic Lyapunov functions, we obtain the dynamics behavior of a stochastic SIRS model. The local asymptotic stability of an ergodic stationary distribution and the extinction or persistence of the system (1.3) are also obtained in this paper.

In the further research, we want to consider the other case of $\mu < \frac{\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2}{2}$. On the other hand, in our following work, we will continue to put our main attention to some more useful models, such as the impulsive perturbations on the system (1.3).

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REFERENCES

[1] B. BUONOMO, A. D'ONOFRIO, D. LACITIGNOLA, *Global stability of an SIR epidemic model with information dependent vaccination*, Mathematical Biosciences **216**, 2 (2008), 9–16.
[2] A. KOROBEINIKOV, *Lyapunov functions and global stability for SIR and SIRS epidemic logical models with nonlinear transmission*, Bulletin of Mathematical Biology **68**, 3 (2006), 615–626.
[3] A. KOROBEINIKOV, G. C. WAKE, *Lyapunov functions and global stability for SIR, SIRS, and SIS epidemiological models*, Applied Mathematics Letters **15**, 8 (2002), 955–960.
[4] A. LAHROUZ, A. SETTATI, *Necessary and sufficient condition for extinction and persistence of SIRS system with random perturbation*, Applied Mathematics and Computation **233**, 3 (2014), 10–19.
[5] E. TORNATORE, S. M. BUCCELLATO, P. VETRO, *Stability of a stochastic SIR system*, Physica A **354**, 15 (2005), 111–126.
[6] O. A. HERWAARDEN, J. GRASMAN, *Stochastic epidemics: major outbreaks and the duration of the endemic period*, J. Math. Biol. **33**, 6 (1995), 581–601.
[7] Y. ZHAO, D. JIANG, D. O'REGAN, *Donal The extinction and persistence of the stochastic SIS epidemic model with vaccination*, Phys. A **392**, 20 (2013), 4916–4927.
[8] H. W. HETHCOTE, *Qualitative analyses of communicable disease models*, Math. Biosci. **28**, 3–4 (1976), 335–356.
[9] X. MAO (Eds), *Stochastic Differential Equations and Applications*, Horwood, Chicheste.
[10] L. J. S. ALLEN, *An introduction to stochastic epidemic models*, Mathematical Epidemiology **683**, 1 (2008), 81–130.
[11] C. G. THOMAS (Eds), *Introduction to Stochastic Differential Equations*, Dekkek, New York.

[12] J. PAN, A. GRAY, D. GREENHALGH, L. HU, X. MAO, *Parameter estimation for the stochastic SIS epidemic model*, Statistical Inference for Stochastic Processes **17**, 1 (2014), 75–98.
[13] Q. YANG, D. JIANG, N. SHI, C. JI, *The ergodicity and extinction of stochastically perturbed SIR and SEIR epidemic models with saturated incidence*, J. Math. Anal. Appl. **388**, 1 (2012), 248–271.
[14] M. M. GAO, D. JIANG, T. HAYAT, A. ALSAEDI, *Threshold behavior of a stochastic Lotka-Volterra food chain chemostat model with jumps*, Phys. A **523**, 1 (2016), 191–203.
[15] Q. LIU, D. JIANG, *The dynamics of a stochastic vaccinated tuberculosis model with treatment*, Phys. A **527**, 12 (2019), 1121274.
[16] K. WANG, Z. TENG, X. ZHANG, *Dynamical behaviors of an Echinococcosis epidemic model with distributed delays*, Math. Biosci. Eng. **14**, 5–6 (2017), 1425–1445.
[17] Z. CHANG, X. MENG, X. LU, *Analysis of a novel stochastic SIRS epidemic model with two different saturated incidence rates*, Phys. A **472**, 1 (2017), 103–112.
[18] R. LIPSTER, *A strong law of large numbers for local martingales*, Stochastics **3**, 3 (1980), 217–228.
[19] Q. LIU, D. JIANG, T. HAYAT, A. ALSAEDI, *Threshold dynamics of a stochastic SIS epidemic model with nonlinear incidence rate*, Phys. A **526**, 1 (2019), 120946.
[20] Q. LIU, D. JIANG, T. HAYAT, A. ALSAEDI, *Stationary distribution and extinction of a stochastic SIRI epidemic model with relapse*, Stochastic Analysis Applications **36**, 1 (2018), 135–151.
[21] X. MAO, G. MARION, E. RENSHAW, *Environmental Brownian noise suppresses explosions in population dynamics*, Stochastic Process. Appl. **97**, 1 (2002), 95–110.
[22] I. OWUSU-MENSAH, L. A. BISMARKODURO, O. S. IYIOLA, *A fractional order approach to modeling and simulations of the novel COVID-19*, Advances in Difference Equations **683**, 1 (2020), 1–21.
[23] R. HAS'MINISKII (Eds), *Stochastic stability of differential equations*, Hor, Chichester.

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Jianguo Sun
School of Science
China University of Petroleum (east China)
266580 Qingdao, P. R. China
e-mail: sunjg616@upc.edu.cn

Miaomiao Gao
School of Science
China University of Petroleum (east China)
266580 Qingdao, P. R. China
e-mail: gaomm1991@126.com