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A NEW SUBCLASS OF CLOSE–TO–CONVEX HARMONIC MAPPINGS

XIAO-YUAN WANG, ZHI-GANG WANG*, JIN-HUA FAN AND ZHEN-YONG HU

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Abstract. In this paper, we introduce and investigate a new subclass of harmonic mappings which satisfy a third-order differential inequality. Such results as close-to-convexity, coefficient bounds, growth estimates, sufficient coefficient condition and convolution properties are derived. Furthermore, we obtain several improved versions of the sharp Bohr radius for harmonic mappings.

1. Introduction

A harmonic mapping in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is a complex-valued function $f = u + iv$ in $\mathbb{D}$, which satisfies the Laplace equation $\Delta f = 4 f_{z\overline{z}} = 0$, where $f_z = (f_x - i f_y)/2$ and $f_{\overline{z}} = (f_x + i f_y)/2$, u and v are real-valued harmonic functions in $\mathbb{D}$. It is convenient to use the canonical representation $f = h + \overline{g}$, where h is the analytic part and g is the co-analytic part of f. The Jacobian J_f of $f = h + \overline{g}$ is given by $J_f = |h'|^2 - |g'|^2$. By Lewy's theorem (see [15]), f is locally univalent and sense-preserving in $\mathbb{D}$, if and only if $J_f(z) > 0$, or equivalently, if $h' \neq 0$ and the dilation $\omega_f = g'/h'$ has the property $|\omega_f| < 1$ in $\mathbb{D}$.

Let $\mathscr{H}$ denote the class of complex valued harmonic functions f defined in $\mathbb{D}$, and normalized by $f(0) = f_z(0) - 1 = 0$. Also, let $\mathscr{H}^0 = \{f \in \mathscr{H} : f_{\overline{z}}(0) = 0\}$. Each function $f \in \mathscr{H}^0$ can be expressed as $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1.1}$$

are analytic in $\mathbb{D}$.

Denote by $\mathscr{S}_{\mathscr{H}}$ the class of functions $f = h + \overline{g}$ that are harmonic, univalent, and sense-preserving in $\mathbb{D}$. Furthermore, let $\mathscr{S}_{\mathscr{H}}^0 = \{f \in \mathscr{S}_{\mathscr{H}} : f_{\overline{z}}(0) = 0\}$. Note that, when $g(z) = 0$, the classical family $\mathscr{S}$ of analytic univalent and normalized functions in $\mathbb{D}$ is a subclass of $\mathscr{S}_{\mathscr{H}}^0$.

For two analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in $\mathscr{S}$, the Hadamard product (or convolution) of f and g is defined by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$.

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* Corresponding author.

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Goodloe [21] considered the Hadamard product of a harmonic function with an analytic function, which is defined as $f\widehat{*}\phi = h*\phi + \overline{g*\phi}$, where $f = h+\overline{g}$ is harmonic function and ϕ is an analytic function in $\mathbb{D}$.

For two functions f and g analytic in $\mathbb{D}$, f is subordinate to g, written as $f(z) \prec g(z)$, if there is an analytic function w satisfying $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. If g is univalent in $\mathbb{D}$, then $f \prec g$ is equivalent to $f(\mathbb{D}) \subset g(\mathbb{D})$ and $f(0) = g(0)$.

In 2013, Ponnusamy *et al.* [37] introduced the following function class:

$$\mathscr{P}_{\mathscr{H}} = \left\{ f = h+\overline{g} \in \mathscr{H} : \Re\left(h'(z)\right) > \left|g'(z)\right| \ (z \in \mathbb{D}) \right\},$$

and $\mathscr{P}^0_{\mathscr{H}} = \mathscr{P}_{\mathscr{H}} \cap \mathscr{H}^0$, they proved that the functions in the class $\mathscr{P}_{\mathscr{H}}$ are close-to-convex. At the same time, Li and Ponnusamy [29] obtained univalency and convexity of the partial sums of for the class $\mathscr{P}^0_{\mathscr{H}}(\alpha)$ defined by

$$\mathscr{P}^0_{\mathscr{H}}(\alpha) = \{ f = h+\overline{g} \in \mathscr{H} : \Re\left(h'(z) - \alpha\right) > \left|g'(z)\right| \\ \text{with } 0 \leqslant \alpha < 1, g'(0) = 0 \text{ for } z \in \mathbb{D} \}.$$

Later, Nagpal and Ravichandran [36] studied the class

$$\mathscr{W}^0_{\mathscr{H}} = \left\{ f = h+\overline{g} \in \mathscr{H} : \Re\left(h'(z) + zh''(z)\right) > \left|g'(z) + zg''(z)\right| \text{ for } z \in \mathbb{D} \right\},$$

and obtained the coefficient bounds for the functions in the class $\mathscr{W}^0_{\mathscr{H}}$.

In 2019, Ghosh and Vasudevarao [20] studied the class $\mathscr{W}^0_{\mathscr{H}}(\alpha)$, where

$$\mathscr{W}^0_{\mathscr{H}}(\alpha) = \left\{ f = h+\overline{g} \in \mathscr{H}^0 : \Re\left(h'(z) + \alpha zh''(z)\right) > \left|g'(z) + \alpha zg''(z)\right| \ (\alpha \geqslant 0; z \in \mathbb{D}) \right\},$$

and they investigated coefficient bounds, growth estimates, convolution, and radius of convexity for the partial sums of the function class.

In 2020, Rajbala and Prajapat [38] studied the class of $\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta)$ defined by

$$\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta) = \left\{ f = h+\overline{g} \in \mathscr{H}^0 : \Re\left(h'(z) + \alpha zh''(z) - \beta\right) > \left|g'(z) + \alpha zg''(z)\right| \ (z \in \mathbb{D}) \right\},$$

where $\alpha \geqslant 0$ and $0 \leqslant \beta < 1$.

Recently, Yaşar and Yalçın [45] studied the class $\mathscr{R}^0_{\mathscr{H}}(\lambda,\delta)$ of functions $f = h+\overline{g} \in \mathscr{H}^0$ that satisfy

$$\Re\left(h'(z) + \lambda zh''(z) + \delta z^2 h'''(z)\right) > \left|g'(z) + \lambda zg''(z) + \delta z^2 g'''(z)\right|,$$

where $\lambda \geqslant \delta \geqslant 0$.

For more recent results involving harmonic mappings, we refer the reader to [30, 41, 43, 44].

Motivated essentially by the work of Rajbala and Prajapat [38], Yaşar and Yalçın [45], we define the following class of close-to-convex harmonic mappings.

DEFINITION 1.1. For $\alpha \geqslant \gamma \geqslant 0$ and $0 \leqslant \beta < 1$, let $\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ denote the class of harmonic mappings $f = h+\overline{g}$, which is defined by

$$\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma) = \Big\{ f = h+\overline{g} \in \mathscr{H}^0 : \Re\left(h'(z)+\alpha z h''(z)+\gamma z^2 h'''(z)-\beta\right) > \left|g'(z)+\alpha z g''(z)+\gamma z^2 g'''(z)\right| \ (z \in \mathbb{D}) \Big\}. \tag{1.2}$$

We observe that, the class $\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ generalizes several classes of harmonic mappings, such as $\mathscr{W}^0_{\mathscr{H}}(\alpha,0,\gamma) := \mathscr{W}^0_{\mathscr{H}}(\alpha,\gamma) = \mathscr{R}^0_{\mathscr{H}}(\lambda,\delta)$, $\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,0) = \mathscr{W}^0_{\mathscr{H}}(\alpha,\beta)$, $\mathscr{W}^0_{\mathscr{H}}(\alpha,0,0) = \mathscr{W}^0_{\mathscr{H}}(\alpha)$, $\mathscr{W}^0_{\mathscr{H}}(0,0,\beta) = \mathscr{P}^0_{\mathscr{H}}(\alpha)$. By putting $\alpha = \delta/\gamma, \beta = \lambda/\gamma, \gamma = (\alpha-1)/2$, where $0 \leqslant \lambda < \gamma \leqslant \delta$, we get the class $R^0_{\mathscr{H}}(\gamma,\delta,\lambda)$ introduced by Çakmak *et al.* [12].

Let $\mathscr{W}(\alpha,\beta,\gamma)$ denote the class of functions $h \in \mathscr{S}$ such that

$$\Re\left(h'(z)+\alpha z h''(z)+\gamma z^2 h'''(z)\right) > \beta \quad (\alpha \geqslant \gamma \geqslant 0; 0 \leqslant \beta < 1).$$

The class $\mathscr{W}(\alpha,\beta,\gamma)$ was considered by Ali *et al.* [9]. we note that the class $\mathscr{W}(\alpha,\beta,0)$ introduced by Gao and Zhou [19] for $\beta < 1$ and $\alpha > 0$. The class $\mathscr{W}(\alpha,0,0)$ studied by Chichra [13] and $\mathscr{W}(1,0,0)$ are starlike in $\mathbb{D}$ proved by Singh and Singh [40].

The Bohr phenomenon was reappeared in the 1990s due to the extensions to holomorphic function in $\mathbb{C}^n$ and to abstract setting [7]. Under this framework, Boas and Khavinson [11] found bounds for Bohr's radius in any complete Reinhard domains.

Let $\mathscr{B}$ be the class of analytic functions f in $\mathbb{D}$ such that $|f(z)| < 1$ for all $z \in \mathbb{D}$, and let $\mathscr{B}_0 = \{f \in \mathscr{B} : f(0) = 0\}$. In 1914, Bohr [10] proved that if $f \in \mathscr{B}$ is of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then the majorant series $M_f(r) = \sum_{n=0}^{\infty} |a_n||z|^n$ of f satisfies

$$M_{f_0}(r) = \sum_{n=1}^{\infty} |a_n||z|^n \leqslant 1-|a_0| = d(f(0),\partial f(\mathbb{D})) \tag{1.3}$$

for all $z \in \mathbb{D}$ with $|z| = r \leqslant 1/3$, where $f_0(z) = f(z)-f(0)$. Bohr actually obtained the inequality (1.3) for $|z| \leqslant 1/6$. Later, Wiener, Riesz and Schur, independently established the Bohr inequality (1.3) for $|z| \leqslant 1/3$ (known as Bohr radius for the class $\mathscr{B}$) and hence proved that $1/3$ is the best possible. Moreover, for $\phi_a(z) = (a-z)/(1-az)$ $(0 \leqslant a < 1)$, it concludes that $M_{\phi_a}(r) > 1$ if and only if $r > 1/(1+2a)$, which for $a \to 1$ shows that $1/3$ is optimal.

In recent years, Bohr inequality and Bohr radius have become an active research field in the theory of univalent functions. The Bohr's phenomenon for the complex-valued harmonic mappings have been widely studied (see [1, 2, 4, 3]). In 2016, Ali *et al.* [8] studied Bohr radius for the starlike log-harmonic mappings. In 2021, Liu and Ponnusamy [33] established a version of multidimensional analogue of the refined Bohr inequality. Bohr-type inequalities for harmonic mappings with a multiple zero at the origin have been discussed by Huang *et al.* [22]. The Bohr radius for intriguing aspects, such as, locally univalent harmonic mappings, lacunary series, k-quasiconformal mappings have been extensively researched in [23, 25]. For more various of classes and Bohr's phenomenon for harmonic mappings, we refer the reader to [6, 18, 27, 32, 42] and the references therein.

In 2017, Kayumov and Ponnusamy [24] derived a modified version of Bohr inequality for analytic functions as follows.

THEOREM A. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be analytic in $\mathbb{D}$, $|f(z)|\leqslant 1$ and S_r denote the area of the disk $|z|<r$ under the harmonic mapping f. Then*

$$B_1(r):=\sum_{n=0}^{\infty}|a_n|r^n+\frac{16}{9}\left(\frac{S_r}{\pi}\right)\leqslant 1 \ \text{for} \ r\leqslant\frac{1}{3},$$

and the values $1/3$ and $16/9$ cannot be improved. Moreover,

$$B_2(r):=|a_0|^2+\sum_{n=1}^{\infty}|a_n|r^n+\frac{8}{9}\left(\frac{S_r}{\pi}\right)\leqslant 1 \ \text{for} \ r\leqslant\frac{1}{2},$$

and the values $1/2$ and $8/9$ cannot be improved.

Similar to Bohr radius, Bohr-Rogosinski radius has also been defined (see [39]) which is described: If $f\in\mathscr{B}$, then for $N\geqslant 1$, we have $|S_N(z)|<1$ in the disk $\mathbb{D}_{1/2}$ and this radius is sharp, where $S_N(z)=\sum_{n=0}^{N}a_nz^n$ denotes the partial sum of f. There is a relevant quantity, which we call the Bohr-Rogosinski sum $R_N^f(z)$ of f defined by

$$R_N^f(z):=|f(z)|+\sum_{n=N}^{\infty}|a_n|r^n \quad (|z|=r). \tag{1.4}$$

It is important to note that for $N=1$, the quantity (1.4) is reduces to the classical Bohr sum in which $f(0)$ is replaced by $|f(z)|$. In 2017, Kayumov and Ponnusamy [24] proved the following interesting result on Bohr-Rogosinski radius for analytic functions.

THEOREM B. *Suppose that $f(z)=\sum_{n=0}^{\infty}a_nz^n$ is analytic in the unit disk $\mathbb{D}$ and $|f(z)|<1$ in $\mathbb{D}$. Then*

$$|f(z)|+\sum_{n=N}^{\infty}|a_n|r^n\leqslant 1 \ \text{for} \ r\leqslant R_N,$$

where R_N is the positive root of the equation

$$2(1+r)r^N-\left(1-r^2\right)=0.$$

The radius R_N is the best possible. Moreover,

$$|f(z)|^2+\sum_{n=N}^{\infty}|a_n|r^n\leqslant 1 \ \text{for} \ r\leqslant R_N',$$

where R_N' is the positive root of the equation

$$(1+r)r^N-\left(1-r^2\right)=0.$$

The radius R_N' is the best possible.

In this paper, we mainly deal with the functions $f = h+\overline{g} \in \mathscr{H}^0$ of the class $\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ which is defined by the third-order differential inequality (1.2). In Section 3, we exhibit the close-to-convexity, coefficient bounds, growth estimates, and sufficient coefficient condition of the class $\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$. Furthermore, we show that this class is closed under convex combination and convolution of its members. In Section 4, we establish various Bohr inequalities for the harmonic functions in a suitable fashion. In Section 5, we give the proofs of the results.

2. Preliminary results

In order to derive our main results, we need following lemmas.

LEMMA 2.1. (See [14]) *Suppose h and g are analytic in $\mathbb{D}$ with $|g'(0)| < |h'(0)|$ and $F_\varepsilon = h+\varepsilon g$ is close-to-convex for each ε $(|\varepsilon| = 1)$, then $f = h+\overline{g}$ is close-to-convex in $\mathbb{D}$.*

The following Lemma (known as Jack-Miller-Mocanu Lemma) contributed by Miller and Mocanu [34, p.19] or [35, p.290].

LEMMA 2.2. *Let $\omega(z) = c_m z^m + c_{m+1} z^{m+1} + \cdots$ be analytic in $\mathbb{D}$ with $c_m \neq 0$, and let $z_0 \neq 0$ be a point of $\mathbb{D}$ such that*

$$|\omega(z_0)| = \max_{|z|\leqslant|z_0|} |\omega(z)|.$$

Then there exists a real number k $(k \geqslant m \geqslant 1)$ such that

$$\frac{z_0\omega'(z_0)}{\omega(z_0)} = k \text{ and } \Re\left(1+\frac{z_0\omega''(z_0)}{\omega'(z_0)}\right) \geqslant k.$$

LEMMA 2.3. *If $f \in \mathscr{W}(\alpha,\beta,\gamma)$ with $\alpha \geqslant \gamma \geqslant 0$, then $\Re(f'(z)) > \beta$ $(0 \leqslant \beta < 1)$, and hence f is close-to-convex in $\mathbb{D}$.*

Proof. If $f \in \mathscr{W}(\alpha,\beta,\gamma)$, then $\Re(\psi(z)) > 0$, where $\psi(z) = f'(z)+\alpha z f''(z)+\gamma z^2 f'''(z)-\beta$. Let w be an analytic function in $\mathbb{D}$ such that $w(0) = 0$ and

$$f'(z) = \frac{1+(1-2\beta)w(z)}{1-w(z)}.$$

To prove the result, we need to show that $|w(z)| < 1$ for all z in $\mathbb{D}$. If not, then by Lemma 2.2, we could find some ξ $(|\xi| < 1)$, such that $|w(\xi)| = 1$ and $\xi w'(\xi) = kw(\xi)$, where $k \geqslant 1$. Since

$$\Re(\psi(\xi)) = \Re\Bigg(\frac{1+(1-2\beta)w(\xi)}{1-w(\xi)} + \frac{2\alpha k(1-\beta)w(\xi)}{(1-w(\xi))^2} + \frac{2(1-\beta)\cdot\left((1-w(\xi))^2 w''(\xi)+2(1-w(\xi))\,(w'(\xi))^2\right)}{(1-w(\xi))^4} - \beta\Bigg)$$

$$
\begin{aligned}
&= (1-\beta)\Re\left(\frac{2\alpha k w(\xi)+\left(1-w^2(\xi)\right)}{(1-w(\xi))^2}+\frac{2\gamma z w'(\xi)}{(1-w(\xi))^2}\cdot\frac{z w''(\xi)}{w'(\xi)}+\frac{4\gamma(z w'(\xi))^2}{(1-w(\xi))^3}\right)\\
&= (1-\beta)\left(\frac{4\alpha k(\Re(w(\xi))-1)}{|1-w(\xi)|^4}+\frac{4\gamma k(k-1)(\Re(w(\xi))-1)}{|1-w(\xi)|^4}\right.\\
&\qquad\left.+\frac{2\gamma k^2\left(w^2(\xi)+\overline{w}^2(\xi)-8\Re(w(\xi))+6\right)}{|1-w(\xi)|^6}\right)\\
&\leqslant 4(1-\beta)\frac{(\Re(w(\xi))-1)(\alpha-\gamma)k}{|1-w|^4}\leqslant 0
\end{aligned}
$$

for $|w(\xi)|=1$. This contradicts the hypotheses. Hence, $|w(z)|<1$, which leads to $\Re(f'(z))>\beta$. $\square$

3. Properties of the class of $\mathscr{W}_{\mathscr{H}}^{0}(\alpha,\beta,\gamma)$

This section is devoted to the results required to obtain the members of close-to-convexity, coefficient inequalities and growth estimates between the classes $\mathscr{W}_{\mathscr{H}}^{0}(\alpha,\beta,\gamma)$ of harmonic mappings and the class $\mathscr{W}(\alpha,\beta,\gamma)$ of analytic functions.

Firstly, we derive a sufficient condition for $f\in\mathscr{H}$ to be close-to-convex.

THEOREM 3.1. *The harmonic mapping* $f=h+\overline{g}\in\mathscr{W}_{\mathscr{H}}^{0}(\alpha,\beta,\gamma)$ *if and only if* $F_{\varepsilon}=h+\varepsilon g\in\mathscr{W}(\alpha,\beta,\gamma)$ *for each* ε $(|\varepsilon|=1)$.

THEOREM 3.2. *The functions in the class* $\mathscr{W}_{\mathscr{H}}^{0}(\alpha,\beta,\gamma)$ *are close-to-convex in* $\mathbb{D}$.

The following results provide sharp coefficient bounds for the functions in $\mathscr{W}_{\mathscr{H}}^{0}(\alpha,\beta,\gamma)$.

THEOREM 3.3. *Let* $f=h+\overline{g}\in\mathscr{W}_{\mathscr{H}}^{0}(\alpha,\beta,\gamma)$ *be of the form* (1.1) *with* $b_1=0$. *Then for* $n\geqslant 2$, *we have*

$$
|b_n|\leqslant\frac{1-\beta}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}. \tag{3.1}
$$

The result (3.1) *is sharp for the function given by*

$$
f(z)=z+\frac{1-\beta}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\overline{z}^n. \tag{3.2}
$$

THEOREM 3.4. *Let* $f = h + \overline{g} \in \mathscr{W}^0_{\mathscr{H}}(\alpha, \beta, \gamma)$ *be of the form* (1.1) *with* $b_1 = 0$. *Then for* $n \geqslant 2$, *we have*

$$|a_n| + |b_n| \leqslant \frac{2(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n},$$
$$||a_n| - |b_n|| \leqslant \frac{2(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n},$$
$$|a_n| \leqslant \frac{2(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}.$$

All these results are sharp for the function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} z^n. \tag{3.3}$$

The following result gives a sufficient condition for a function f belonging to the class $\mathscr{W}^0_{\mathscr{H}}(\alpha, \beta, \gamma)$.

THEOREM 3.5. *Let* $f = h + \overline{g} \in \mathscr{H}^0$, *where* h *and* g *are of the form* (1.1). *If*

$$\sum_{n=2}^{\infty} n[1 + (n-1)(\alpha + \gamma(n-2))]\,(|a_n| + |b_n|) \leqslant 1 - \beta, \tag{3.4}$$

then $f \in \mathscr{W}^0_{\mathscr{H}}(\alpha, \beta, \gamma)$.

COROLLARY 3.1. *Let* $f = h + \overline{g} \in \mathscr{H}^0$, $\alpha \geqslant 1$ *and* $0 \leqslant \beta < 1$. *If*

$$\sum_{n=2}^{\infty} n^2[3 - n + \alpha(n-1)]\,(|a_n| + |b_n|) \leqslant 2(1-\beta),$$

then $f \in \mathscr{W}^0_{\mathscr{H}}(\alpha, \beta, (\alpha - 1)/2)$.

The following result gives sharp growth theorem for the class $\mathscr{W}^0_{\mathscr{H}}(\alpha, \beta, \gamma)$.

THEOREM 3.6. *If* $f = h + \overline{g} \in \mathscr{W}^0_{\mathscr{H}}(\alpha, \beta, \gamma)$, *then*

$$\begin{aligned} |z| + 2\sum_{n=2}^{\infty} \frac{(-1)^{n-1}(1-\beta)|z|^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} &\leqslant |f(z)| \\ &\leqslant |z| + 2\sum_{n=2}^{\infty} \frac{(1-\beta)|z|^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n}. \end{aligned} \tag{3.5}$$

Both the inequalities are sharp when

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2(1-\beta)}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1 - \alpha + 2\gamma)n} z^n$$

or its rotations.

Finally, we shall prove that the class $\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ is closed under convex combinations and convolutions.

THEOREM 3.7. *The class $\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ is closed under convex combinations.*

THEOREM 3.8. *If functions f_1 and f_2 belong to $\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$, then $f_1 * f_2 \in \mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$.*

THEOREM 3.9. *Let $f \in \mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ and $\phi \in \mathscr{A}$ with $\Re\left(\frac{\phi(z)}{z}\right) > \frac{1}{2}$ for $z \in \mathbb{D}$, then $f \widehat{*} \phi \in \mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$.*

COROLLARY 3.2. *Suppose that $f \in \mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ and $\phi \in \mathscr{K}$, then $f \widehat{*} \phi \in \mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$.*

REMARK 1. (1) The process of the proofs of the results in this section we can refer to references Liu and Yang [31], Yaşar and Yalçın [45], Ghosh and Vasudevarao [20] and Rajbala and Prajapat [38].

(2) As an application, we also can construct harmonic polynomials involving Gaussian hypergeometric function which belong to the considered class $\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$. The method can be found in Yaşar and Yalçın [45] and Rajbala and Prajapat [38], we omit it here.

4. Bohr's phenomenon for the class $\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$

The main aim of this section is to study the Bohr phenomenon for the class $\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ for $\alpha \geqslant \gamma \geqslant 0$ and $0 \leqslant \beta < 1$. We find the improved Bohr radius as well as Bohr-Rogosinski inequality for functions in the class $\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$. In this section, we state the main results of the paper, and in next section, we shall give the proofs of main results.

Before stating the main results, we recall the definition of polylogarithm. The polylogarithm function $\mathrm{Li}_s(z)$ is defined by the power series, which is also a Dirichlet series in s, $\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \quad (|z| < 1)$ is valid for arbitrary complex order s. It can be extended to $|z| \geqslant 1$ by the process of analytic continuation. The special case $s = 1$ involves Taylor series of the ordinary natural logarithm, $\mathrm{Li}_1(z) = -\log(1-z)$, while the special cases $s = 2$ and $s = 3$ are called the dilogarithm (also referred to as Spence's function) and trilogarithm, respectively. The definition and the name of the polylogarithm function come from the fact that it may also be defined as the repeated integral of itself: $\mathrm{Li}_{s+1}(z) = \int_0^z \frac{\mathrm{Li}_s(t)}{t} dt$, thus the dilogarithm is an integral involving the logarithm $\mathrm{Li}_2(z) = -\int_0^z \log(1-u)\frac{du}{u} \quad (z \in \mathbb{C}\backslash[1,\infty))$, and so on. The polylogarithm is a special case of the Lerch transcendent [17]. The structural properties of polylogarithms we can refer the book [28].

Using Theorem 3.4 and Theorem 3.6, considering power of the coefficient $|a_n| + |b_n|$, we prove the following sharp Bohr radius for the class $\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$.

THEOREM 4.1. *Let* $f \in \mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ *for* $\alpha \geqslant \gamma \geqslant 0, 0 \leqslant \beta < 1$ *be of the form* (1.1). *Then for* $n \geqslant 2$ *and* $p \geqslant 1$,

$$|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\,|z|^n + \sum_{n=2}^{\infty} (|a_n| + |b_n|)^p\,|z|^{pn} \leqslant d(f(0), \partial f(\mathbb{D})) \tag{4.1}$$

for $|z| = r \leqslant r_p(\alpha,\beta,\gamma) := r_p$, *where* r_p *is the unique root of*

$$r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} + \sum_{n=2}^{\infty} \left(\frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n}\right)^p$$
$$= 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n}. \tag{4.2}$$

Here r_p *is the best possible.*

For particular choices of $\alpha = 2$, $\beta = 0$, $\gamma = 1/2$ and $p = 2$, we have the following corollaries of Theorem 4.1.

COROLLARY 4.1. *Let* $f \in \mathscr{W}^0_{\mathscr{H}}(2,0,1/2)$ *and be of the form* (1.1). *Then for* $|z| = r$,

$$|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\,|z|^n + \sum_{n=2}^{\infty} (|a_n| + |b_n|)^2\,|z|^{2n} \leqslant d(f(0), \partial f(\mathbb{D}))$$

for $r \leqslant r_2(2,0,1/2) := r_2 \approx 0.585660$, *where* r_2 *is the unique root of the equation*

$$4\left[4\left(\frac{1}{r^2}+3\right)\mathrm{Li}_2\left(r^2\right) - 8\mathrm{Li}_3\left(r^2\right) + 4\mathrm{Li}_4\left(r^2\right) - r^2 + 16\log\left(1-r^2\right) - \frac{16\log\left(1-r^2\right)}{r^2} - 20\right]$$
$$+ 4\mathrm{Li}_2(r) + r - 2(r_2+2) + \frac{4(r-1)\log(1-r)}{r} - 3 - \frac{\pi^2}{3} + 8\log 2 = 0$$

in $(0,1)$. *Here* $r_2 \approx 0.585660$ *is the best possible.*

COROLLARY 4.2. *Let* $f \in \mathscr{W}^0_{\mathscr{H}}(2,0,1/2)$ *be of the form* (1.1). *Then for* $|z| = r$,

$$|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)^2\,|z|^{2n} \leqslant d(f(0), \partial f(\mathbb{D}))$$

for $r \leqslant r_2^*(2,0,1/2) := r_2^* \approx 0.713982$, *where* r_2^* *is the unique root of the equation*

$$r + 4\left[4\left(\frac{1}{r^2}+3\right)\mathrm{Li}_2\left(r^2\right) - 8\mathrm{Li}_3\left(r^2\right) + 4\mathrm{Li}_4\left(r^2\right) - r^2 + 16\log\left(1-r^2\right)\right.$$
$$\left. - \frac{16\log\left(1-r^2\right)}{r^2} - 20\right] - 3 - \frac{\pi^2}{3} + 8\log 2 = 0$$

in $(0,1)$. *Here* $r_2^* \approx 0.713982$ *is the best possible.*

REMARK 2. For a particular choice of $\alpha=2$, $\beta=1/2$, $\gamma=1/2$ and $p=2$, we obtain $r_2(2,1/2,1/2)\approx 0.737687$; For $\alpha=2$, $\beta=0$, $\gamma=1/2$ and $p=3$, we get $r_3(2,0,1/2)\approx 0.593252$.

THEOREM 4.2. *Let $f\in\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ for $\alpha\geqslant\gamma\geqslant 0, 0\leqslant\beta<1$ be of the form* (1.1)*. Then for $n\geqslant 2$,*

$$|f(z)|+\sum_{n=2}^{\infty}(|a_n|+|b_n|)|z|^n\leqslant d(f(0),\partial f(\mathbb{D})) \tag{4.3}$$

for $|z|=r\leqslant\widetilde{r}(\alpha,\beta,\gamma):=\widetilde{r}$, where $\widetilde{r}$ is the unique root of

$$r+\sum_{n=2}^{\infty}\frac{4(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}-1-\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}=0.$$

Here $\widetilde{r}$ is the best possible.

For a particular choice of α, β, γ, we have the following corollary of Theorem 4.2.

COROLLARY 4.3. *Let $f\in\mathscr{W}^0_{\mathscr{H}}(2,0,1/2)$ be of the form* (1.1)*. Then for $|z|=r$,*

$$|f(z)|+\sum_{n=2}^{\infty}(|a_n|+|b_n|)|z|^n\leqslant d(f(0),\partial f(\mathbb{D}))$$

for $r\leqslant\widetilde{r}_*\approx 0.521468$, where $\widetilde{r}_*$ is the unique root of the equation

$$8\mathrm{Li}_2(r)+r-4(r+2)+\frac{8(r-1)\log(1-r)}{r}-\frac{\pi^2}{3}-3+8\log 2=0$$

in $(0,1)$. Here $\widetilde{r}_*\approx 0.521468$ is the best possible.

THEOREM 4.3. *Let $f\in\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ for $\alpha\geqslant\gamma\geqslant 0, 0\leqslant\beta<1$ be of the form* (1.1)*. Then for integers $m\geqslant 1$, $n,N\geqslant 2$, we have*

$$|f(z^m)|+\sum_{n=N}^{\infty}(|a_n|+|b_n|)|z|^n\leqslant d(f(0),\partial f(\mathbb{D})) \tag{4.4}$$

for $|z|=r\leqslant R_{m,N}(\alpha,\beta,\gamma):=R_{m,N}$, where $R_{m,N}$ is the unique root of

$$r^m+\sum_{n=2}^{\infty}\frac{2(1-\beta)r^{mn}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}+\sum_{n=N}^{\infty}\frac{2(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}$$
$$-1-\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}=0.$$

Here $R_{m,N}$ is the best possible.

THEOREM 4.4. *Let* $f \in \mathcal{W}^0_{\mathcal{H}}(\alpha,\beta,\gamma)$ *for* $\alpha \geqslant \gamma \geqslant 0, 0 \leqslant \beta < 1$ *be given by* (1.1). *Then*

(i)

$$|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\,|z|^n + \frac{S_r}{\pi} \leqslant d(f(0), \partial f(\mathbb{D}))$$

for $|z| = r \leqslant r_f(\alpha,\beta,\gamma) := r_f$, *where* r_f *is the unique root of the equation*

$$\begin{aligned} r^2 + r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} \\ + \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 n r^{2n}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} \\ -1 - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} = 0 \end{aligned}$$

in $(0,1)$. *Here* r_f *is the best possible.*

(ii)

$$|f(z)|^2 + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\,|z|^n + \left(\frac{S_r}{\pi}\right)^2 \leqslant d(f(0), \partial f(\mathbb{D}))$$

for $|z| = r \leqslant r_f^*(\alpha,\beta,\gamma) := r_f^*$, *where* r_f^* *is the unique root of the equation*

$$\begin{aligned} \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n}\right)^2 \\ + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} \\ + \left(r^2 + \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 n r^{2n}}{[\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n]^2}\right)^2 \\ -1 - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} = 0 \end{aligned}$$

in $(0,1)$. *Here* r_f^* *is the best possible.*

For particular choices of α, β, γ, we have the following corollary of Theorem 4.4.

COROLLARY 4.4. *Let* $f \in \mathcal{W}^0_{\mathcal{H}}(\alpha,\beta,\gamma)$ *be given by* (1.1). *Then*

(i) for $\alpha = 2$, $\beta = 0$, $\gamma = 1/2$,

$$|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\,|z|^n + \frac{S_r}{\pi} \leqslant d(f(0), \partial f(\mathbb{D}))$$

for $|z| = r \leqslant \dot{r}_f(2,0,1/2) := \dot{r}_f \approx 0.451007$, *where* $\dot{r}_f$ *is the unique root of the equation*

$$4\left(-\frac{4}{r^2}-8\right)\mathrm{Li}_2\left(r^2\right)+16\mathrm{Li}_3\left(r^2\right)+4\mathrm{Li}_2(r)-3r^2+\frac{48\log\left(1-r^2\right)}{r^2}$$
$$-48\log\left(1-r^2\right)+r-2(r+2)+\frac{4(r-1)\log(1-r)}{r}-\frac{\pi^2}{3}+61+8\log 2=0$$

in $(0,1)$. *Here* $\dot{r}_f \approx 0.451007$ *is the best possible.*

(ii) for $\alpha=2,\ \beta=0,\ \gamma=1/2,$

$$|f(z)|^2+\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)|z|^n+\left(\frac{S_r}{\pi}\right)^2 \leqslant d(f(0),\partial f(\mathbb{D}))$$

for $|z| = r \leqslant \ddot{r}_f(2,0,1/2) := \ddot{r}_f \approx 0.566259$, *where* $\ddot{r}_f$ *is the unique root of the equation*

$$\left[4\left(-\frac{4}{r^2}-8\right)\mathrm{Li}_2\left(r^2\right)+16\mathrm{Li}_3\left(r^2\right)-3r^2+\frac{48\log\left(1-r^2\right)}{r^2}-48\log\left(1-r^2\right)+64\right]^2$$
$$+4\mathrm{Li}_2(r)+\left[-4\mathrm{Li}_2(r)+r+\left(\frac{4}{r}-4\right)\log(1-r)+4\right]^2$$
$$-2(r+2)+\frac{4(r-1)\log(1-r)}{r}-\frac{\pi^2}{3}-3+8\log 2=0$$

in $(0,1)$. *Here* $\ddot{r}_f \approx 0.566259$ *is the best possible.*

THEOREM 4.5. *Let* $f \in \mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ *be given by* (1.1).

(i)

$$|z|+|h(z)|+\sum_{n=2}^{\infty}|a_n||z|^n \leqslant d(f(0),\partial f(\mathbb{D}))$$

for $|z| = r \leqslant r_h(\alpha,\beta,\gamma) := r_h$, *where* r_h *is the unique root of the equation*

$$2r+\sum_{n=2}^{\infty}\frac{4(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}-1$$
$$-\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}=0 \tag{4.5}$$

in $(0,1)$. *Here* r_h *is the best possible.*

(ii)

$$|z|+|g(z)|+\sum_{n=2}^{\infty}|b_n||z|^n \leqslant d(f(0),\partial f(\mathbb{D}))$$

for $|z| = r \leqslant r_g(\alpha,\beta,\gamma) := r_g$*, where* r_g *is the unique root of the equation*

$$r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} - 1 - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} = 0 \tag{4.6}$$

in $(0,1)$*. Here* r_g *is the best possible.*

REMARK 3. For more calculations, the radii are $r_h(2,0,1/2) \approx 0.331050$, $r_h(2,1/2,1/2) \approx 0.404384$, $r_g(2,0,1/2) \approx 0.671758$, $r_g(2,1/2,1/2) \approx 0.787798$.

REMARK 4. The results in this section yields the results of the class $\mathscr{W}^0_{\mathscr{H}}(\alpha,0,0)$ in [5].

5. Proofs of the main results

Proof of Theorem 4.1. Let $f \in \mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ be given by (1.1). Then, by Theorem 3.4 and Theorem 3.6, it is evident that the Euclidean distance $d(f(0),\partial f(\mathbb{D}))$ between $f(0)$ and the boundary of $f(\mathbb{D})$ is

$$d(f(0),\partial f(\mathbb{D})) = \liminf_{|z|\to 1} |f(z) - f(0)| \geqslant 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n}. \tag{5.1}$$

Let $\Theta_1 : [0,1] \to \mathbb{R}$ be defined by

$$\Theta_1(r) = r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} + \sum_{n=2}^{\infty} \left(\frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} \right)^p - 1 - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n}. \tag{5.2}$$

Clearly, the function $\Theta_1(r)$ is continuous on $[0,1]$ and differentiable on $(0,1)$. Since

$$\left| \sum_{n=2}^{\infty} \frac{(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} \right| \leqslant \frac{1}{2},$$

we get

$$\Theta_1(0) = -1 - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} < 0.$$

On the other hand, since for $n \geqslant 2$,

$$\sum_{n=2}^{\infty} \frac{1-\beta}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} \geqslant \sum_{n=2}^{\infty} \frac{(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n},$$

it follows that

$$\Theta_1(1)=\sum_{n=2}^{\infty}\frac{2(1-\beta)}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}+2^p\sum_{n=2}^{\infty}\frac{(1-\beta)^p}{[\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n]^p}$$
$$-\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}$$
$$\geqslant 2^p\sum_{n=2}^{\infty}\frac{(1-\beta)^p}{[\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n]^2}>0.$$

Therefore, $\Theta_1(0)\Theta_1(1)<0$, and then, by the intermediate value theorem, the function $\Theta_1(r)$ has roots in $(0,1)$. In order to show that $\Theta_1(r)$ has exactly one root in $(0,1)$, it is sufficient to show that Θ_1 is strictly monotonic function in $(0,1)$. It follows that

$$\Theta_1'(r)=1+\sum_{n=2}^{\infty}\frac{2(1-\beta)nr^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}$$
$$+\sum_{n=2}^{\infty}\frac{np(1-\beta)^p2^pr^{np-1}}{[\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n]^p}>0$$

for all $r\in(0,1)$, which shows that $\Theta_1(r)$ is strictly increasing function. Therefore, $\Theta_1(r)$ has a unique root in $(0,1)$, say $r_p(\alpha,\beta,\gamma):=r_p$. Thus, the function $\Theta_1(r_p)=0$ and hence from (5.2), we obtain

$$r_p+\sum_{n=2}^{\infty}\frac{2(1-\beta)r_p^n}{(\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}+\sum_{n=2}^{\infty}\frac{2^p(1-\beta)^pr_p^{np}}{[\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n]^p}$$
$$=1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}. \tag{5.3}$$

To show that r_p is the best possible, considering the function $f=f_{(\alpha,\beta,\gamma)}$ defined by (3.3). It holds for the function $f_{(\alpha,\beta,\gamma)}\in\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ and for $f=f_{(\alpha,\beta,\gamma)}$ and

$$d(f(0),\partial f(\mathbb{D}))=1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}. \tag{5.4}$$

It is obvious that

$$r+\sum_{n=2}^{\infty}\frac{2(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}+\sum_{n=2}^{\infty}\frac{2^p(1-\beta)^pr^{np}}{(\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n)^p}$$
$$\leqslant 1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}$$

for $r\leqslant r_p$. Depending on (5.3) and (5.4) for the function $f=f_{(\alpha,\beta,\gamma)}$ and $r=r_p$, we show that

$$\begin{aligned}
&|z|+\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,|z|^n+\sum_{n=2}^{\infty}(|a_n|+|b_n|)^p\,|z|^{np}\\
&=r+\sum_{n=2}^{\infty}\frac{2(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\\
&\quad+\sum_{n=2}^{\infty}\frac{2^p(1-\beta)^p r^{np}}{(\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n)^p}\\
&=r_p+\sum_{n=2}^{\infty}\frac{2(1-\beta)r_p^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\\
&\quad+\sum_{n=2}^{\infty}\frac{2^p(1-\beta)^p r_p^{np}}{[\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n]^p}\\
&=1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\\
&=d(f(0),\partial f(\mathbb{D})).
\end{aligned}$$

Therefore, r_p is the best possible. The proof of the theorem is completed. □

Proof of Corollary 4.1. Let $f\in\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ be given by (1.1). Then in view of Theorem 3.4 and Theorem 3.6, and (5.1), for $|z|=r$ and $\alpha=2,\beta=0,\gamma=1/2$, we obtain

$$|z|+\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,|z|^n+\sum_{n=2}^{\infty}(|a_n|+|b_n|)^2\,|z|^{2n}\leqslant r+\sum_{n=2}^{\infty}\frac{4r^n}{n^3+n^2}+\sum_{n=2}^{\infty}\left(\frac{4r^n}{n^3+n^2}\right)^2. \tag{5.5}$$

It follows that

$$\begin{aligned}
\sum_{n=2}^{\infty}\frac{4r^n}{n^3+n^2}&=\sum_{n=2}^{\infty}\frac{4r^n}{n^2}+\sum_{n=2}^{\infty}\frac{4r^n}{n+1}-\sum_{n=2}^{\infty}\frac{4r^n}{n}\\
&=-4r+4\sum_{n=1}^{\infty}\frac{r^n}{n^2}+\frac{4}{r}\left(-\frac{r^2}{2}-r+\sum_{n=1}^{\infty}\frac{r^n}{n}\right)+4r-4\sum_{n=1}^{\infty}\frac{r^n}{n}\\
&=-4r+4\mathrm{Li}_2(r)+\frac{4}{r}\left[-\frac{r^2}{2}-r-\log(1-r)\right]+4r+4\log(1-r)\\
&=4\mathrm{Li}_2(r)-2(r+2)+\frac{4(r-1)\log(1-r)}{r}.
\end{aligned}$$

On the other hand, we see that

$$\begin{aligned}
\sum_{n=2}^{\infty}\left(\frac{4r^n}{n^3+n^2}\right)^2&=\sum_{n=2}^{\infty}\frac{16r^{2n}}{n^4}+\sum_{n=2}^{\infty}-\frac{32r^{2n}}{n^3}+\sum_{n=2}^{\infty}\frac{48r^{2n}}{n^2}+\sum_{n=2}^{\infty}-\frac{64r^{2n}}{n}\\
&\quad+\sum_{n=2}^{\infty}\frac{16r^{2n}}{(n+1)^2}+\sum_{n=2}^{\infty}\frac{64r^{2n}}{n+1}.
\end{aligned}$$

By the definition of dilogarithm function Li_s, it follows that

$$\begin{cases} \sum_{n=2}^{\infty} \frac{16r^{2n}}{n^4} = -16r^2 + 16\mathrm{Li}_4(r^2), \\ \sum_{n=2}^{\infty} -\frac{32r^{2n}}{n^3} = 32r^2 + 32\mathrm{Li}_3(r^2), \\ \sum_{n=2}^{\infty} \frac{48r^{2n}}{n^2} = -48r^2 + 48\mathrm{Li}_2(r^2), \\ \sum_{n=2}^{\infty} -\frac{64r^{2n}}{n} = 64r^2 + 64\log(1-r^2), \\ \sum_{n=2}^{\infty} \frac{16r^{2n}}{(n+1)^2} = \frac{16}{r^2}\left[-\frac{r^4}{4} - r^2 + \mathrm{Li}_2\left(r^2\right)\right], \\ \sum_{n=2}^{\infty} \frac{64r^{2n}}{n+1} = \frac{64}{r^2}\left[-\frac{r^4}{2} - r^2 - \log\left(1-r^2\right)\right]. \end{cases} \tag{5.6}$$

Therefore, by (5.6), we deduce that

$$\begin{aligned} &\sum_{n=2}^{\infty}\left(\frac{4r^n}{n^3+n^2}\right)^2 \\ &= \left[-16r^2 + 16\mathrm{Li}_4(r^2)\right] + \left[32r^2 + 32\mathrm{Li}_3(r^2) + (-48r^2 + 48\mathrm{Li}_2(r^2))\right] \\ &\quad + \left[64r^2 + 64\log(1-r^2)\right] + \frac{16}{r^2}\left[-\frac{r^4}{4} - r^2 + \mathrm{Li}_2\left(r^2\right)\right] + \frac{64}{r^2}\left[-\frac{r^4}{2} - r^2 - \log\left(1-r^2\right)\right] \\ &= 4\left[4\left(\frac{1}{r^2}+3\right)\mathrm{Li}_2\left(r^2\right) - 8\mathrm{Li}_3\left(r^2\right) + 4\mathrm{Li}_4\left(r^2\right) - r^2 + 16\log\left(1-r^2\right)\right. \\ &\quad \left. - \frac{16\log\left(1-r^2\right)}{r^2} - 20\right]. \end{aligned}$$

Hence, from (5.1) and (5.5), we get

$$\begin{aligned} &|z| + \sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)|z|^n + \sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right)^2|z|^{2n} \\ &= 4\left[4\left(\frac{1}{r^2}+3\right)\mathrm{Li}_2\left(r^2\right) - 8\mathrm{Li}_3\left(r^2\right) + 4\mathrm{Li}_4\left(r^2\right) - r^2 + 16\log\left(1-r^2\right)\right. \\ &\quad \left. - \frac{16\log\left(1-r^2\right)}{r^2} - 20\right] + 4\mathrm{Li}_2(r) + r - 2(r+2) + \frac{4(r-1)\log(1-r)}{r}. \end{aligned}$$

Furthermore, we show that

$$\begin{aligned}&4\left[4\left(\frac{1}{r^2}+3\right)\mathrm{Li}_2\left(r^2\right)-8\mathrm{Li}_3\left(r^2\right)+4\mathrm{Li}_4\left(r^2\right)-r^2+16\log\left(1-r^2\right)\right.\\&\quad\left.-\frac{16\log\left(1-r^2\right)}{r^2}-20\right]+4\mathrm{Li}_2(r)+r-2(r+2)+\frac{4(r-1)\log(1-r)}{r}\\&\leqslant 1+\left(2+\frac{\pi^2}{3}-8\log 2\right)\end{aligned}$$

for $r\leqslant r_2(2,0,1/2):=r_2$, where $r_2\approx 0.585660$ is a root of $F_1(r)=0$ in $(0,1)$ and $F_1:[0,1]\to\mathbb{R}$ is defined by

$$\begin{aligned}F_1(r):=4&\left[4\left(\frac{1}{r^2}+3\right)\mathrm{Li}_2\left(r^2\right)-8\mathrm{Li}_3\left(r^2\right)+4\mathrm{Li}_4\left(r^2\right)-r^2+16\log\left(1-r^2\right)\right.\\&\left.-\frac{16\log\left(1-r^2\right)}{r^2}-20\right]+4\mathrm{Li}_2(r)+r-2(r_2+2)\\&+\frac{4(r-1)\log(1-r)}{r}-3-\frac{\pi^2}{3}+8\log 2.\end{aligned}$$

An analogous calculation as in the proof of Theorem 4.1, we can show that the function $F_1(r)$ has the unique root $r_2\approx 0.585660$ in $(0,1)$. Hence, it yields

$$\begin{aligned}&4\left[4\left(\frac{1}{r_2^2}+3\right)\mathrm{Li}_2\left(r_2^2\right)-8\mathrm{Li}_3\left(r_2^2\right)+4\mathrm{Li}_4\left(r_2^2\right)-r_2^2+16\log\left(1-r_2^2\right)-\frac{16\log\left(1-r_2^2\right)}{r_2^2}-20\right]\\&\quad+4\mathrm{Li}_2(r_2)+r_2-2(r_2+2)+\frac{4(r_2-1)\log(1-r_2)}{r_2}-3-\frac{\pi^2}{3}+8\log 2=0,\end{aligned}$$

or equivalently,

$$\begin{aligned}&4\left[4\left(\frac{1}{r_2^2}+3\right)\mathrm{Li}_2\left(r_2^2\right)-8\mathrm{Li}_3\left(r_2^2\right)+4\mathrm{Li}_4\left(r_2^2\right)-r^2+16\log\left(1-r_2^2\right)-\frac{16\log\left(1-r_2^2\right)}{r_2^2}-20\right]\\&\quad+4\mathrm{Li}_2(r_2)+r_2-2(r_2+2)+\frac{4(r_2-1)\log(1-r_2)}{r_2}=1+\left(2+\frac{\pi^2}{3}-8\log 2\right).\end{aligned} \tag{5.7}$$

To show that r_2 is the best possible, considering the function $f=f_{(2,0,1/2)}$ defined by

$$f_{(2,0,1/2)}(z)=z+\sum_{n=2}^{\infty}\frac{4z^n}{n^3+n^2}. \tag{5.8}$$

In view of (5.1), we see that

$$d(f(0),\partial f(\mathbb{D}))=1+\sum_{n=2}^{\infty}\frac{4(-1)^{n-1}}{n^3+n^2}=1+\left(2+\frac{\pi^2}{3}-8\log 2\right). \tag{5.9}$$

By (5.5), (5.7) and (5.9), for $f = f_{(2,0,1/2)}$ and $r = r_2$, it implies that

$$\begin{aligned}
&|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\,|z|^n + \sum_{n=2}^{\infty} (|a_n| + |b_n|)^2\,|z|^{2n} \\
&= r + 4\left[4\left(\frac{1}{r_2^2}+3\right)\mathrm{Li}_2\left(r_2^2\right) - 8\mathrm{Li}_3\left(r^2\right) + 4\mathrm{Li}_4\left(r_2^2\right) - r_2^2\right. \\
&\quad \left. + 16\log\left(1-r_2^2\right) - \frac{16\log\left(1-r_2^2\right)}{r_2^2} - 20\right] \\
&\quad + 4\mathrm{Li}_2(r_2)_2 - 2(r_2+2) + \frac{4(r_2-1)\log(1-r_2)}{r_2} \\
&= 1 + \left(2 + \frac{\pi^2}{3} - 8\log 2\right) \\
&= d(f_{(2,0,1/2)}(0), \partial f_{(2,0,1/2)}(\mathbb{D})).
\end{aligned}$$

Therefore, r_2 is the best possible. □

Proof of Corollary 4.2. Let $f \in \mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ be given by (1.1). Then in view of Theorem 3.4 and Theorem 3.6, and (5.1), for $|z| = r$ and $\alpha = 2$, $\beta = 0$, $\gamma = 1/2$, we obtain

$$\begin{aligned}
|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)^2\,|z|^{2n} &\leqslant r + \sum_{n=2}^{\infty} \frac{16r^{2n}}{(n^3+n^2)^2} \\
&= 4\left[4\left(\frac{1}{r^2}+3\right)\mathrm{Li}_2\left(r^2\right) - 8\mathrm{Li}_3\left(r^2\right) + 4\mathrm{Li}_4\left(r^2\right) + r - r^2\right. \\
&\quad \left. + 16\log\left(1-r^2\right) - \frac{16\log\left(1-r^2\right)}{r^2} - 20\right].
\end{aligned} \tag{5.10}$$

Without loss of generality, we show that

$$\begin{aligned}
&r + 4\left[4\left(\frac{1}{r^2}+3\right)\mathrm{Li}_2\left(r^2\right) - 8\mathrm{Li}_3\left(r^2\right) + 4\mathrm{Li}_4\left(r^2\right) - r^2 + 16\log\left(1-r^2\right)\right. \\
&\quad \left. - \frac{16\log\left(1-r^2\right)}{r^2} - 20\right] \leqslant 1 + \left(2 + \frac{\pi^2}{3} - 8\log 2\right)
\end{aligned}$$

for $r \leqslant r_2^*(2,0.5,0) := r_2^*$, where $r_2^* \approx 0.713982$ is a root of $F_2(r) = 0$ in $(0,1)$, and $F_2 : [0,1] \to \mathbb{R}$ is defined by

$$\begin{aligned}
F_2(r) &:= 4\left[4\left(\frac{1}{r^2}+3\right)\mathrm{Li}_2\left(r^2\right) - 8\mathrm{Li}_3\left(r^2\right) + 4\mathrm{Li}_4\left(r^2\right) + r - r^2\right. \\
&\quad \left. + 16\log\left(1-r^2\right) - \frac{16\log\left(1-r^2\right)}{r^2} - 20\right] - 3 - \frac{\pi^2}{3} + 8\log 2.
\end{aligned}$$

By using the same argument as the proof of Theorem 4.1, we see that $F_2(r)$ has a unique root $r_2^* \approx 0.713982$ in $(0,1)$ and r_2^* is the best possible. □

Proof of Theorem 4.2. Let $f \in \mathscr{W}_{\mathscr{H}}^0(\alpha,\beta,\gamma)$ be given by (1.1). Then in view of Theorem 3.4, Theorem 3.6 and (5.1), for $|z| = r$, we obtain

$$\begin{aligned}
&|f(z)| + \sum_{n=2}^{\infty}(|a_n|+|b_n|)\,|z|^n \\
&\leqslant |z| + \sum_{n=2}^{\infty}\frac{2(1-\beta)|z|^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n} + \sum_{n=2}^{\infty}\frac{2(1-\beta)|z|^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n} \\
&= r + \sum_{n=2}^{\infty}\frac{4(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}.
\end{aligned} \tag{5.11}$$

Furthermore, we know that

$$r + \sum_{n=2}^{\infty}\frac{4(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n} \leqslant 1 + \sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}$$

for $r \leqslant \widetilde{r}(\alpha,\beta,\gamma) := \widetilde{r}$, where $\widetilde{r}$ is a root of $\Theta_2(r) = 0$ in $(0,1)$ and $\Theta_2 : [0,1] \to \mathbb{R}$ is defined by

$$\begin{aligned}
\Theta_2(r) := r &+ \sum_{n=2}^{\infty}\frac{4(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n} - 1 \\
&- \sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}.
\end{aligned}$$

By observing that $\Theta_2(r)$ is continuous on $[0,1]$ and differentiable on $(0,1)$,

$$\Theta_2(0) = -1 - \sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n} < 0,$$

and

$$\begin{aligned}
\Theta_2(1) &= \sum_{n=2}^{\infty}\frac{4(1-\beta)}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n} - \sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n} \\
&= \sum_{n=2}^{\infty}\frac{(4-2(-1)^{n-1})(1-\beta)}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n} > 0,
\end{aligned}$$

thus, Θ_2 has a root in $(0,1)$. By noting that

$$\Theta_2'(r) = 1 + \sum_{n=2}^{\infty}\frac{4(1-\beta)nr^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n} > 0 \ \text{ for } n \geqslant 2, \tag{5.12}$$

the function $\Theta_2(r)$ is strictly increasing function in $(0,1)$. Therefore, the function Θ_2 has a unique root $\widetilde{r}$ in $(0,1)$, that is $\Theta_2(\widetilde{r})=0$, or equivalently,

$$\widetilde{r}+\sum_{n=2}^{\infty}\frac{4(1-\beta)\widetilde{r}^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}=1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}. \tag{5.13}$$

To show that $\widetilde{r}$ is the best possible, we consider the function $f=f_{(\alpha,\beta,\gamma)}$ defined by (3.3). Clearly, the function $f_{(\alpha,\beta,\gamma)}$ belongs to the class $\mathscr{W}_{\mathscr{H}}^0(\alpha,\beta,\gamma)$. By means of (5.4), (5.12) and (5.13), for $f=f_{(\alpha,\beta,\gamma)}$ and $r=\widetilde{r}$, we see that

$$\begin{aligned}|f_{(\alpha,\beta,\gamma)}(z)|+\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,|z|^n&=\widetilde{r}+\sum_{n=2}^{\infty}\frac{4(1-\beta)\widetilde{r}^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\\&=1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\\&=d(f(0),\partial f(\mathbb{D})).\end{aligned}$$

This shows that $\widetilde{r}$ is the best possible. □

Proof of Corollary 4.3. Let $f\in\mathscr{W}_{\mathscr{H}}^0(\alpha,\beta,\gamma)$ be given by (1.1). Then for $\alpha=2$, $\beta=0$, $\gamma=1/2$, in view of Theorem 3.4, Theorem 3.6 and (5.11), we obtain

$$|f(z)|+\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,|z|^n\leqslant r+4\sum_{n=2}^{\infty}\frac{2r^n}{n^3+n^2}, \tag{5.14}$$

it follows that

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{2r^n}{n^3+n^2}&=\sum_{n=2}^{\infty}\frac{2r^n}{n^2}+\sum_{n=2}^{\infty}\frac{2r^n}{n+1}-\sum_{n=2}^{\infty}\frac{2r^n}{n}\\&=-2r+2\sum_{n=1}^{\infty}\frac{r^n}{n^2}+\frac{2}{r}\left(-\frac{r^2}{2}-r+\sum_{n=1}^{\infty}\frac{r^n}{n}\right)+2r-2\sum_{n=1}^{\infty}\frac{r^n}{n}\\&=-2r+2\mathrm{Li}_2(r)+\frac{2}{r}\left[-\frac{r^2}{2}-r-\log(1-r)\right]+2r+2\log(1-r)\\&=2\mathrm{Li}_2(r)-(r+2)+\frac{2(r-1)\log(1-r)}{r}.\end{aligned}$$

On the other hand, we know that

$$\sum_{n=2}^{\infty}\frac{2(-1)^{n-1}}{n^3+n^2}=1+\frac{\pi^2}{6}-4\log 2.$$

In view of (5.1) and (5.14), we have

$$|f(z)|+\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,|z|^n\leqslant r+4\left[2\mathrm{Li}_2(r)-(r+2)+\frac{2(r-1)\log(1-r)}{r}\right].$$

Moreover,

$$r+4\left[2\mathrm{Li}_2(r)-(r+2)+\frac{2(r-1)\log(1-r)}{r}\right]\leqslant 1+2\left(1+\frac{\pi^2}{6}-4\log 2\right).$$

for $r\leqslant \widetilde{r}_*(2,0,1/2):=\widetilde{r}_*$, where $\widetilde{r}_*$ is a root of $F_3(r)=0$ in $(0,1)$ and $F_3:[0,1]\to\mathbb{R}$ is defined by

$$\begin{aligned}F_3(r):&=r+4\left[2\mathrm{Li}_2(r)-(r+2)+\frac{2(r-1)\log(1-r)}{r}\right]-1-2\left(1+\frac{\pi^2}{6}-4\log 2\right)\\&=8\mathrm{Li}_2(r)+r-4(r+2)+\frac{8(r-1)\log(1-r)}{r}-\frac{\pi^2}{3}-3+8\log 2.\end{aligned}$$

By using the same argument as in the proof of Theorem 4.2, we can show that $F_3(r)$ has a unique root $\widetilde{r}_*\approx 0.521468$ and $\widetilde{r}_*\approx 0.521468$ is the best possible. □

Proof of Theorem 4.3. Let $f\in\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$ be given by (1.1). Then, by Theorem 3.4, Theorem 3.6, and (5.1), for $|z|=r$, we obtain

$$\begin{aligned}&|f(z^m)|+\sum_{n=N}^{\infty}(|a_n|+|b_n|)\,|z|^n\\&\leqslant |z|^m+\sum_{n=2}^{\infty}\frac{2(1-\beta)|z|^{mn}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}+\sum_{n=N}^{\infty}\frac{2(1-\beta)|z|^{n}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\\&=r^m+\sum_{n=2}^{\infty}\frac{2(1-\beta)r^{mn}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}+\sum_{n=N}^{\infty}\frac{2(1-\beta)r^{n}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}.\end{aligned}\tag{5.15}$$

It follows that

$$\begin{aligned}&r^m+\sum_{n=2}^{\infty}\frac{2(1-\beta)r^{mn}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}+\sum_{n=N}^{\infty}\frac{2(1-\beta)r^{n}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\\&\qquad\leqslant 1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\end{aligned}$$

for $r\leqslant R_{m,N}(\alpha)$, where $R_{m,N}(\alpha)$ is a root of $\Theta_3(r)=0$ in $(0,1)$, and $\Theta_3:[0,1]\to\mathbb{R}$ is defined by

$$\begin{aligned}\Theta_3(r):=r^m&+\sum_{n=2}^{\infty}\frac{2(1-\beta)r^{mn}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\\&+\sum_{n=N}^{\infty}\frac{2(1-\beta)r^{n}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}-1\\&-\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}.\end{aligned}$$

We deduce that $\Theta_3(0)\Theta_3(1) < 0$. Further, $\Theta_3(r)$ is strictly increasing basing on the fact that

$$\Theta_3'(r) = mr^{m-1} + \sum_{n=2}^{\infty} \frac{2(1-\beta)mnr^{mn-1}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} + \sum_{n=N}^{\infty} \frac{2(1-\beta)nr^{n-1}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} > 0.$$

The function Θ_3 is differentiable and strictly increasing on $(0,1)$, which asserts that $\Theta_3(r)$ has a unique root in $(0,1)$, say $R_{m,N}(\alpha,\beta,\gamma) := R_{m,N}$. Therefore, we have $\Theta_3(R_{m,N}) = 0$, or equivalently,

$$R_{m,N}^m + \sum_{n=2}^{\infty} \frac{2(1-\beta)R_{m,N}^{mn}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} + \sum_{n=N}^{\infty} \frac{2(1-\beta)R_{m,N}^{n}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} = 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n}. \tag{5.16}$$

To show that $R_{m,N}$ is the best possible, we consider the function $f = f_{(\alpha,\beta,\gamma)}$ defined by (3.3). In view of (5.4), (5.15) and (5.16), for $f = f_{(\alpha,\beta,\gamma)}$ and $z = R_{m,N}$, we obtain

$$\begin{aligned}
&|f_{(\alpha,\beta,\gamma)}(z^m)| + \sum_{n=N}^{\infty} (|a_n| + |b_n|)\,|z|^n \\
&= R_{m,N}^m + \sum_{n=2}^{\infty} \frac{2(1-\beta)R_{m,N}^{mn}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} \\
&\quad + \sum_{n=N}^{\infty} \frac{2(1-\beta)R_{m,N}^{n}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} \\
&= 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n} \\
&= d(f_{(\alpha,\beta,\gamma)}(0), \partial f_{(\alpha,\beta,\gamma)}(\mathbb{D})).
\end{aligned}$$

This shows that $R_{m,N}$ is the best possible. □

Proof of Theorem 4.4. For $f \in \mathscr{W}_{\mathscr{H}}^0(\alpha,\beta,\gamma)$, the Jacobian of f is given by

$$J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2.$$

It is well-known that (see [15, p.113]) the area of the disk $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ under the harmonic mapping $f = h + \overline{g}$ is

$$S_r = \iint_{\mathbb{D}_r} J_f(z)dxdy = \iint_{\mathbb{D}_r} \left(|h'(z)|^2 - |g'(z)|^2\right) dxdy = \pi \sum_{n=1}^{\infty} n(|a_n| + |b_n|)(|a_n| - |b_n|)r^{2n}. \tag{5.17}$$

In view of Theorem 3.4, and (5.17), we have

$$\begin{aligned}\frac{S_r}{\pi} &= \frac{1}{\pi}\iint\limits_{\mathbb{D}_r} \left(|h'(z)|^2 - |g'(z)|^2\right) dxdy \\ &= r^2 + \sum_{n=2}^{\infty} n\left(|a_n|^2 - |b_n|^2\right) r^{2n} \\ &= r^2 + \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 n r^{2n}}{[\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n]^2}.\end{aligned} \tag{5.18}$$

(i) By virtue of Theorem 3.4, Theorem 3.6 and (5.18) for $|z| = r$, we obtain

$$\begin{aligned}|z| + \sum_{n=2}^{\infty}(|a_n| + |b_n|)|z|^n + \frac{S_r}{\pi} \leqslant r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n} \\ + r^2 + \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 n r^{2n}}{[\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n]^2}.\end{aligned} \tag{5.19}$$

Moreover,

$$\begin{aligned}r^2 + r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n} \\ + \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 n r^{2n}}{[\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n]^2} \\ \leqslant 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n}.\end{aligned}$$

for $r \leqslant r_f(\alpha,\beta,\gamma) := r_f$, where r_f is the root of $\Theta_4(r) = 0$, where $\Theta_4 : [0,1] \to \mathbb{R}$ is defined by

$$\begin{aligned}\Theta_4(r) := r^2 + r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n} \\ + \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 n r^{2n}}{[\gamma n^3 + (\alpha - 3\gamma)n^2 + (1-\alpha+2\gamma)n]^2} - 1 \\ - \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n}.\end{aligned}$$

It is easy to see that $\Theta_4(0)\Theta_4(1) < 0$ and $\Theta_4'(r) > 0$ for $r \in (0,1)$. Then, the function

Θ_4 has a unique root r_f in $(0,1)$. Therefore, we have

$$\begin{aligned} r_f^2 + r_f + \sum_{n=2}^{\infty} \frac{2(1-\beta)r_f^n}{\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n} \\ + \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 n r_f^{2n}}{[\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n]^2} \\ = 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n}. \end{aligned} \tag{5.20}$$

To show that r_f is the best possible, we consider the function $f = f_{(\alpha,\beta,\gamma)}$ given by (3.3). By (5.4), (5.19) and (5.20), for $f = f_{(\alpha,\beta,\gamma)}$ and $r = r_f$, we see that

$$\begin{aligned} &|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\,|z|^n + \frac{S_{r_f}}{\pi} \\ &= r_f^2 + r_f + \sum_{n=2}^{\infty} \frac{2(1-\beta)r_f^n}{\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n} \\ &\quad + \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 n r_f^{2n}}{[\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n]^2} \\ &= 1 + \sum_{n=2}^{\infty} \frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n} \\ &= d(f_{(\alpha,\beta,\gamma)}(0), \partial f_{(\alpha,\beta,\gamma)}(\mathbb{D})). \end{aligned}$$

Therefore, r_f is the best possible. This completes the proof of (i).

(ii) In view of Theorem 3.4, Theorem 3.6 and (5.18) for $|z| = r$, we obtain

$$\begin{aligned} &|f(z)|^2 + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\,|z|^n + \left(\frac{S_r}{\pi}\right)^2 \\ &\leqslant \left(|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)|z|^n\right)^2 + \sum_{n=2}^{\infty} (|a_n| + |b_n|)|z|^n \\ &\quad + \left(r^2 + \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 n r^{2n}}{[\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n]^2}\right)^2 \\ &\leqslant \left(r + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n}\right)^2 \\ &\quad + \sum_{n=2}^{\infty} \frac{2(1-\beta)r^n}{\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n} \\ &\quad + \left(r^2 + \sum_{n=2}^{\infty} \frac{4(1-\beta)^2 n r^{2n}}{[\gamma n^3 + (\alpha-3\gamma)n^2 + (1-\alpha+2\gamma)n]^2}\right)^2. \end{aligned} \tag{5.21}$$

It follows that

$$\left(r+\sum_{n=2}^{\infty}\frac{2(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\right)^2 + \sum_{n=2}^{\infty}\frac{2(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n} + \left(r^2+\sum_{n=2}^{\infty}\frac{4(1-\beta)^2 n r^{2n}}{[\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n]^2}\right)^2 \leqslant 1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}$$

for $r\leqslant r_f^*(\alpha,\beta,\gamma):=r_f^*$, where r_f^* is a root of $\Theta_5(r)=0$ in $(0,1)$, and $\Theta_5:[0,1]\to\mathbb{R}$ is defined by

$$\Theta_5(r)=\left(r+\sum_{n=2}^{\infty}\frac{2(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\right)^2 + \sum_{n=2}^{\infty}\frac{2(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n} + \left(r^2+\sum_{n=2}^{\infty}\frac{4(1-\beta)^2 n r^{2n}}{[\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n]^2}\right)^2-1 - \sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}.$$

By using the same argument as in the proof of Theorem 4.1, we can easily show that $\Theta_5(r)$ has a unique root r_f^*, and r_f^* is the best possible. □

Proof of Corollary 4.4. (i) Let $f\in\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$, then for $|z|=r$ and $\alpha=2$, $\gamma=1/2$, $\beta=0$, using Theorem 3.4, we obtain

$$|z|+\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,|z|^n+\frac{S_r}{\pi}\leqslant r^2+r+\sum_{n=2}^{\infty}\frac{4r^n}{n^3+n^2}+\sum_{n=2}^{\infty}\frac{16nr^{2n}}{(n^3+n^2)^2}. \tag{5.22}$$

It follows from (5.22) that

$$\sum_{n=2}^{\infty}\frac{4r^n}{n^3+n^2}=4\mathrm{Li}_2(r)-2(r+2)+\frac{4(r-1)\log(1-r)}{r},$$

and

$$\sum_{n=2}^{\infty}\frac{4(-1)^{n-1}}{n^3+n^2}=\frac{\pi^2}{3}+2-8\log 2.$$

On the other hand, we find that

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{16nr^{2n}}{(n^3+n^2)^2}&=\sum_{n=2}^{\infty}\frac{16r^{2n}}{n^3}+\sum_{n=2}^{\infty}-\frac{32r^{2n}}{n^2}+\sum_{n=2}^{\infty}\frac{48r^{2n}}{n}+\sum_{n=2}^{\infty}-\frac{16r^{2n}}{(n+1)^2}+\sum_{n=2}^{\infty}-\frac{48r^{2n}}{n+1}\\&=4\left[\left(-\frac{4}{r^2}-8\right)\mathrm{Li}_2\left(r^2\right)+4\mathrm{Li}_3\left(r^2\right)-r^2\right.\\&\quad\left.+\frac{12\log\left(1-r^2\right)}{r^2}-12\log\left(1-r^2\right)+16\right].\end{aligned}$$

Therefore, we have

$$\begin{aligned}&r^2+r+\sum_{n=2}^{\infty}\frac{4r^n}{n^3+n^2}+\sum_{n=2}^{\infty}\frac{16nr^{2n}}{(n^3+n^2)^2}\\&=r^2+r+4\mathrm{Li}_2(r)-2(r+2)+\frac{4(r-1)\log(1-r)}{r}+4\left[\left(-\frac{4}{r^2}-8\right)\mathrm{Li}_2\left(r^2\right)\right.\\&\qquad\left.+4\mathrm{Li}_3\left(r^2\right)-r^2+\frac{12\log\left(1-r^2\right)}{r^2}-12\log\left(1-r^2\right)+16\right]\\&=4\left(-\frac{4}{r^2}-8\right)\mathrm{Li}_2\left(r^2\right)+16\mathrm{Li}_3\left(r^2\right)+4\mathrm{Li}_2(r)-3r^2+\frac{48\log\left(1-r^2\right)}{r^2}\\&\qquad-48\log\left(1-r^2\right)+r-2(r+2)+\frac{4(r-1)\log(1-r)}{r}+64.\end{aligned}\tag{5.23}$$

It is easy to see that

$$\begin{aligned}&4\left(-\frac{4}{r^2}-8\right)\mathrm{Li}_2\left(r^2\right)+16\mathrm{Li}_3\left(r^2\right)+4\mathrm{Li}_2(r)-3r^2+\frac{48\log\left(1-r^2\right)}{r^2}\\&\qquad-48\log\left(1-r^2\right)+r-2(r+2)+\frac{4(r-1)\log(1-r)}{r}+64\\&\leqslant 1+\left(\frac{\pi^2}{3}+2-8\log 2\right)\end{aligned}$$

for $r\leqslant \dot{r}_f(2,0,1/2):=\dot{r}_f$, where $\dot{r}_f$ is root of $F_4(r)=0$ in $(0,1)$ and $F_4:[0,1]\to\mathbb{R}$ is defined by

$$\begin{aligned}F_4(r):=&4\left(-\frac{4}{r^2}-8\right)\mathrm{Li}_2\left(r^2\right)+16\mathrm{Li}_3\left(r^2\right)+4\mathrm{Li}_2(r)-3r^2+\frac{48\log\left(1-r^2\right)}{r^2}\\&-48\log\left(1-r^2\right)+r-2(r+2)+\frac{4(r-1)\log(1-r)}{r}-\frac{\pi^2}{3}+61+8\log 2.\end{aligned}$$

By using the standard argument as in the proof of Theorem 4.4, we can show that $F_4(r)$ has a unique root $\dot{r}_f\approx 0.451007$ in $(0,1)$, and $\dot{r}$ is the best possible. This completes the proof of (i).

(ii) By Theorem 3.4, for $|z|=r$, it implies that

$$\begin{aligned}
&|f(z)|^2+\sum_{n=2}^{\infty}(|a_n|+|b_n|)|z|^n+\left(\frac{S_r}{\pi}\right)^2\\
&\leqslant\left(r+\sum_{n=2}^{\infty}\frac{4r^n}{n^3+n^2}\right)^2+\sum_{n=2}^{\infty}\frac{4r^n}{n^3+n^2}+\left(r^2+\sum_{n=2}^{\infty}\frac{16nr^{2n}}{(n^3+n^2)^2}\right)^2\\
&=\left[4\left(-\frac{4}{r^2}-8\right)\mathrm{Li}_2\left(r^2\right)+16\mathrm{Li}_3\left(r^2\right)-3r^2+\frac{48\log\left(1-r^2\right)}{r^2}-48\log\left(1-r^2\right)+64\right]^2\\
&\quad+4\mathrm{Li}_2(r)+\left[-4\mathrm{Li}_2(r)+r+\left(\frac{4}{r}-4\right)\log(1-r)+4\right]^2\\
&\quad-2(r+2)+\frac{4(r-1)\log(1-r)}{r}.
\end{aligned}\tag{5.24}$$

It follows that

$$\begin{aligned}
&\left[4\left(-\frac{4}{r^2}-8\right)\mathrm{Li}_2\left(r^2\right)+16\mathrm{Li}_3\left(r^2\right)-3r^2+\frac{48\log\left(1-r^2\right)}{r^2}-48\log\left(1-r^2\right)+64\right]^2\\
&\quad+4\mathrm{Li}_2(r)+\left[-4\mathrm{Li}_2(r)+r+\left(\frac{4}{r}-4\right)\log(1-r)+4\right]^2-2(r+2)+\frac{4(r-1)\log(1-r)}{r}\\
&\leqslant 1+\left(\frac{\pi^2}{3}+2-8\log 2\right)
\end{aligned}$$

for $r\leqslant\ddot{r}_f(2,0,1/2):=\ddot{r}_f$, where $\ddot{r}_f$ is a root of $F_5(r)=0$ in $(0,1)$ and $F_5:[0,1]\to\mathbb{R}$ is defined by

$$\begin{aligned}
F_5:=&\left[4\left(-\frac{4}{r^2}-8\right)\mathrm{Li}_2\left(r^2\right)+16\mathrm{Li}_3\left(r^2\right)-3r^2+\frac{48\log\left(1-r^2\right)}{r^2}-48\log\left(1-r^2\right)+64\right]^2\\
&\quad+4\mathrm{Li}_2(r)+\left[-4\mathrm{Li}_2(r)+r+\left(\frac{4}{r}-4\right)\log(1-r)+4\right]^2\\
&\quad\quad-2(r+2)+\frac{4(r-1)\log(1-r)}{r}-\frac{\pi^2}{3}-3+8\log 2.
\end{aligned}$$

Furthermore, we can deduce that $F_5(r)$ has a unique root in $(0,1)$. Let $\ddot{r}_f$ be the root of $F_5(r)$. Then we have $\ddot{r}_f\approx 0.566259$ and $\ddot{r}_f$ is the best possible. This completes the proof of (ii). $\square$

Proof of Theorem 4.5. (i) Let $f\in\mathscr{W}^0_{\mathscr{H}}(\alpha,\beta,\gamma)$, then for $|z|=r$, by Theorem 3.4, if

$$2r+\sum_{n=2}^{\infty}\frac{4(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\leqslant 1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n},$$

we obtain

$$|z|+|h(z)|+\sum_{n=2}^{\infty}|a_n||z|^n \leqslant d(f(0),\partial f(\mathbb{D})). \tag{5.25}$$

Let $\Theta_6:[0,1]\to\mathbb{R}$ be defined by

$$\Theta_6(r):=2r+\sum_{n=2}^{\infty}\frac{4(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}-1 \\ -\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}.$$

It is not difficult to show that Θ_6 has a unique root in $(0,1)$. Let $r_h(\alpha,\beta,\gamma):=r_h$ be the root of $\Theta_6(r)$ and hence,

$$2r_h+\sum_{n=2}^{\infty}\frac{4(1-\beta)r_h^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}=1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}. \tag{5.26}$$

It needs to show that r_h is the best possible. To prove this, considering the function $f=f_{(\alpha,\beta,\gamma)}=h_{(\alpha,\beta,\gamma)}+\overline{g_{(\alpha,\beta,\gamma)}}$ given by (3.3). By using (5.4), (5.25) and (5.26), for $f=f_{(\alpha,\beta,\gamma)}$ and $z=r_h$ shows that

$$\begin{aligned}|z|+|h_{(\alpha,\beta,\gamma)}(z)|+\sum_{n=2}^{\infty}|a_n||z|^n &= 2r_h+\sum_{n=2}^{\infty}\frac{4(1-\beta)r_h^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\\ &=1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\\ &=d(f_{(\alpha,\beta,\gamma)}(0),\partial f_{(\alpha,\beta,\gamma)}(\mathbb{D})),\end{aligned}$$

which shows that r_h is the best possible. This completes the proof of (i).

(ii) Let $\Theta_7:[0,1]\to\mathbb{R}$ be defined by

$$\Theta_7(r):=r+\sum_{n=2}^{\infty}\frac{2(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}-1 \\ -\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}.$$

We can show that $\Theta_6(r)$ has a unique root in $(0,1)$. For convenience, we denote by $r_g(\alpha,\beta,\gamma):=r_g$. Therefore,

$$r_g(\alpha)+\sum_{n=2}^{\infty}\frac{2(1-\beta)r_g^n(\alpha)}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}=1+\sum_{n=2}^{\infty}\frac{(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}. \tag{5.27}$$

For $|z|=r$, by Theorem 3.3, if

$$r+\sum_{n=2}^{\infty}\frac{2(1-\beta)r^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\leqslant 1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}$$

with $r \leqslant r_g$, we obtain

$$|z|+|g(z)|+\sum_{n=2}^{\infty}|b_n||z|^n \leqslant d(f(0),\partial f(\mathbb{D})). \tag{5.28}$$

In order to show that r_g is the best possible, we consider the function $f = f^*_{(\alpha,\beta,\gamma)}$ defined by (3.3). In view of (5.1), it is easy to see that

$$d(f^*_{(\alpha,\beta,\gamma)}(0),\partial f^*_{(\alpha,\beta,\gamma)}(\mathbb{D})) = 1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}. \tag{5.29}$$

By using (5.27), (5.28) and (5.29), for $f = f^*_{(\alpha,\beta,\gamma)}$ and $z = r_g$, we know that

$$\begin{aligned}|z|+|g(z)|+\sum_{n=2}^{\infty}|b_n||z|^n &= r_g+\sum_{n=2}^{\infty}\frac{2(1-\beta)r_g^n}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\\ &= 1+\sum_{n=2}^{\infty}\frac{2(1-\beta)(-1)^{n-1}}{\gamma n^3+(\alpha-3\gamma)n^2+(1-\alpha+2\gamma)n}\\ &= d(f^*_{(\alpha,\beta,\gamma)}(0),\partial f^*_{(\alpha,\beta,\gamma)}(\mathbb{D})).\end{aligned}$$

Therefore, we deduce that r_g is the best possible. □

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Xiao-Yuan Wang
School of Mathematics and Statistics
Nanjing University of Science and Technology
Nanjing 210094, Jiangsu, P. R. China
e-mail: mewangxiaoyuan@163.com

Zhi-Gang Wang
School of Mathematics and Statistics
Hunan First Normal University
Changsha 410205, Hunan, P. R. China
e-mail: wangmath@163.com

Jin-Hua Fan
School of Mathematics and Statistics
Nanjing University of Science and Technology
Nanjing 210094, Jiangsu, P. R. China
e-mail: jinhuafan@hotmail.com

Zhen-Yong Hu
School of Mathematics and Statistics
Nanjing University of Science and Technology
Nanjing 210094, Jiangsu, P. R. China
e-mail: huzhenyongad@163.com