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CONSTRUCTION OF THE KANTOROVICH VARIANT OF THE BERNSTEIN–CHLODOVSKY OPERATORS BASED ON PARAMETER α

BO-YONG LIAN AND QING-BO CAI*

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Abstract. In this article, a new family of kantorovich variant of Chlodovsky operators is introduced. The authors establish some approximation theorems, such as a direct approximation by means of the Ditzian-Totik modulus of smoothness, a global approximation theorem in terms of second order modulus of continuity and so on. Furthermore, a voronovskaja type asymptotic estimate formula is presented. Finally, the rate of convergence for some absolutely continuous functions having a derivative equivalent to a bounded variation function is obtained.

1. Introduction

For $0 \leqslant \alpha \leqslant 1$ and $x \in [0,1]$, Chen et al. [1] introduced a new family of generalized Bernstein operators as follows:

$$T_{n,\alpha}(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{(\alpha)}(x), \tag{1}$$

where

$$\begin{aligned} p_{1,0}^{(\alpha)}(x) &= 1-x, \\ p_{1,1}^{(\alpha)}(x) &= x, \\ p_{n,k}^{(\alpha)}(x) &= \left[\binom{n-2}{k}(1-\alpha)x + \binom{n-2}{k-2}(1-\alpha)(1-x) + \binom{n}{k}\alpha x(1-x)\right] x^{k-1}(1-x)^{n-k-1} \end{aligned}$$

for $n \geqslant 2$ and $\binom{n}{k} = 0$ $(k > n)$. When $\alpha = 1$, the operators $T_{n,\alpha}$ reduces to the classical Bernstein operators.

In [1], the authors studied many approximaiton properties of $T_{n,\alpha}$ such as uniform convergence, rate of convergence in terms of modulus of continuity, voronovskaja-type

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* Corresponding author.

asymptotic formula, and shape preserving properties. After that, the variant of the operators $T_{n,\alpha}$, such as Kantorovich type operators, Durrmeyer type operators, Complex type operators, Stancu type operators, q-Kantorovich operators, have been studied by a lot of researchers, see for examples [2, 3, 4, 5, 6].

In 1937, in order to generalize the Bernstein operators, Chlodovsky [7] introduced the operators $C_n(f,x)$, which are defined by

$$C_n(f,x) = \sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right) p_{n,k}\left(\frac{x}{b_n}\right), \tag{2}$$

where $x \in [0,b_n]$, $p_{n,k}(\frac{x}{b_n}) = \binom{n}{k}(\frac{x}{b_n})^k(1-\frac{x}{b_n})^{n-k}$, and $(b_n)_{n=1}^{\infty}$ is a sequence of increasing positive numbers with the properties $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} b_n/n = 0$.

Many scholars have done a lot of relevant research work on $C_n(f,x)$ and the related operators, we can see references [8, 9, 10, 11] and some other studies about positive linear operators [12, 13, 14, 15, 16, 17].

Base on the operators of (1) and (2), Smuc [18] proposed a new family of Chlodovsky operators $C_{n,\alpha}(f,x)$ in the following way:

$$C_{n,\alpha}(f,x) = \sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right) p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right). \tag{3}$$

When $\alpha = 1$, the operators $C_{n,\alpha}$ reduces to the Chlodovsky operators C_n. When $b_n = 1$, the operators $C_{n,\alpha}$ reduces to the operators T_n. In [18], the author studied some results concerning uniform convergence and estimates of the degree of approximation.

To approximate the integrable functions on $[0,b_n]$, we construct the kantorovich variant of the operators (3) which are defined by

$$CK_{n,\alpha}(f,x) = \frac{n+1}{b_n} \sum_{k=0}^{n} p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right) \int_{\frac{kb_n}{n+1}}^{\frac{(k+1)b_n}{n+1}} f(t)dt. \tag{4}$$

Also, $CK_{n,\alpha}$ reduce to the operators discussed by Mohiuddine [2] and Pych-Taberska [19, 20] for $b_n = 1$ and $\alpha = 1$ respectively. For more details about the kantorivich operators, we refer to [21, 22].

2. Preliminaries

LEMMA 1. [18] *For $x \in [0,b_n]$ and $C_{n,\alpha}(f,x)$ defined by (3), we have*

$$C_{n,\alpha}(1,x) = 1,$$

$$C_{n,\alpha}(t,x) = x,$$

$$C_{n,\alpha}(t^2,x) = x^2 + \frac{n+2(1-\alpha)}{n^2} \cdot x(b_n - x).$$

LEMMA 2. *For $x \in [0, b_n]$, we have*

$$CK_{n,\alpha}(1,x) = 1, \tag{5}$$

$$CK_{n,\alpha}(t,x) = x + \frac{b_n - 2x}{2n+2}, \tag{6}$$

$$CK_{n,\alpha}(t^2,x) = x^2 + \frac{n(4b_n x - 3x^2 - 6) + b_n^2 + 6(1-\alpha)x(b_n - x) - 3}{3(n+1)^2}. \tag{7}$$

Proof. By Lemma 1, we have

$$CK_{n,\alpha}(1,x) = \frac{n+1}{b_n} \sum_{k=0}^{n} p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right) \int_{\frac{kb_n}{n+1}}^{\frac{(k+1)b_n}{n+1}} 1dt = C_{n,\alpha}(1,x) = 1.$$

$$\begin{aligned} CK_{n,\alpha}(t,x) &= \frac{n+1}{b_n} \sum_{k=0}^{n} p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right) \int_{\frac{kb_n}{n+1}}^{\frac{(k+1)b_n}{n+1}} t dt \\ &= \frac{n}{n+1} \sum_{k=0}^{n} \frac{kb_n}{n} p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right) + \frac{b_n}{2n+2} \sum_{k=0}^{n} p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right) \\ &= \frac{n}{n+1} C_{n,\alpha}(t,x) + \frac{b_n}{2n+2} C_{n,\alpha}(1,x) \\ &= x + \frac{b_n - 2x}{2n+2}. \end{aligned}$$

$$\begin{aligned} CK_{n,\alpha}(t^2,x) &= \frac{n+1}{b_n} \sum_{k=0}^{n} p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right) \int_{\frac{kb_n}{n+1}}^{\frac{(k+1)b_n}{n+1}} t^2 dt \\ &= \left(\frac{n}{n+1}\right)^2 \sum_{k=0}^{n} \left(\frac{kb_n}{n}\right)^2 p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right) + \frac{nb_n}{(n+1)^2} \sum_{k=0}^{n} \frac{kb_n}{n} p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right) \\ &\quad + \frac{b_n^2}{3(n+1)^2} \sum_{k=0}^{n} p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right) \\ &= \left(\frac{n}{n+1}\right)^2 C_{n,\alpha}(t^2,x) + \frac{nb_n}{(n+1)^2} C_{n,\alpha}(t,x) + \frac{b_n^2}{3(n+1)^2} C_{n,\alpha}(1,x) \\ &= x^2 + \frac{3nx(2b_n - 3x) - 3x^2 + 6(1-\alpha)x(b_n - x) + b_n^2}{3(n+1)^2}. \quad \square \end{aligned}$$

By Lemma 2 and Cauchy Schwarz inequality, we get

$$CK_{n,\alpha}(t-x,x) = \frac{b_n - 2x}{2n+2}, \tag{8}$$

$$CK_{n,\alpha}((t-x)^2,x) = \frac{3x(b_n - x)(n+1-2\alpha) + b_n^2}{3(n+1)^2} = \eta_{n\alpha}^2(x). \tag{9}$$

$$CK_{n,\alpha}(|t-x|,x) \leqslant \sqrt{CK_{n,\alpha}((t-x)^2,x)} \cdot \sqrt{CK_{n,\alpha}(1,x)} = \eta_{n\alpha}(x). \tag{10}$$

Using the same methods, and after some easy but tedious computations, we also can obtain the following result:

$$CK_{n,\alpha}((t-x)^4,x) = O\left(\frac{b_n^2}{n^2}\right). \tag{11}$$

Let $C_B[0,\infty)$ denote the space of all real-valued bounded and uniformly continuous functions f on $[0,\infty)$, endowed with the norm $\|f\| = sup_{x\in[0,\infty)}|f|$.

LEMMA 3. *For $f \in C_B[0,\infty)$, $x \in [0,\infty)$, the following inequalities hold*

$$\|CK_{n,\alpha}(f,x)\| \leqslant \|f\|. \tag{12}$$

Proof. Since $CK_{n,\alpha}(1,x) = 1$, we get

$$\|CK_{n,\alpha}(f,x)\| \leqslant CK_{n,\alpha}(1,x) \cdot \|f\| = \|f\|. \quad \square$$

Let

$$\vartheta_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right) = \sum_{k=0}^{n} \frac{n+1}{b_n} p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right)\chi_k(t)$$

and

$$\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right) = \int_0^t \vartheta_{n,\alpha}\left(\frac{x}{b_n},\frac{s}{b_n}\right) ds,$$

where $\chi_k(t)$ is the characteristic function of the interval $[\frac{kb_n}{n+1},\frac{(k+1)b_n}{n+1}]$ with respect to $I = [0,b_n]$. By the Lebesgue-Stieltjes integral representations, we have

$$CK_{n,\alpha}(f,x) = \int_0^{b_n} f(t)\vartheta_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right) dt = \int_0^{b_n} f(t) d_t\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right). \tag{13}$$

LEMMA 4. *(i) For $0 \leqslant y < x < b_n$, there holds*

$$\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{y}{b_n}\right) = \int_0^y \vartheta_{n,\alpha}\left(\frac{x}{b_n},\frac{s}{b_n}\right) ds \leqslant \frac{1}{(x-y)^2}\eta_{n\alpha}^2(x). \tag{14}$$

(ii) For $0 < x < z < b_n$, there holds

$$1 - \lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{z}{b_n}\right) = \int_z^1 \vartheta_{n,\alpha}\left(\frac{x}{b_n},\frac{s}{b_n}\right) ds \leqslant \frac{1}{(x-z)^2}\eta_{n\alpha}^2(x). \tag{15}$$

Proof. (i) By (9) and (13), we get

$$\begin{aligned}
\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{y}{b_n}\right) &= \int_0^y \vartheta_{n,\alpha}(\frac{x}{b_n},\frac{t}{b_n})dt \\
&\leqslant \int_0^y \left(\frac{t-x}{x-y}\right)^2 d_t\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right) \\
&\leqslant \frac{1}{(x-y)^2}\int_0^{b_n}(t-x)^2 d_t\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right) \\
&= \frac{1}{(x-y)^2} CK_{n,\alpha}((t-x)^2,x) \\
&= \frac{1}{(x-y)^2}\eta_{n\alpha}^2(x).
\end{aligned}$$

(ii) Using a similar method, we can get (ii) easily. □

3. Main results

THEOREM 1. *Let* $f \in C_B[0,\infty)$*, then*

$$\lim_{n\to\infty} CK_{n,\alpha}(f,x) = f(x),$$

uniformly in each compact subset of $[0,\infty)$.

Proof. From (5) and (9), we get

$$\lim_{n\to\infty} CK_{n,\alpha}(1,x) = 1,$$

$$\lim_{n\to\infty} CK_{n,\alpha}((t-x)^2,x) = 0.$$

Hence by Theorem 3.2 of [23], we get Theorem 1 immediately. □

Now we give the rate of convergence of the operators by means of the modulus of continuity which is denoted by $\omega(f;\delta)$.

Let $f \in C_B[0,\infty)$ and $\forall x_1,x_2 \in [0,\infty)$, the definition of the modulus of continuity of f is given by

$$\omega(f;\delta) = \sup_{|x_1-x_2|\leqslant\delta} |f(x_1)-f(x_2)|.$$

THEOREM 2. *For* $f \in C_B[0,b_n]$ *and* $x \in [0,b_n]$*, we have*

$$\left|CK_{n,\alpha}(f,x) - f(x)\right| \leqslant 2\omega\left(f;\eta_{n\alpha}(x)\right). \tag{16}$$

Proof. In view of $CK_{n,\alpha}(1,x)=1$,

$$\begin{aligned}|CK_{n,\alpha}(f,x)-f(x)| &= \left|\sum_{k=0}^{n}\left[f\left(\frac{kb_n}{n}\right)-f(x)\right]p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right)\right| \\ &\leqslant \sum_{k=0}^{n}\left|f\left(\frac{kb_n}{n}\right)-f(x)\right|p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right) \\ &\leqslant \sum_{k=0}^{n}\omega\left(f;\left|\frac{kb_n}{n}-x\right|\right)p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right).\end{aligned}$$

As we know $\omega(f;\lambda\delta)\leqslant(1+\lambda)\omega(f;\delta)$ for $\lambda>0$, so

$$\begin{aligned}\omega\left(f;\left|\frac{kb_n}{n}-x\right|\right) &= \omega\left(f;\frac{\left|\frac{kb_n}{n}-x\right|}{\eta_{n\alpha}(x)}\cdot\eta_{n\alpha}(x)\right) \\ &\leqslant \left(1+\frac{\left|\frac{kb_n}{n}-x\right|}{\eta_{n\alpha}(x)}\right)\cdot\omega\left(f;\eta_{n\alpha}(x)\right).\end{aligned}$$

Then

$$\begin{aligned}|CK_{n,\alpha}(f,x)-f(x)| &\leqslant \sum_{k=0}^{n}\left(1+\frac{\left|\frac{kb_n}{n}-x\right|}{\eta_{n\alpha}(x)}\right)\cdot\omega\left(f;\eta_{n\alpha}(x)\right)\cdot p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right) \\ &= \left(1+\frac{1}{\eta_{n\alpha}(x)}\sum_{k=0}^{n}\left|\frac{kb_n}{n}-x\right|p_{n,k}^{(\alpha)}\left(\frac{x}{b_n}\right)\right)\cdot\omega\left(f;\eta_{n\alpha}(x)\right) \\ &= \left(1+\frac{1}{\eta_{n\alpha}(x)}\cdot CK_{n,\alpha}(|t-x|,x)\right)\cdot\omega\left(f;\eta_{n\alpha}(x)\right) \\ &\leqslant 2\omega\left(f;\eta_{n\alpha}(x)\right).\end{aligned}$$

The last inequality is obtained by (10). □

REMARK 1. When $b_n=1$, Theorem 2 is the form of the Theorem 1 of Mohiuddine [2].

For $t>0$ and $W^2[0,\infty)=\{g\in C_B[0,\infty): g''\in C_B[0,\infty)\}$, the appropriate Peetre's K-functional is defined by

$$K_2(f,t)=\inf_{g\in W^2[0,\infty)}\{\|f-g\|+t\|g''\|\}.$$

Let

$$\omega_2(f,t)=\sup_{0<|h|\leqslant t}\ \sup_{x,x+h,x+2h\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|,$$

where ω_2 is the second order modulus of continuity of $f\in C_B[0,\infty)$.

From [24], there exists an absolute constant $A>0$, such that

$$K_2(f,t)\leqslant A\omega_2(f,\sqrt{t}). \tag{17}$$

THEOREM 3. *For* $f \in C_B[0,b_n]$, *then there exists an absolute constant* $A > 0$, *such that*

$$|CK_{n,\alpha}(f,x) - f(x)| \leqslant A\omega_2 \left(f, \sqrt{\frac{1}{8}\left(\eta_{n\alpha}^2(x) + \left(\frac{b_n - 2x}{2n+2}\right)^2 \right)} \right) + \omega\left(f; \frac{|b_n - 2x|}{2n+2} \right). \tag{18}$$

Proof. For $f \in C_B[0,b_n]$, we consider the following auxiliary operators

$$\widehat{CK}_{n,\alpha}(f,x) = CK_{n,\alpha}(f,x) + f(x) - f\left(x + \frac{b_n - 2x}{2n+2} \right). \tag{19}$$

By Lemma 2, we get

$$\widehat{CK}_{n,\alpha}(1,x) = 1, \quad \widehat{CK}_{n,\alpha}(t,x) = x. \tag{20}$$

Let $g \in W^2$. By Taylor's expansion, we get

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$

By (20), we have

$$\widehat{CK}_{n,\alpha}(g,x) = g(x) + \widehat{CK}_{n,\alpha}\left(\int_x^t (t-u)g''(u)du, x \right).$$

So

$$\begin{aligned} \widehat{CK}_{n,\alpha}(g,x) - g(x) &= CK_{n,\alpha}\left(\int_x^t (t-u)g''(u)du, x \right) \\ &\quad - \int_x^{x+\frac{b_n-2x}{2n+2}} \left(x + \frac{b_n - 2x}{2n+2} - u \right) g''(u)du. \end{aligned}$$

As we know

$$\int_x^t (t-u)g''(u)du \leqslant \frac{\|g''\|}{2}(t-x)^2,$$

then

$$\begin{aligned} \left| \widehat{CK}_{n,\alpha}(g,x) - g(x) \right| &\leqslant \frac{\|g''\|}{2} CK_{n,\alpha}\left((t-x)^2; x \right) + \frac{\|g''\|}{2}\left(\frac{b_n - 2x}{2n+2} \right)^2 \\ &= \frac{\|g''\|}{2}\left(\eta_{n\alpha}^2(x) + \left(\frac{b_n - 2x}{2n+2} \right)^2 \right). \end{aligned}$$

Since the definition of $\widehat{CK}_{n,\alpha}(f,x)$ and Lemma 3, we know

$$\left| \widehat{CK}_{n,\alpha}(f,x) \right| \leqslant 3\|f\|.$$

Later, we have

$$\begin{aligned}|CK_{n,\alpha}(f,x)-f(x)| &\leqslant |\widehat{CK}_{n,\alpha}(f-g,x)|+|\widehat{CK}_{n,\alpha}(g,x)-g(x)|+|f-g| \\ &\quad +\left|f\left(\frac{b_n-2x}{2n+2}\right)-f(x)\right| \\ &\leqslant 4\|f-g\|+\frac{\|g''\|}{2}\left(\eta_{n\alpha}^2(x)+\left(\frac{b_n-2x}{2n+2}\right)^2\right) \\ &\quad +\omega\left(f;\frac{|b_n-2x|}{2n+2}\right).\end{aligned}$$

Taking the infimum on the right hand side over all $g\in W^2$, we obtain

$$|CK_{n,\alpha}(f,x)-f(x)|\leqslant 4K_2\left(f,\frac{1}{8}\left(\eta_{n\alpha}^2(x)+\left(\frac{b_n-2x}{2n+2}\right)^2\right)\right)+\omega\left(f;\frac{|b_n-2x|}{2n+2}\right).$$

By (17), we get (18) immediately. This completes the proof of Theorem 3. □

REMARK 2. When $b_n=1$, Theorem 3 is the form of the Theorem 2 of Mohiuddine [2].

Let $\phi(x)=\sqrt{x}$ and $f\in C_B[0,\infty)$, the first order Ditzian-Totik modulus of smoothness and corresponding K-functional are given by

$$\omega_\phi(f,t)=\sup_{0<h\leqslant t}\left|f\left(x+\frac{h\phi(x)}{2}\right)-f\left(x-\frac{h\phi(x)}{2}\right)\right|,\ \ x\pm\frac{h\phi(x)}{2}\in[0,\infty),$$

and

$$K_\phi(f,t)=\inf_{g\in W_\phi[0,\infty)}\{\|f-g\|+t\|\phi g'\|\}(t>0),$$

respectively. Here $W_\phi[0,\infty)=\{g:g\in AC[0,\infty),\|\phi g'\|<\infty\}$ means that g is differentiable and absolutely continuous on every compact subset of $[0,\infty)$. By [25], there exists a constant $B>0$ such that

$$K_\phi(f,t)\leqslant B\omega_\phi(f,t). \tag{21}$$

THEOREM 4. *For $f\in C_B(0,\infty)$, then there exists an absolute constant $B>0$, such that*

$$|CK_{n,\alpha}(f,x)-f(x)|\leqslant B\omega_\phi\left(f,\frac{\eta_{n\alpha}(x)}{\sqrt{x}}\right). \tag{22}$$

Proof. Applying the operators $C_{n,\alpha}(\cdot,x)$ to the representation

$$g(t)=g(x)+\int_x^t g'(u)du,$$

we have

$$CK_{n,\alpha}(g,x) = g(x) + CK_{n,\alpha}\left(\int_x^t g'(u)du, x\right).$$

For any $x,t \in (0,\infty)$, we can get

$$\left|\int_x^t g'(u)du\right| = \left|\int_x^t \frac{g'(u)\phi(u)}{\phi(u)}du\right| \leqslant \|\phi g'\| \left|\int_x^t \frac{1}{\phi(u)}du\right| \leqslant 2\|\phi g'\| \frac{|t-x|}{\phi(x)}.$$

By (10), we have

$$\begin{aligned}|CK_{n,\alpha}(g,x) - g(x)| &\leqslant 2\|\phi g'\| \cdot \phi^{-1}(x) \cdot CK_{n,\alpha}(|t-x|,x)\\ &\leqslant 2\|\phi g'\| \cdot \phi^{-1}(x) \cdot \eta_{n\alpha}(x).\end{aligned}$$

Thus

$$\begin{aligned}|CK_{n,\alpha}(f,x) - f(x)| &\leqslant |CK_{n,\alpha}(f-g,x)| + |f-g| + |CK_{n,\alpha}(g,x) - g(x)|\\ &\leqslant 2\|f-g\| + 2\|\phi g'\| \cdot \phi^{-1}(x) \cdot \eta_{n\alpha}(x).\end{aligned}$$

For all $g \in W_\phi(0,\infty)$, taking the infimum on the right hand side, we can get

$$|CK_{n,\alpha}(f,x) - f(x)| \leqslant 2K_\phi\left(f, \phi^{-1}(x) \cdot \eta_{n\alpha}(x)\right).$$

By (21) and the above inequality, we get (22) immediately. □

As we know, a function f belongs to the Lipschitz class $Lip_M(\beta)$ $(0 < \beta \leqslant 1, M > 0)$ if the inequality

$$|f(t) - f(x)| \leqslant M|t-x|^\beta$$

holds for all $t,x \in R$. Now we compute the rate of convergence of the operators $C_{n,\alpha}(f,x)$ for the Lipschitz class functions.

THEOREM 5. *For $x \in [0,\infty)$ and $f \in Lip_M(\beta) \bigcap C_B[0,\infty)$, we have*

$$\left|CK_{n,\alpha}(f,x) - f(x)\right| \leqslant M\left[\eta_{n\alpha}(x)\right]^\beta. \tag{23}$$

Proof. Applying the Hölder inequality with $p = \frac{2}{\beta}$, $q = \frac{2}{2-\beta}$, we get

$$\begin{aligned}\left|CK_{n,\alpha}(f,x)(f,x) - f(x)\right| &\leqslant CK_{n,\alpha}\left(|f(t) - f(x)|, x\right)\\ &\leqslant M \cdot CK_{n,\alpha}\left(|t-x|^\beta, x\right)\\ &\leqslant M \cdot \left[CK_{n,\alpha}((t-x)^2,x)\right]^{\beta/2} \cdot \left[CK_{n,\alpha}(1,x)\right]^{(2-\beta)/2}\\ &= M\left[\eta_{n\alpha}(x)\right]^\beta.\end{aligned}$$

The last equation is obtained by (5) and (9). □

Now, we give a Voronovskaja type asymptotic formula for the operators $CK_{n,\alpha}(f,x)$.

THEOREM 6. *Let $f \in C_B[0,\infty)$, if f'' exists at a point $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} \frac{n}{b_n}\left[CK_{n,\alpha}(f,x) - f(x)\right] = \frac{1}{2}f'(x) + \frac{x}{6}f''(x). \tag{24}$$

Proof. By Taylor's expansion, we may write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \phi(t;x)(t-x)^2,$$

where $\phi(t;x)$ is the Peano form of the remainder, and $\phi(t;x) \in C[0,\infty), \lim_{t\to x}\phi(t;x) = 0$.

By applying the operators $C_{n,\alpha}(f,x)$ to the above relation, we obtain

$$\begin{aligned}\frac{n}{b_n}\left[CK_{n,\alpha}(f,x) - f(x)\right] = \frac{n}{b_n}f'(x)CK_{n,\alpha}(t-x,x) + \frac{n}{2b_n}f''(x)CK_{n,\alpha}\left((t-x)^2,x\right)\\ + \frac{n}{b_n}CK_{n,\alpha}\left(\phi(t;x)(t-x)^2,x\right).\end{aligned} \tag{25}$$

By the Cauchy-Schwartz inequality, we have

$$CK_{n,\alpha}\left(\phi(t;x)(t-x)^2,x\right) \leqslant \sqrt{CK_{n,\alpha}\left(\phi^2(t;x),x\right)} \cdot \sqrt{CK_{n,\alpha}\left((t-x)^4,x\right)}.$$

Observe that $\phi^2(x;x) = 0$ and $\phi^2(t;x) \in C[0,\infty)$, then it follows from Theorem 1 that

$$\lim_{n\to\infty} CK_{n,\alpha}\left(\phi^2(t;x),x\right) = \phi^2(x;x) = 0.$$

From (11), we know $\sqrt{CK_{n,\alpha}\left((t-x)^4,x\right)} = O(\frac{b_n}{n})$, which implies that

$$\lim_{n\to\infty} \frac{n}{b_n}CK_{n,\alpha}\left(\phi(t;x)(t-x)^2,x\right) = 0. \tag{26}$$

From (8) and (9), we have

$$\lim_{n\to\infty} \frac{n}{b_n}f'(x)CK_{n,\alpha}(t-x,x) = \frac{1}{2}f'(x). \tag{27}$$

and

$$\lim_{n\to\infty} \frac{n}{2b_n}f''(x)C_{n,\alpha}\left((t-x)^2,x\right) = \frac{x}{6}f''(x). \tag{28}$$

Theorem 6 now follows from (25)–(28). □

Finally, we would like to study the rate of convergence of $CK_{n,\alpha}(f,x)$ for an absolutely continuous functions f having a derivative f' to a functions of bounded variation on $[0,\infty)$.

We say a function $f \in DBV[0,\infty)$, if f satisfies

$$f(x) = f(0) + \int_0^x h(t)dt,$$

where $h \in BV[0,\infty)$, i.e., h is a function of bounded variation on every finite subinterval of $[0,\infty)$. As for the approximation of operators to this kind of functions, we can refer to [26, 27, 28, 29].

THEOREM 7. *Let* $f \in DBV[0,\infty)$. *If* $h(x+)$ *and* $h(x-)$ *exist at a fixed point* $x \in (0,b_n)$, *then we have*

$$\left|CK_{n,\alpha}(f,x) - f(x) - \frac{\tau_1(b_n-2x)}{4n+2}\right| \leqslant \frac{|\tau_2|}{2}\eta_{n\alpha}(x) + \frac{2\eta_{n\alpha}^2(x)b_n}{x(b_n-x)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x+\frac{b_n-x}{k}}(\varphi_x) + \frac{b_n}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{b_n-x}{\sqrt{n}}}(\varphi_x),$$

where $\tau_1 = h(x+)+h(x-), \tau_2 = h(x+)-h(x-)$,

$$\varphi_x(t) = \begin{cases} h(t)-h(x+), \; x<t\leqslant b_n; \\ \quad 0, \qquad\qquad t=x; \\ h(t)-h(x-), \; 0\leqslant t<x. \end{cases}$$

Proof. Let f satisfy the conditions of Theorem 7, by using Bojanic-Cheng's method [26], we have

$$f(t)-f(x) = \int_x^t h(u)du \tag{29}$$

and $h(u)$ can be expressed as

$$h(u) = \frac{\tau_1}{2} + \varphi_x(u) + \frac{\tau_2}{2}sign(u-x) + \delta_x(u)\left[h(x) - \frac{\tau_1}{2}\right], \tag{30}$$

where

$$\delta_x(u) = \begin{cases} 1, \; u=x; \\ 0, \; u\neq x. \end{cases}$$

$$sign(x) = \begin{cases} 1, \;\; x>0; \\ 0, \;\; x=0; \\ -1, \; x<0. \end{cases}$$

Since $\int_x^t sign(u-x)du = |t-x|$ and $\int_x^t \delta_x(u)du = 0$, we have

$$CK_{n,\alpha}(f,x) - f(x) = CK_{n,\alpha}(f(t)-f(x),x) = CK_{n,\alpha}(\int_x^t h(u)du,x)$$
$$= \frac{\tau_1}{2}CK_{n,\alpha}(t-x,x) + \frac{\tau_2}{2}CK_{n,\alpha}(|t-x|,x) + CK_{n,\alpha}(\int_x^t \varphi_x(u)du,x).$$

By the expression of (8) and (10)), we have

$$\left|CK_{n,\alpha}(f,x) - f(x) - \frac{\tau_1(b_n-2x)}{4n+2}\right| \leqslant \frac{|\tau_2|}{2}\eta_{n\alpha}(x) + \left|CK_{n,\alpha}(\int_x^t \varphi_x(u)du,x)\right|. \tag{31}$$

Next, we estimate another item $CK_{n,\alpha}(\int_x^t \varphi_x(u)du,x)$.

By the Lebesgue-Stieltjes integral representations of (13), the last term of (31) can be expressed as

$$CK_{n,\alpha}\left(\int_x^t \varphi_x(u)du,x\right)=\int_0^{b_n}(\int_x^t \varphi_x(u)du)d_t\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right)=\Sigma_1+\Sigma_2, \tag{32}$$

where

$$\Sigma_1=\int_0^x(\int_x^t \varphi_x(u)du)d_t\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right),$$

$$\Sigma_2=\int_x^{b_n}(\int_x^t \varphi_x(u)du)d_t\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right).$$

Applying the integration by parts and noticing $\lambda_{n,\alpha}(\frac{x}{b_n},0)=0$, $\int_x^x \varphi_x(u)du=0$, we get

$$\begin{aligned}\Sigma_1&=\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right)\int_x^t \varphi_x(u)du\Big|_0^x-\int_0^x\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right)\varphi_x(t)dt\\&=-\int_0^x\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right)\varphi_x(t)dt=-(\int_0^{x-\frac{x}{\sqrt{n}}}+\int_{x-\frac{x}{\sqrt{n}}}^x)\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right)\varphi_x(t)dt.\end{aligned}$$

Thus, it follows that

$$\left|\Sigma_1\right|\leqslant\int_0^{x-\frac{x}{\sqrt{n}}}\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right)\bigvee_t^x(\varphi_x)dt+\int_{x-\frac{x}{\sqrt{n}}}^x\lambda_{n,\alpha}\left(\frac{x}{b_n},\frac{t}{b_n}\right)\bigvee_t^x(\varphi_x)dt.$$

From Lemma 4 (i) and $0\leqslant\lambda_{n,\alpha}(\frac{x}{b_n},\frac{t}{b_n})\leqslant 1$, we get

$$\left|\Sigma_1\right|\leqslant\eta_{n\alpha}^2(x)\int_0^{x-\frac{x}{\sqrt{n}}}\frac{\bigvee_t^x(\varphi_x)}{(x-t)^2}dt+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(\varphi_x). \tag{33}$$

Putting $t=x-\frac{x}{u}$ for the integral of (33), we get

$$\int_0^{x-\frac{x}{\sqrt{n}}}\frac{\bigvee_t^x(\varphi_x)}{(x-t)^2}dt=\frac{1}{x}\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{u}}^x(\varphi_x)du\leqslant\frac{2}{x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^x(\varphi_x). \tag{34}$$

From (33) and (34), it follows that

$$\left|\Sigma_1\right|\leqslant\frac{2\eta_{n\alpha}^2(x)}{x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^x(\varphi_x)+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^x(\varphi_x). \tag{35}$$

From Lemma 4 (ii), using the same method, we also get

$$\left|\Sigma_2\right|\leqslant\frac{2\eta_{n\alpha}^2(x)}{b_n-x}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{b_n-x}{k}}(\varphi_x)+\frac{b_n-x}{\sqrt{n}}\bigvee_x^{x+\frac{b_n-x}{\sqrt{n}}}(\varphi_x). \tag{36}$$

Theorem 7 now follows from (31), (32), (35) and (36). □

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Bo-Yong Lian
Department of Mathematics
Yang-en University
Quanzhou 362014, Fujian, China
e-mail: lianboyong@163.com

Qing-Bo Cai
School of Mathematics and Computer Science
Quanzhou Normal University
Quanzhou 362000, Fujian, China
e-mail: qbcai@126.com