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# MULTILINEAR COMMUTATORS RELATED TO MAXIMAL FUNCTION ON MORREY–BANACH SPACE AND ITS APPLICATION

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*Abstract.* The authors get the equivalent conditions for the boundedness of the commutators generated by the multilinear maximal functions and the BMO functions on Morrey-Banach space. As applications, we obtain the equivalent conditions for the boundedness of such operators on Morrey spaces with variable exponets and Morrey-Lorentz space which are all new results in the multi-linear case. Moreover, as far as we know, the results of this paper seem to be new even for the one-linear case.

## 1. Introduction

The Hardy-littlewood maximal function $M(f)(x)$, which is defined as

$$M(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|dy$$

plays important roles in harmonic analysis. In [24], Muckenhoupt introduced the $A_p$ weight class and gave the characterization of $A_p$ by using the weighted boundedness of $M$ on $L^p_\omega$ with $1<p<\infty$ and $L^p_\omega$ denotes the weighted $L^p$ space.

In the past twenty years, the multilinear Calderón-Zygmund theory was developed a lot and studied by many authors. Grafakos and Torres [10] introduced the multilinear Calderón-Zygmund operator and studied the bounedness of such operators. Later in 2009, Lerner et. al. [20] introduced a new kind of $A_{\vec{P}}$ weight class defined as follows.

DEFINITION 1.1. ([20]) Suppose that each $\omega_j$ is a non-negative and locally integrable function. Denote $\vec{\omega}=(\omega_1,\cdots,\omega_m)$ and $\nu_{\vec{\omega}}(x)=\prod_{j=1}^m\omega_j(x)^{p/p_j}$ with $1<p_1,\cdots,p_m<\infty$ and $1/p=1/p_1+\cdots+1/p_m$.

We say the vector-value function $\vec{\omega}$ satisfies the $A_{\vec{P}}$ condition with $\vec{P}=(p_1,\cdots,p_m)$ if for any ball $B\subset\mathbb{R}^n$, there is

$$\sup_B\left(\frac{1}{|B|}\int_B\nu_{\vec{\omega}}(x)dx\right)^{1/p}\prod_{j=1}^m\left(\frac{1}{|B|}\int_B\omega_j(x)^{1-p_j'}dx\right)^{1/p_j'}<\infty,$$

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where $p'_j$ is defined by $p'_j = \frac{p_j}{p_j - 1}$.

In [20], Lerner et al. introduced the following multilinear maximal function $\mathscr{M}(\vec{f})(x)$ defined as

$$\mathscr{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i$$

with any ball $Q \subset \mathbb{R}^n$.

Lerner at al. gave the characterization of $A_{\vec{P}}$ weight class by the weighted boundedness of $\mathscr{M}(\vec{f})(x)$ from $L^{p_1}_{\omega_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}_{\omega_m}(\mathbb{R}^n)$ to $L^p_{\nu_{\vec{\omega}}}(\mathbb{R}^n)$ with $\frac{1}{p} = \sum\limits_{i=1}^{m} \frac{1}{p_i}$ and $p_i > 1$.

For any ball $B \subset \mathbb{R}^n$, the mean oscillation space BMO($\mathbb{R}^n$) is defined as

$$\mathrm{BMO}(\mathbb{R}^n) = \left\{ b \in L_{\mathrm{loc}}(\mathbb{R}^n) : \|b\|_{\mathrm{BMO}} := \sup_B \frac{1}{|B|} \int_B |b(y) - b_B| dy < \infty \right\}$$

with $b_B = \frac{1}{|B|} \int_B b(x) dx$.

For $Q^m = \underbrace{Q \times \cdots \times Q}_{m}$ with any ball $Q \subset \mathbb{R}^n$, the commutator generated by the BMO function and the multilinear maximal funtions $[\vec{b}, \mathscr{M}](\vec{f})(x)$ and $\mathscr{M}_{\vec{b}}(\vec{f})(x)$ are defined as follows, respectively.

$$[\vec{b}, \mathscr{M}](\vec{f})(x) = \sum_{j=1}^{m} [\vec{b}, M]_j(\vec{f})(x), \qquad \mathscr{M}_{\vec{b}}(\vec{f})(x) = \sum_{j=1}^{m} \mathscr{M}^{b_j}(\vec{f})(x).$$

Here

$$[\vec{b}, M]_j(\vec{f})(x) = b_j(x)\mathscr{M}(\vec{f})(x) - \mathscr{M}(f_1, \cdots, f_{j-1}, b_j f_j, f_{j+1}, \cdots, f_m)(x)$$

and

$$\mathscr{M}^{b_j}(\vec{f})(x) = \sup_{Q \ni x} \frac{1}{|Q|^m} \int_{Q^m} |b_j(x) - b_j(y_j)| \prod_{i=1}^{m} |f_i(y_i)| d\vec{y},$$

where $\vec{y} = (y_1, \cdots, y_m)$.

For the case $m = 1$, we denote the following two kinds of commutators related to Hardy-Littlewood maximal function.

$$[b, M](f)(x) = b(x)M(f)(x) - M(bf)(x)$$

and

$$M_b(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)| dy.$$

For the study of $[b, M](f)(x)$, one may see [1, 3] et al. to find more details. Moreover, the commutators $[\vec{b}, \mathscr{M}](\vec{f})(x)$ and $\mathscr{M}_{\vec{b}}(\vec{f})(x)$ were studied in [33, 34] et al.

In [3], Bastero, Milman and Ruiz [3] got equivalent conditions for the boundedness of $[b, M](f)(x)$ on $L^p(\mathbb{R}^n)$ as follows.

THEOREM A. ([3]) *Suppose that $b$ is a real valued, locally integrable function in $\mathbb{R}^n$. Then, we have the following three equivalent assertions.*

*(i) The commutator $[b,M]$ is bounded on $L^p$ for $1<p<\infty$.*

*(ii) $b$ belongs to BMO and $b^- \in L^\infty$ with $b^- = -\min\{b(x),0\}$.*

*(iii) For $p\in(1,\infty)$, there is*

$$\sup_Q \frac{1}{|Q|^{\frac{1}{p}}}\left(\int_Q |b(x)-M_Q(b)(x)|^p dx\right)^{1/p} < \infty$$

*with* $M_Q(b)(x) = \sup_{Q_0\ni x, Q_0\subset Q}\left(\frac{1}{|Q_0|}\int_{Q_0}|b(t)|dt\right)$.

Theorem A was extended by many authors. For example, in [35], Zhang and Wu studied the equivalent boundedness conditions of $[b,M]$ on Lebesgue space with variable exponents.

It is well known that the Morrey space plays important roles in harmonic analysis and PDE. The classical Morrey space was usually attributed to C. B. Morrey. In fact, it was introduced by Campanato, Peetre and Brudneii in the 1960s, independently. Here, we would like to mention that in 1938, Morrey [23] studied some integral inequalities which is very useful in the connection with the Hölder regularity of solutions related to nonlinear elliptic and parabolic operators and the definition of Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is given as follows.

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f\in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, t>0}\left(\frac{1}{t^\lambda}\int_{B(x,t)}|f(y)|^p dy\right)^{1/p} < \infty \right\},$$

where $0\leqslant\lambda<n$, $1\leqslant p<\infty$ and $B(x,t)$ is the ball centered at $x$ with its radius $r$. Obviously, $L^{p,\lambda}$ becomes $L^p$ if we choose $\lambda=0$. Moreover, readers may see [27] to find more details about the Morrey space.

For the study of commutators generated by the maximal function on Morrey type spaces, Xie [29] proved the equivalent conditions for the boundedness of $[b,M](f)(x)$ on $L^{p,\lambda}(\mathbb{R}^n)$. Recently, Yu, Zhang and Li [32] improve the results of [29, Theorem 7] with $[\vec{b},\mathscr{M}](\vec{f})(x)$ on the product Morrey spaces.

On the other hand, we find that the $L^p$ spaces, the $L^p$ space with variable exponents and the Lorentz space are all the special case of the Banach function spaces (see the definition of the Banach function space in Section 2). In this paper, we would like to unify the above results in a general way. The main result of this paper can be stated as follows.

THEOREM 1.2. *Let $b_i$ be a real valued, locally integrable function in $\mathbb{R}^n$ with $i=1,\cdots,m$. Suppose that $X_i$ and $X$ are Banach function spaces. Moreover, we assume that $X_i\in\mathbb{M}$, $X\in\mathbb{M}'$ and $u_i$ satisfies the $\mathbb{W}^1_{X_i}$ condition. If $\mathscr{M}(\vec{f})(x)$ and $\mathscr{M}^{b_i}(\vec{f})(x)$ are bounded from $X_1\times\cdots X_m$ to $X$ with $\|\chi_B\|_X \sim \prod_{i=1}^m \|\chi_B\|_{X_i}$ for any ball $B\subset\mathbb{R}^n$, the following conditions are equivalent.*

*(I) The commutator* $[\vec{b},\mathscr{M}]$ *is bounded from* $M^{u_1}_{X_1}(\mathbb{R}^n)\times\cdots M^{u_m}_{X_m}(\mathbb{R}^n)$ *to* $M^u_X(\mathbb{R}^n)$ *with* $u=u_1\cdots u_m$.

*(II)* $b_i$ *is in BMO and* $b_i^-$ *belongs to* $L^\infty$.

*(III) Define* $M_Q(b_i)(x)=\sup_{Q_0\ni x, Q_0\subset Q}\frac{1}{|Q_0|}\int_{Q_0}|b_i(t)|dt$, *for any* $i=1,2,\ldots,m$, *there is*

$$\sup_Q\frac{\|(M_Q(b_i)-b_i)\chi_Q\|_X}{\prod_{i=1}^m\|\chi_Q\|_{X_i}}<\infty.$$

*Here, the definitions of Banach function spaces,* $M^u_X(\mathbb{R}^n)$, $\mathbb{W}^1_{X_i}$ *condition and the set of* $\mathbb{M}$ *and* $\mathbb{M}'$ *will be introduced in the next section.*

This paper is organized as follows. In Section 2, we will give the definitions of the Banach function space and the Morrey-Banach function space. Moreover, some properties and definitions related to these function spaces will also be given. In Section 3, we will give the boundedness of $\mathscr{M}(\vec{f})(x)$ and $\mathscr{M}_{\vec{b}}(\vec{f})(x)$ on product Morrey-Banach spaces and prove Theorem 1.2. In Sections 4 and 5, we will give the applications of Theorem 1.2 on Morrey spaces with variable exponents and Morrey-Lorentz space, respectively. Moreover, we will point out that the main results in Sections 4 and 5 are also new even in the linear case as far as we know.

## 2. Preliminaries

Denote that $\mathscr{M}(\mathbb{R}^n)$ is the set which consists of all the spaces of Lebesgue measurable functions in $\mathbb{R}^n$. Moroever, for any open ball $B(z,r)=\{x\in\mathbb{R}^n:|x-z|<r\}$ with its center $z\in\mathbb{R}^n$ and radius $r>0$, we denote $\mathbb{B}=\{B(z,r):z\in\mathbb{R}^n, r>0\}$.

DEFINITION 2.1. ([4]) We say that a Banach space $X\subset\mathscr{M}(\mathbb{R}^n)$ belongs to a Banach function space (B.f.s.) on $\mathbb{R}^n$ if it satisfies

(i) $\|f\|_X=0\Leftrightarrow f=0$ a.e.

(ii) $|g|\leqslant|f|$ a.e. $\Rightarrow$ $\|g\|_X\leqslant\|f\|_X$.

(iii) $0\leqslant f_n\uparrow f$ a.e. $\Rightarrow$ $\|f_n\|_X\uparrow\|f\|_X$.

(iv) $\chi_E\in\mathscr{M}(\mathbb{R}^n)$ and $|E|<\infty$ $\Rightarrow$ $\chi_E\in X$.

(v) $\chi_E\in\mathscr{M}(\mathbb{R}^n)$ and $|E|<\infty$ $\Rightarrow$ $\int_E|f(x)|dx<C_E\|f\|_X$, $\forall f\in X$ for some $C_E>0$.

REMARK 2.2. [4, 9] tells us that the Lorentz space, the Orlicz space and the Lebesgue space with variable exponent are the special case of Banach function space. Moreover, for any B.f.s. $X$, there is $X\in L^1_{\text{loc}}(\mathbb{R}^n)$ from (v) of Definition 2.1.

The duality theory for B.f.s. can be stated as follows.

DEFINITION 2.3. ([4, Chapter 1, Definitions 2.1. and 2.3]) Suppose that $X$ is a B.f.s. Then, we denote

$$\|f\|_{X'}=\sup\left\{\left|\int_{\mathbb{R}^n}f(t)g(t)dt\right|:g\in X,\|g\|_X\leqslant 1\right\}<\infty. \tag{2.1}$$

Here, we say the $X'$ is the associate space of $X$ and is the collection of all $f \in \mathscr{M}(\mathbb{R}^n)$ satisfying (2.1).

From [4], we know that if $X$ is a B.f.s, then $X'$ is also a B.f.s. Moreover, the following Hölder inequality for $X$ holds.

THEOREM 2.4. ([4, Chapter 1, Theorem 2.4]) *Let $X$ be a B.f.s. Then for any $f \in X$ and $g \in X'$, there is*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leqslant \|f\|_X \|g\|_{X'}.$$

DEFINITION 2.5. ([18]) For any B.f.s. $X$, we denote $X \in \mathbb{M}$ if $M(f)(x)$ is bounded on $X$. Similarly, we denote $X \in \mathbb{M}'$ if the $M(f)(x)$ is bounded on $X'$, which is the dual space of $X$.

Next, we give the definition of Morrey-Banach spaces following from [15, 18, 19].

DEFINITION 2.6. For any B.f.s. X, the Morrey-Banach space is defined as

$$M_X^u(\mathbb{R}^n) = \left\{ f \in M_X^u(\mathbb{R}^n) : \|f\|_{M_X^u(\mathbb{R}^n)} = \sup_{y \in \mathbb{R}^n, r>0} \frac{1}{u(y,r)} \|\chi_{B(y,r)} f\|_X < \infty \right\},$$

with all $f \in \mathscr{M}(\mathbb{R}^n)$, $B = B(y,r) \in \mathbb{B}$ and $u(y,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function.

For the study of integral operator and its commutator on $M_X^u(\mathbb{R}^n)$, one may see the paper [15, 18, 19] to find more details. Moreover, Ho [16] also got the boundedness for commutators of singular integral operator on weak Morrey-Banach space.

For any weight function $u(y,r)$, we introduce the $\mathbb{W}_X^1$ class following from [15, 18, 19] with some modifications.

DEFINITION 2.7. Let $X$ be a B.f.s. We say that a Lebesgue measurable function, $u(x,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, belongs to $u \in \mathbb{W}_X^1$ if there exits a constant $C > 0$ such that for any $x_1, x_2 \in \mathbb{R}^n$ and $r_1, r_2 \in \mathbb{R}^+$, $u$ fulfills

$$\frac{\|\chi_{B(x_1,r_1)}\|_X}{u(x_1,r_1)} \leqslant \frac{\|\chi_{B(x_2,r_2)}\|_X}{u(x_2,r_2)} \quad \text{if} \quad u(x_1,r_1) \leqslant u(x_2,r_2), \tag{2.2}$$

and

$$\sum_{j=0}^{\infty} (j+1) \frac{\|\chi_{B(x,r)}\|_X}{\|\chi_{B(x,2^{j+1}r)}\|_X} u(x,2^{j+1}r) \leqslant Cu(x,r). \tag{2.3}$$

From (2.2), for any fixed $Q_0 = Q(x_0,r_0) \in \mathbb{B}$ and any $B = B(x,r) \in \mathbb{B}$, if $u(x_0,r_0) \geqslant u(x,r)$, there is

$$\frac{1}{u(x,r)} \|\chi_{Q_0}\chi_B\|_X \leqslant \frac{1}{u(x,r)} \|\chi_B\|_X \leqslant \frac{1}{u(x_0,r_0)} \|\chi_{Q_0}\|_X.$$

For the case $u(x_0,r_0) < u(x,r)$, we have

$$\frac{1}{u(x,r)} \|\chi_{Q_0}\chi_B\|_X \leqslant \frac{1}{u(x,r)} \|\chi_{Q_0}\|_X \leqslant \frac{1}{u(x_0,r_0)} \|\chi_{Q_0}\|_X.$$

Thus, it is easy to see that $\chi_{Q_0} \in M_X^u(\mathbb{R}^n)$ and we have

$$\|\chi_{Q_0}\|_{M_X^u(\mathbb{R}^n)} \leqslant \frac{1}{u(x_0,r_0)}\|\chi_{Q_0}\|_X, \tag{2.4}$$

which implies $M_X^u(\mathbb{R}^n)$ is non-trivial with $u \in \mathbb{W}_X^1$.

REMARK 2.8. If we choose $X = L^p$ with $p > 1$ and $u(y,r) = r^{\frac{\lambda}{p}}$ with $0 < \lambda < n$, then $M_X^u(\mathbb{R}^n)$ becomes the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$. Moreover, it is easy to check $u(y,r) = r^{\frac{\lambda}{p}}$ satisfies (2.2) and (2.3) with $X = L^p$.

## 3. Proof of Theorem 1.2

Before proving the main results of this section, we give some lemmas.

LEMMA 3.1. ([18]) *Let $X$ be a B.f.s. If $X \in \mathbb{M}\bigcup\mathbb{M}'$, there exists a positive constant $C > 1$, such that*

$$|B| \leqslant \|\chi_B\|_X \|\chi_B\|_{X'} \leqslant C|B|$$

*for any $B \in \mathbb{B}$.*

*For any $B = B(x,t) \in \mathbb{B}$, we have $\|\chi_B\|_{X'} \leqslant \|\chi_{2B}\|_{X'}$ with $2B = B(x,2t)$. If $X \in \mathbb{M}\bigcup\mathbb{M}'$, then using Lemma 3.1, we have*

$$\|\chi_{2B}\|_X \leqslant C\frac{|2B|}{\|\chi_{2B}\|_{X'}} \leqslant C\frac{|B|}{\|\chi_B\|_{X'}} \leqslant C\|\chi_B\|_X. \tag{3.1}$$

LEMMA 3.2. ([12]) *Let $X \in \mathbb{M}'$. Then, the norms*

$$\|f\|_{BMO_X} := \sup_{B\in\mathbb{B}} \frac{\|\chi_B(f-f_B)\|_X}{\|\chi_B\|_X}$$

*and $\|\cdot\|_{BMO}$ are mutually equivalent.*

Next, we will give the boundedness of $\mathscr{M}(\vec{f})(x)$ and $\mathscr{M}_{\vec{b}}(\vec{f})(x)$ on the product Morrey-Banach spaces under different conditions and we will prove the following two lemmas respectively for the sake of completeness.

LEMMA 3.3. *Suppose that for any $i : 1 \leqslant i \leqslant m$, $X, X_i$ are B.f.s. with $\|\chi_B\|_X \leqslant C\prod_{i=1}^{m}\|\chi_B\|_{X_i}$ for any ball $B \in \mathbb{R}^n$ and $X_i \in \mathbb{M}\bigcup\mathbb{M}'$. If $\mathscr{M}(\vec{f})(x)$ is bounded from $X_1 \times \cdots \times X_m$ to $X$, then $\mathscr{M}(\vec{f})(x)$ is bounded from $M_{X_1}^{u_1}(\mathbb{R}^n) \times \cdots M_{X_m}^{u_m}(\mathbb{R}^n)$ to $M_X^u(\mathbb{R}^n)$ with $u = \prod_{i=1}^{m} u_i$ and $u_i \in \mathbb{W}_{X_i}^1$.*

*Proof.* From [33], we know that for any $z \in \mathbb{R}^n$, there is $\mathscr{M}_c(\vec{f})(z) \sim \mathscr{M}(\vec{f})(z)$ with

$$\mathscr{M}_c(\vec{f})(z) = \sup_{r>0}\frac{1}{|Q(z,r)|^m}\int_{(Q(z,r))^m}\prod_{i=1}^{m}|f_i(y_i)|d\vec{y}.$$

Thus, we may only consider $\mathscr{M}_c(\vec{f})(z)$ throughout this lemma.

For any ball $B=B(x,t)$ centered at $x$ with radius $t>0$, we denote $2B=B(x,2t)$. Split each $f_i=f_i^0+f_i^\infty$ with $f_i^0=f_i\chi_{2B}$ and $f_i^\infty=f_i\chi_{(2B)^c}$. Then, for any $z\in B(x,t)$, we obtain

$$\|\mathscr{M}_c(\vec{f})\chi_B\|_X\leqslant\|\mathscr{M}_c(\vec{f}^0)\chi_B\|_X+\sum{}'\|\mathscr{M}_c(f_1^{\alpha_1},\cdots,f_m^{\alpha_m})\chi_B\|_X:=I+II.$$

where $\alpha_1,\cdots,\alpha_m\in\{0,\infty\}$ and each term in the sum $\sum'$ contains at least one $\alpha_i=\infty$.

From [18], we know that for any $x\in\mathbb{R}^n$ and $t>0$, there is $u_i(x,2t)\leqslant Cu_i(x,t)$ if $u_i(x,t)$ satisfies (2.3). Thus, by the assumption that $\mathscr{M}(\vec{f})(z)$ is bounded from $X_1\times\cdots X_m$ to $X$, we have

$$\begin{aligned}I=\|\mathscr{M}_c(\vec{f}^0)\chi_{B(x,t)}\|_X&\leqslant\|f_1\chi_{B(x,2t)}\|_{X_1}\cdots\|f_m\chi_{B(x,2t)}\|_{X_m}\\&\leqslant C\prod_{i=1}^m\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\prod_{i=1}^m(u_i(x,2t))\leqslant C\prod_{i=1}^m\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}u(x,t).\end{aligned}$$

For $II$, without loss of generality, it suffices to prove the following two inequalities.

$$\|\mathscr{M}_c(f_1^0,f_2^\infty,f_3^\infty,\cdots,f_m^\infty)\chi_{B(x,t)}\|_X\leqslant C\prod_{i=1}^m\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}u(x,t)\tag{3.2}$$

and

$$\|\mathscr{M}_c(f_1^\infty,f_2^\infty,f_3^\infty,\cdots,f_m^\infty)\chi_{B(x,t)}\|_X\leqslant C\prod_{i=1}^m\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}u(x,t).\tag{3.3}$$

To give the proofs of (3.2) and (3.3), we should show the following two estimates which will be very useful in the proof of this lemma and the next lemma.

$$\int_{\mathbb{R}^n}|f_i^0(y_i)|dy_i\leqslant C\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\frac{u_i(x,t)t^n}{\|\chi_B\|_{X_i}}\tag{3.4}$$

and

$$\sup_{r>0}\frac{1}{|Q(z,r)|^k}\int_{Q(z,r)}|f_i^\infty(y_i)|dy_i\leqslant C\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\frac{u_i(x,t)t^{-(k-1)n}}{\|\chi_B\|_{X_i}}\tag{3.5}$$

for any $k,i\in\mathbb{N}$.

Using the Hölder inequality on Banach function space (Theorem 2.4), the fact $u_i(x,2t)\leqslant Cu_i(x,t)$ and $\|\chi_{2B}\|_{X_i}\geqslant\|\chi_B\|_{X_i}$, (3.4) is obviously true since

$$\begin{aligned}\int_{\mathbb{R}^n}|f_i^0(y_i)|dy_i&\leqslant\|f_i\chi_{2B}\|_{X_i}\|\chi_{2B}\|_{X_i'}\leqslant C\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}u_i(x,2t)\|\chi_{2B}\|_{X_i'}\\&\leqslant C\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\frac{u_i(x,2t)t^n}{\|\chi_{2B}\|_{X_i}}\leqslant C\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\frac{u_i(x,t)t^n}{\|\chi_B\|_{X_i}}.\end{aligned}$$

Next, we give the proof of (3.5).

For any $k,i\in\mathbb{N}$, as $z\in B(x,t)$ and $y_i\in(2B)^c\bigcap Q(z,r)$, we have

$$\begin{aligned}\sup_{r>0}\frac{1}{|Q(z,r)|^k}\int_{Q(z,r)}|f_i^\infty(y_i)|dy_i&=\sup_{r>0}\frac{1}{|Q(z,r)|^k}\int_{Q(z,r)\cap(2B)^c}|f_i(y_i)|dy_i\\&\leqslant C\sup_{r>t}\int_{Q(z,r)\bigcap(2B)^c}\frac{|f_i(y_i)|}{|y_i-z|^{kn}}dy_i\\&\leqslant C\int_{(2B)^c}\frac{|f_i(y_i)|}{|y_i-x|^{kn}}dy_i.\end{aligned}$$

Decompose

$$\int_{(2B)^c}\frac{|f_i(y_i)|}{|y_i-x|^{kn}}dy_i\leqslant\sum_{j=1}^{\infty}\int_{2^{j+1}B\backslash 2^jB}\frac{|f_i(y_i)|}{|y_i-x|^{kn}}dy_i.$$

Then, it is easy to see $\frac{2^jt}{2}\leqslant|x-y_i|\leqslant\frac{2^{j+1}t\sqrt{n}}{2}$. Thus, we conclude that there exists a constant $C$ depending on the dimension $n$, such that

$$\frac{1}{|x-y_i|^{kn}}\sim\frac{C}{|2^{j+1}B|^k}. \tag{3.6}$$

Then, using (2.3), (3.6), Lemma 3.1 and the Hölder inequality on Banach function space, there is

$$\begin{aligned}\sup_{r>0}\frac{1}{|Q(z,r)|^k}\int_{Q(z,r)}|f_i^\infty(y_i)|dy_i&\leqslant\sum_{j=1}^{\infty}\int_{2^{j+1}B\backslash 2^jB}|x-y_i|^{-kn}|f_i(y_i)|dy_i\\&\leqslant C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^k}\int_{2^{j+1}B\backslash 2^jB}|f_i(y_i)|dy_i\\&\leqslant C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^k}\|f_i\chi_{2^{j+1}B}\|_{X_i}\|\chi_{2^{j+1}B}\|_{X_i'}\\&\leqslant C\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}B|^k}\|f_i\|_{M^{u_i}_{X_i}(\mathbb{R}^n)}u_i(x,2^{j+1}t)\frac{|2^{j+1}B|}{\|\chi_{2^{j+1}B}\|_{X_i}}\\&\leqslant C\|f_i\|_{M^{u_i}_{X_i}(\mathbb{R}^n)}\sum_{j=1}^{\infty}\frac{u_i(x,2^{j+1}t)}{|2^{j+1}B|^{(k-1)}\|\chi_{2^{j+1}B}\|_{X_i}}\\&\leqslant C\|f_i\|_{M^{u_i}_{X_i}(\mathbb{R}^n)}t^{-(k-1)n}\sum_{j=1}^{\infty}2^{-n(j+1)}\frac{u_i(x,2^{j+1}t)}{\|\chi_{2^{j+1}B}\|_{X_i}}\\&\leqslant C\|f_i\|_{M^{u_i}_{X_i}(\mathbb{R}^n)}t^{-(k-1)n}\sum_{j=1}^{\infty}(j+1)\frac{u_i(x,2^{j+1}t)}{\|\chi_{2^{j+1}B}\|_{X_i}}\\&\leqslant C\|f_i\|_{M^{u_i}_{X_i}(\mathbb{R}^n)}t^{-(k-1)n}\frac{u_i(x,t)}{\|\chi_{B(x,t)}\|_{X_i}},\end{aligned}$$

which implies (3.5) is true.

Now, we will show the proofs of (3.2) and (3.3). Here, we would like to point out that the multi-version of the Hölder inequality on Banach function space is unknown. Thus, we decompose $\mathscr{M}_c(f_1^0, f_2^\infty, f_3^\infty, \cdots, f_m^\infty)(z)$ as

$$\mathscr{M}_c(f_1^0, f_2^\infty, f_3^\infty, \cdots, f_m^\infty)(z) \leqslant \int_{\mathbb{R}^n} |f_1^0(y_1)|dy_1 \sup_{r>0} \frac{1}{|Q(z,r)|^2} \int_{Q(z,r)} |f_2^\infty(y_2)|dy_2 \times \prod_{i=3}^m \sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r)} |f_i^\infty(y_i)|dy_3.$$

Then, using (3.4)–(3.5), there is

$$\begin{aligned}
&\big\|\mathscr{M}_c(f_1^0, f_2^\infty, f_3^\infty, \cdots, f_m^\infty)\chi_{B(x,t)}\big\|_X \\
&\leqslant C\left\|\prod_{i=1}^2 \left(\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)} \frac{u_i(x,t)}{\|\chi_B\|_{X_i}}\right) t^n t^{-n} \prod_{i=3}^m \left(\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)} \frac{u_i(x,t)}{\|\chi_B\|_{X_i}}\right) \chi_{B(x,t)}\right\|_X \\
&\leqslant C \frac{u(x,t)}{\prod_{i=1}^m \|\chi_B\|_{X_i}} \prod_{i=1}^m \|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)} \|\chi_{B(x,t)}\|_X \\
&\leqslant C \prod_{i=1}^m \|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)} u(x,t).
\end{aligned}$$

Similarly, there is

$$\begin{aligned}
&\|\mathscr{M}_c(f_1^\infty, f_2^\infty, f_3^\infty, \cdots, f_m^\infty)\chi_{B(x,t)}\|_X \\
&\leqslant C\left\|\prod_{i=1}^m \left(\sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r)} |f_i^\infty(y_i)|dy_i\right) \chi_{B(x,t)}\right\|_X \\
&\leqslant C \prod_{i=1}^m \|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)} \frac{\prod_{i=1}^m u_i(x,t)}{\prod_{i=1}^m \|\chi_B\|_{X_i}} \|\chi_{B(x,t)}\|_X \\
&= C \prod_{i=1}^m \|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)} u(x,t).
\end{aligned}$$

Consequently, we finish the proof of Lemma 3.3 according to the definition of $M_X^u(\mathbb{R}^n)$. □

Next, we would like to show the bounedness of $\mathscr{M}_{\vec{b}}(\vec{f})(x)$ on the product Morrey-Banach spaces. In this case, we need the condition $X_i \in \mathbb{M}$ and $X \in \mathbb{M}'$. Moreover, we have the following lemma.

LEMMA 3.4. *Suppose that for any $i : 1 \leqslant i \leqslant m$, $X, X_i$ are B.f.s. with $X_i \in \mathbb{M}$ and $X \in \mathbb{M}'$. If $\mathscr{M}^{b_i}(\vec{f})(x)$ is bounded from $X_1 \times \cdots X_m$ to $X$ with $b_i \in BMO$*

*and* $\|\chi_B\|_X \leqslant C\prod_{i=1}^m \|\chi_B\|_{X_i}$ *for any ball* $B \in \mathbb{R}^n$. *Then,* $\mathcal{M}_{\vec{b}}(\vec{f})(x)$ *is bounded from* $M^{u_1}_{X_1}(\mathbb{R}^n)\times\cdots M^{u_m}_{X_m}(\mathbb{R}^n)$ *to* $M^u_X(\mathbb{R}^n)$ *with* $u=\prod_{i=1}^m u_i$, $u_i \in \mathbb{W}^1_{X_i}$ *and* $b_i \in BMO$.

*Proof.* From [33], we know that for any $z\in\mathbb{R}^n$, we have $\mathcal{M}^{b_i}_c(\vec{f})(z) \sim \mathcal{M}^{b_i}(\vec{f})(z)$ where

$$\mathcal{M}^{b_i}_c(\vec{f})(z) = \sup_{r>0}\frac{1}{|Q(z,r)|^m}\int_{(Q(z,r))^m}|b_i(z)-b_i(y_i)|\prod_{i=1}^m|f_i(y_i)|d\vec{y}.$$

By the definition of $\mathcal{M}_{\vec{b}}(\vec{f})(z)$, without loss of generality, we may only consider $\mathcal{M}^{b_1}_c(\vec{f})(z)$ throughout this lemma as the other cases can be treated in a similar way.

Use the same notations as in the proof of Lemma 3.3, for any ball $B=B(x,t)$, we may split each $f_i = f_i^0 + f_i^\infty$. Then, we obtain

$$\|\mathcal{M}^{b_1}_c(\vec{f})\chi_B\|_X \leqslant \|\mathcal{M}^{b_1}_c(\vec{f}^0)\chi_B\|_X + \sum{}'\|\mathcal{M}^{b_1}_c(f_1^{\alpha_1},\cdots,f_m^{\alpha_m})\chi_B\|_X := III + IV.$$

where $\alpha_1,\cdots,\alpha_m \in \{0,\infty\}$ and each term in the sum $\sum'$ contains at least one $\alpha_i=\infty$.

For $III$, from the assumption that $\mathcal{M}^{b_1}(\vec{f})(x)$ is bounded from $X_1\times\cdots X_m$ to $X$ and the fact $u(x,2t)\leqslant Cu(x,t)$, we have

$$\begin{aligned}\|\mathcal{M}^{b_1}_c(\vec{f}^0)\chi_{B(x,t)}\|_X &\leqslant \|\mathcal{M}^{b_1}(\vec{f}^0)\|_X \leqslant C\|f_1\chi_{B(x,2t)}\|_{X_1}\cdots\|f_m\chi_{B(x,2t)}\|_{X_m}\\ &\leqslant C\prod_{i=1}^m\|f_i\|_{M^{u_i}_{X_i}(\mathbb{R}^n)}u(x,2t) \leqslant C\prod_{i=1}^m\|f_i\|_{M^{u_i}_{X_i}(\mathbb{R}^n)}u(x,t).\end{aligned}$$

For $IV$, without loss of generality, it suffices to show the following four inequalities.

$$\|\mathcal{M}^{b_1}_c(f_1^0,f_2^0,f_3^\infty,\cdots,f_m^\infty)\chi_{B(x,t)}\|_X \leqslant C\|b_1\|_{\mathrm{BMO}}\prod_{i=1}^m\|f_i\|_{M^{u_i}_{X_i}(\mathbb{R}^n)}u(x,t), \tag{3.7}$$

$$\|\mathcal{M}^{b_1}_c(f_1^0,f_2^\infty,f_3^\infty,\cdots,f_m^\infty)\chi_{B(x,t)}\|_X \leqslant C\|b_1\|_{\mathrm{BMO}}\prod_{i=1}^m\|f_i\|_{M^{u_i}_{X_i}(\mathbb{R}^n)}u(x,t), \tag{3.8}$$

$$\|\mathcal{M}^{b_1}_c(f_1^\infty,f_2^0,f_3^\infty,\cdots,f_m^\infty)\chi_{B(x,t)}\|_X \leqslant C\|b_1\|_{\mathrm{BMO}}\prod_{i=1}^m\|f_i\|_{M^{u_i}_{X_i}(\mathbb{R}^n)}u(x,t) \tag{3.9}$$

and

$$\|\mathcal{M}^{b_1}_c(f_1^\infty,f_2^\infty,f_3^\infty,\cdots,f_m^\infty)\chi_{B(x,t)}\|_X \leqslant C\|b_1\|_{\mathrm{BMO}}\prod_{i=1}^m\|f_i\|_{M^{u_i}_{X_i}(\mathbb{R}^n)}u(x,t). \tag{3.10}$$

First, we would like to give the following two estimates which will be very useful in the proof of the above four inequalities. That is, for any $z\in B=B(x,t)$, there is

$$\int_{\mathbb{R}^n}|b_1(z)-b_1(y_1)||f_1^0(y_1)|dy_1 \leqslant C\|f_1\|_{M^{u_1}_{X_1}(\mathbb{R}^n)}\frac{u_1(x,t)t^n}{\|\chi_B\|_{X_1}}(|b_1(z)-(b_1)_B|+\|b_1\|_{\mathrm{BMO}}) \tag{3.11}$$

and

$$\left\|\left(|b_1(\cdot)-(b_1)_B|+\|b_1\|_{\mathrm{BMO}}\right)\chi_{B(x,t)}\right\|_X \leqslant C\|b_1\|_{\mathrm{BMO}}\|\chi_{B(x,t)}\|_X. \tag{3.12}$$

To prove (3.11), we decompose $\int_{\mathbb{R}^n}|b_1(z)-b_1(y_1)||f_1^0(y_1)|dy_1$ as

$$\begin{aligned}&\int_{\mathbb{R}^n}|b_1(z)-b_1(y_1)||f_1^0(y_1)|dy_1\\ &\leqslant \int_{\mathbb{R}^n}|b_1(z)-(b_1)_B||f_1^0(y_1)|dy_1+\int_{\mathbb{R}^n}|b_1(y_1)-(b_1)_B||f_1^0(y_1)|dy_1.\end{aligned}$$

For $\int_{\mathbb{R}^n}|b_1(z)-(b_1)_B||f_1^0(y_1)|dy_1$, we have the following estimates from (3.4).

$$\begin{aligned}&\int_{\mathbb{R}^n}|b_1(z)-(b_1)_B||f_1^0(y_1)|dy_1=|b_1(z)-(b_1)_B|\int_{\mathbb{R}^n}|f_1^0(y_1)|dy_1\\ &\leqslant C|b_1(z)-(b_1)_B|\|f_1\|_{M_{X_1}^{u_1}(\mathbb{R}^n)}\frac{u_1(x,t)t^n}{\|\chi_B\|_{X_1}}.\end{aligned}$$

For $\int_{\mathbb{R}^n}|b_1(y_1)-(b_1)_B||f_1^0(y_1)|dy_1$, [18] tells us that if $X_1\in\mathbb{M}$, there is

$$\|(b_1(\cdot)-(b_1)_B)\chi_B\|_{X_1'}\leqslant C\|b_1\|_{\mathrm{BMO}}\|\chi_B\|_{X_1'}.$$

Then, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}|b_1(y_1)-(b_1)_B||f_1^0(y_1)|dy_1\leqslant C\|(b_1(\cdot)-(b_1)_B)\chi_B\|_{X_1'}\|f_1\chi_B\|_{X_1}\\ &\leqslant C\|b_1\|_{\mathrm{BMO}}\|\chi_B\|_{X_1'}\|f_1\|_{M_{X_1}^{u_1}(\mathbb{R}^n)}u_1(x,t)\\ &\leqslant C\|b_1\|_{\mathrm{BMO}}\|f_1\|_{M_{X_1}^{u_1}(\mathbb{R}^n)}\frac{u_1(x,t)t^n}{\|\chi_B\|_{X_1}}.\end{aligned}$$

Thus, we deduce that (3.11) is true.

To prove (3.12), we can easily get the following estimates from Lemma 3.2 with the condition $X\in\mathbb{M}'$.

$$\begin{aligned}&\left\|\left(|b_1(\cdot)-(b_1)_B|+\|b_1\|_{\mathrm{BMO}}\right)\chi_{B(x,t)}\right\|_X\\ &\leqslant C\left(\|(b_1(\cdot)-(b_1)_B)\chi_{B(x,t)}\|_X+\|b_1\|_{\mathrm{BMO}}\|\chi_{B(x,t)}\|_X\right)\\ &\leqslant C\|b_1\|_{\mathrm{BMO}}\|\chi_{B(x,t)}\|_X\end{aligned}$$

and we finish the proof of (3.12).

Now, we give the proof of (3.7). Using (3.4)–(3.5) and (3.11), there is

$$\begin{aligned}&\mathscr{M}_c^{b_1}(f_1^0,f_2^0,f_3^\infty,\cdots,f_m^\infty)(z)\leqslant\int_{\mathbb{R}^n}|b_1(z)-b_1(y_1)||f_1^0(y_1)|dy_1\int_{\mathbb{R}^n}|f_2^0(y_2)|dy_2\\ &\quad\times\sup_{r>0}\frac{1}{|Q(z,r)|^3}\int_{Q(z,r)}|f_3^\infty(y_3)|dy_3\prod_{i=4}^m\sup_{r>0}\frac{1}{|Q(z,r)|}\int_{Q(z,r)}|f_i^\infty(y_i)|dy_i\\ &\leqslant\int_{\mathbb{R}^n}|b_1(z)-b_1(y_1)||f_1^0(y_1)|dy_1t^{-n}\prod_{i=2}^m\left(\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\frac{u_i(x,t)}{\|\chi_B\|_{X_i}}\right)\\ &\leqslant C\left(|b_1(z)-(b_1)_B|+\|b_1\|_{\mathrm{BMO}}\right)\prod_{i=1}^m\left(\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\frac{u_i(x,t)}{\|\chi_B\|_{X_i}}\right).\end{aligned}$$

Then, we have

$$\begin{aligned}&\|\mathscr{M}_c^{b_1}(f_1^0,f_2^0,f_3^\infty,\cdots,f_m^\infty)\chi_{B(x,t)}\|_X\\&\leqslant C\left\|(|b_1(\cdot)-(b_1)_B|+\|b_1\|_{\mathrm{BMO}})\,\chi_{B(x,t)}\right\|_X\prod_{i=1}^m\left(\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\frac{u_i(x,t)}{\|\chi_B\|_{X_i}}\right).\end{aligned}$$

Using (3.12), we get

$$\begin{aligned}&\left\|(|b_1(\cdot)-(b_1)_B|+\|b_1\|_{\mathrm{BMO}})\,\chi_{B(x,t)}\right\|_X\prod_{i=1}^m\left(\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\frac{u_i(x,t)}{\|\chi_B\|_{X_i}}\right)\\&\leqslant C\|b_1\|_{\mathrm{BMO}}\|\chi_{B(x,t)}\|_X\prod_{i=1}^m\left(\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\frac{u_i(x,t)}{\|\chi_B\|_{X_i}}\right)\\&\leqslant C\|b_1\|_{\mathrm{BMO}}\prod_{i=1}^m\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}u(x,t).\end{aligned}\tag{3.13}$$

Thus, we obtain

$$\|\mathscr{M}_c^{b_1}(f_1^0,f_2^0,f_3^\infty,\cdots,f_m^\infty)\chi_{B(x,t)}\|_X\leqslant C\|b_1\|_{\mathrm{BMO}}\prod_{i=1}^m\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}u(x,t)$$

and we finish the proof of (3.7).

Next, we show the proof of (3.8). Using (3.5) and (3.11), there is

$$\begin{aligned}&\mathscr{M}_c^{b_1}(f_1^0,f_2^\infty,f_3^\infty,\cdots,f_m^\infty)(z)\leqslant\int_{\mathbb{R}^n}|b_1(z)-b_1(y_1)||f_1^0(y_1)|dy_1\\&\quad\times\sup_{r>0}\frac{1}{|Q(z,r)|^2}\int_{Q(z,r)}|f_2^\infty(y_2)|dy_2\prod_{i=3}^m\sup_{r>0}\frac{1}{|Q(z,r)|}\int_{Q(z,r)}|f_i^\infty(y_i)|dy_i\\&\leqslant C\prod_{i=1}^m\left(\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\frac{u_i(x,t)}{\|\chi_B\|_{X_i}}\right)(|b_1(z)-(b_1)_B|+\|b_1\|_{\mathrm{BMO}}).\end{aligned}$$

Then, we have the following estimates according to (3.13).

$$\begin{aligned}&\|\mathscr{M}_c^{b_1}(f_1^0,f_2^\infty,f_3^\infty,\cdots,f_m^\infty)\chi_{B(x,t)}\|_X\\&\leqslant C\prod_{i=1}^m\left(\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\frac{u_i(x,t)}{\|\chi_B\|_{X_i}}\right)\left\|(|b_1(\cdot)-(b_1)_B|+\|b_1\|_{\mathrm{BMO}})\,\chi_{B(x,t)}\right\|_X\\&\leqslant C\|b_1\|_{\mathrm{BMO}}\prod_{i=1}^m\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}u(x,t),\end{aligned}$$

which finishes the proof of (3.8).

For the proofs of (3.9)–(3.10), we would like to show the following estimates.

$$\begin{aligned}&\sup_{r>0}\frac{1}{|Q(z,r)|^k}\int_{Q(z,r)}|b_1(z)-b_1(y_1)||f_1^\infty(y_1)|dy_1\\&\leqslant C\|f_1\|_{M_{X_1}^{u_1}(\mathbb{R}^n)}\frac{u_1(x,t)t^{-(k-1)n}}{\|\chi_B\|_{X_1}}(|b_1(z)-(b_1)_B|+\|b_1\|_{\mathrm{BMO}})\end{aligned}\tag{3.14}$$

with $k \in \mathbb{N}^+$.

Since

$$\begin{aligned}
&\sup_{r>0} \frac{1}{|Q(z,r)|^k} \int_{Q(z,r)} |b_1(z) - b_1(y_1)||f_1^{\infty}(y_1)|dy_1 \\
&\leqslant \sup_{r>0} \frac{1}{|Q(z,r)|^k} \int_{Q(z,r)\bigcap(2B)^c} |b_1(z) - b_1(y_1)||f_1(y_1)|dy_1 \\
&\leqslant C \int_{(2B)^c} \frac{|b_1(z) - b_1(y_1)||f_1(y_1)|}{|x-y_1|^{kn}} dy_1 \\
&\leqslant C|b_1(z) - (b_1)_B| \int_{(2B)^c} \frac{|f_1(y_1)|}{|x-y_1|^{kn}} dy_1 \\
&\quad + \int_{(2B)^c} \frac{|b_1(y_1) - (b_1)_B||f_1(y_1)|}{|x-y_1|^{kn}} dy_1.
\end{aligned}$$

For $\int_{(2B)^c} \frac{|f_1(y_1)|}{|x-y_1|^{kn}} dy_1$, by the Hölder inequality on Banach function space and (2.3), we have

$$\begin{aligned}
&\int_{(2B)^c} \frac{|f_1(y_1)|}{|x-y_1|^{kn}} dy_1 \\
&\leqslant \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|f_1(y_1)|}{|x-y_1|^{kn}} dy_1 \\
&\leqslant C \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^k} \int_{2^{j+1}B} |f_1(y_1)| dy_1 \\
&\leqslant C \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^k} \|f_1 \chi_{2^{j+1}B}\|_{X_1} \|\chi_{2^{j+1}B}\|_{X_1'} \\
&\leqslant C \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^k} \|f_1\|_{M_{X_1}^{u_1}(\mathbb{R}^n)} \frac{u_1(x, 2^{j+1}t)|2^{j+1}B|}{\|\chi_{2^{j+1}B}\|_{X_1}} \\
&\leqslant C \|f_1\|_{M_{X_1}^{u_1}(\mathbb{R}^n)} \sum_{j=1}^{\infty} \frac{t^{-n(k-1)}}{2^{(j+1)n(k-1)}} \frac{u_1(x, 2^{j+1}t)}{\|\chi_{2^{j+1}B}\|_{X_1}} \\
&\leqslant C \|f_1\|_{M_{X_1}^{u_1}(\mathbb{R}^n)} t^{-n(k-1)} \sum_{j=1}^{\infty} (j+1) \frac{1}{2^{(j+1)n(k-1)}} \frac{u_1(x, 2^{j+1}t)}{\|\chi_{2^{j+1}B}\|_{X_1}} \\
&\leqslant C \|f_1\|_{M_{X_1}^{u_1}(\mathbb{R}^n)} t^{-n(k-1)} \frac{u_1(x,t)}{\|\chi_B\|_{X_1}}.
\end{aligned}$$

For $\int_{(2B)^c} \frac{|b_1(y_1)-(b_1)_B||f_1(y_1)|}{|x-y_1|^{kn}} dy_i$, Ho [18] also proved that if $X_1 \in \mathbb{M}$, there is

$$\|(b_1(\cdot) - (b_1)_B)\chi_{2^{j+1}B}\|_{X_1'} \leqslant C(j+1)\|b_1\|_{\mathrm{BMO}} \|\chi_{2^{j+1}B}\|_{X_1'}$$

for any $j \in \mathbb{N}$.

Thus, using (2.3) again, we have

$$
\begin{aligned}
&\int_{(2B)^c} \frac{|b_1(y_1)-(b_1)_B||f_1(y_1)|}{|x-y_1|^{kn}}dy_1 \\
&\leqslant \sum_{j=1}^{\infty} \int_{(2^{j+1}B)\setminus(2^jB)} \frac{|b_1(y_1)-(b_1)_B||f_1(y_1)|}{|x-y_1|^{kn}}dy_i \\
&\leqslant C\sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^k} \int_{(2^{j+1}B)} |b_1(y_1)-(b_1)_B||f_1(y_1)|dy_1 \\
&\leqslant C\sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^k} \|f_1\chi_{2^{j+1}B}\|_{X_1} \|(b_1(\cdot)-(b_1)_B)\chi_{2^{j+1}B}\|_{X_1'} \\
&\leqslant C\sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^k} \|f_1\|_{M^{u_1}_{X_1}(\mathbb{R}^n)} u_1(x,2^{j+1}t)(j+1)\|b_1\|_{\mathrm{BMO}} \|\chi_{2^{j+1}B}\|_{X_1'} \\
&\leqslant C\|b_1\|_{\mathrm{BMO}} \|\|f_1\|_{M^{u_1}_{X_1}(\mathbb{R}^n)} t^{-n(k-1)} \sum_{j=1}^{\infty} \frac{(j+1)}{2^{(j+1)n(k-1)}} \frac{u_1(x,2^{j+1}t)}{\|\chi_{2^{j+1}B}\|_{X_1}} \\
&\leqslant C\|b_1\|_{\mathrm{BMO}} \|\|f_1\|_{M^{u_1}_{X_1}(\mathbb{R}^n)} t^{-n(k-1)} \sum_{j=1}^{\infty} (j+1) \frac{u_1(x,2^{j+1}t)}{\|\chi_{2^{j+1}B}\|_{X_1}} \\
&\leqslant C\|b_1\|_{\mathrm{BMO}} \|\|f_1\|_{M^{u_1}_{X_1}(\mathbb{R}^n)} t^{-n(k-1)} \frac{u_1(x,t)}{\|\chi_B\|_{X_1}}.
\end{aligned}
$$

Thus, we prove that (3.14) is true.

By (3.4)–(3.5) and (3.14), there is

$$
\begin{aligned}
&\mathscr{M}_c^{b_1}(f_1^{\infty}, f_2^0, f_3^{\infty}, \cdots, f_m^{\infty})(z) \\
&\leqslant \prod_{i=3}^{m} \sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r)} |f_i^{\infty}(y_i)|dy_i \\
&\quad \times \sup_{r>0} \frac{1}{|Q(z,r)|^2} \int_{Q(z,r)} |b_1(z)-b_1(y_1)||f_1^{\infty}(y_1)|dy_1 \int_{\mathbb{R}^n} |f_2^0(y_2)|dy_2 \\
&\leqslant C\left(|b_1(z)-(b_1)_B| + \|b_1\|_{\mathrm{BMO}}\right) \prod_{i=1}^{m} \left( \|f_i\|_{M^{u_i}_{X_i}(\mathbb{R}^n)} \frac{u_i(x,t)}{\|\chi_B\|_{X_i}} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathscr{M}_c^{b_1}(f_1^{\infty}, f_2^{\infty}, f_3^{\infty}, \cdots, f_m^{\infty})(z) \\
&\leqslant \sup_{r>0} \frac{1}{|Q(z,r)|} \int_{\mathbb{R}^n} |b_1(z)-b_1(y_1)||f_1^{\infty}(y_1)|dy_1 \\
&\quad \times \prod_{i=2}^{m} \sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r)} |f_i^{\infty}(y_i)|dy_i \\
&\leqslant C\left(|b_1(z)-(b_1)_B| + \|b_1\|_{\mathrm{BMO}}\right) \prod_{i=1}^{m} \left( \|f_i\|_{M^{u_i}_{X_i}(\mathbb{R}^n)} \frac{u_i(x,t)}{\|\chi_B\|_{X_i}} \right).
\end{aligned}
$$

Then, we have the following two estimates according to (3.13).

$$\begin{aligned}
&\|\mathscr{M}_c^{b_1}(f_1^\infty, f_2^0, f_3^\infty, \cdots, f_m^\infty)\chi_{B(x,t)}\|_X \\
&\leqslant C\left\|\left(|b_1(\cdot)-(b_1)_B|+\|b_1\|_{\mathrm{BMO}}\right)\chi_{B(x,t)}\right\|_X \prod_{i=1}^m \left(\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\frac{u_i(x,t)}{\|\chi_B\|_{X_i}}\right) \\
&\leqslant C\|b_1\|_{\mathrm{BMO}} \prod_{i=1}^m \left(\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)} u(x,t)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathscr{M}_c^{b_1}(f_1^\infty, f_2^\infty, f_3^\infty, \cdots, f_m^\infty)\chi_{B(x,t)}\|_X \\
&\leqslant C\left\|\left(|b_1(\cdot)-(b_1)_B|+\|b_1\|_{\mathrm{BMO}}\right)\chi_{B(x,t)}\right\|_X \prod_{i=1}^m \left(\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\frac{u_i(x,t)}{\|\chi_B\|_{X_i}}\right) \\
&\leqslant C\|b_1\|_{\mathrm{BMO}} \prod_{i=1}^m \|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)} u(x,t),
\end{aligned}$$

which imply (3.9) and (3.10) are true.

Combing the estimates of $III$ and $IV$, we finish the proof of Lemma 3.4 according to the definition of $M_X^u(\mathbb{R}^n)$. □

LEMMA 3.5. ([33]) *Let $\vec{b}=(b_1,\cdots,b_m)$ and $\vec{f}=(f_1,\cdots,f_m)$ be two collections of locally integrable function, then*

$$|[\vec{b},\mathscr{M}](\vec{f})(x)| \leqslant \mathscr{M}_{\vec{b}}(\vec{f})(x) + 2\left(\sum_{i=1}^m b_i^-(x)\right)\mathscr{M}(\vec{f})(x).$$

### 3.1. Proof of Theorem 1.2

Now, we are ready to give the proof of Theorem 1.2.

First, using Lemmas 3.3-3.5, we know that (II) implies (I).

Next, we show (I)$\Rightarrow$ (III).

That is, we need to prove for any ball $Q=Q(z,r)$, there is

$$\sup_Q \frac{\|(M_Q(b_i)-b_i)\chi_Q\|_X}{\prod_{i=1}^m \|\chi_Q\|_{X_i}} < \infty.$$

Here $M_Q(b_i)(x) = \sup_{Q_0 \ni x, Q_0 \subset Q} \frac{1}{|Q_0|}\int_{Q_0}|b_i(t)|dt$.

From [3], we know that $M_Q(b_i) \geqslant b_i$ with $x\in Q$. Then, using (ii) of Definition 2.1, we have

$$\|(M_Q(b_i)-b_i)\chi_Q\|_X \leqslant \|\sum_{i=1}^m (M_Q(b_i)-b_i(\cdot))\chi_Q\|_X.$$

From [33], we have $\mathscr{M}(\chi_Q,\dots,b_i\chi_Q,\cdots,\chi_Q)(x)=M_Q(b_i)(x)$ for $x\in Q(z,r)$ and any $1\leqslant i\leqslant m$.

Then, choosing $f_i(x)=\chi_Q\in X_i$ for any $Q=Q(z,r)\subset\mathbb{R}^n$, we have

$$\sum_{i=1}^{m}(M_Q(b_i)(x)-b_i(x))=[\vec{b},\mathscr{M}](\vec{f})(x).$$

Using (2.4), we obtain

$$\|[\vec{b},\mathscr{M}](\vec{f})\|_{M_X^u(\mathbb{R}^n)}\leqslant C\prod_{i=1}^{m}\|f_i\|_{M_{X_i}^{u_i}(\mathbb{R}^n)}\leqslant C\prod_{i=1}^{m}\left(\frac{\|\chi_Q\|_{X_i}}{u_i(z,r)}\right)=C\frac{\prod_{i=1}^{m}\|\chi_Q\|_{X_i}}{u(z,r)}.$$

Thus, by the definition of the Morrey-Banach space, we get

$$\|(M_Q(b_i)-b_i)\chi_Q\|_X\leqslant\|[\vec{b},\mathscr{M}](\vec{f})\chi_Q\|_X\leqslant\|[\vec{b},\mathscr{M}](\vec{f})\|_{M_X^u(\mathbb{R}^n)}u(z,r)\leqslant C\prod_{i=1}^{m}\|\chi_Q\|_{X_i},$$

which implies

$$\sup_Q\frac{\|(M_Q(b_i)-b_i)\chi_Q\|_X}{\prod_{i=1}^{m}\|\chi_Q\|_{X_i}}<\infty$$

and we finish the proof of (I)$\Rightarrow$ (III).

Finally, we give the proof of (III) $\rightarrow$ (II).

Let

$$E=\{x\in Q:b_i(x)\leqslant(b_i)_Q\}\qquad\text{and}\qquad F=\{x\in Q:b_i(x)>(b_i)_Q\}.$$

Then, we have

$$\int_E|(b_i)_Q-b_i(x)|dx=\int_F|b_i(x)-(b_i)_Q|dx.$$

Thus, using the condition in (III) and the fact $b_i(x)\leqslant(b_i)_Q\leqslant M_Q(b_i)(x)$ with $x\in E\subset Q$, there is

$$\begin{aligned}\frac{1}{|Q|}\int_Q|b_i(x)-(b_i)_Q|dx&=\frac{2}{|Q|}\int_E|b_i(x)-(b_i)_Q|dx\\&=\frac{2}{|Q|}\int_E((b_i)_Q-b_i(x))dx\\&\leqslant\frac{2}{|Q|}\int_{\mathbb{R}^n}(M_Q(b_i)(x)-b_i(x))\chi_Q(x)dx\\&\leqslant\frac{2}{|Q|}\|(M_Q(b_i)-b_i)\chi_Q\|_X\|\chi_Q\|_{X'}\\&\leqslant C\frac{\prod_{i=1}^{m}\|\chi_Q\|_{X_i}\|\chi_Q\|_{X'}}{|Q|}\leqslant C,\end{aligned}$$

which tells us that $b_i \in \mathrm{BMO}$.

From [3], we know that

$$0 \leqslant b_i^- \leqslant M_Q(b_i) - b_i.$$

Moreover, we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q (M_Q(b_i)(x) - b_i(x))dx &\leqslant \frac{1}{|Q|}\|(M_Q(b_i)-b_i)\chi_Q\|_X \|\chi_Q\|_{X'} \\ &\leqslant \frac{1}{|Q|}\prod_{i=1}^m \|\chi_Q\|_{X_i}\frac{|Q|}{\|\chi_Q\|_X} \leqslant C,\end{aligned}$$

which implies $(b_i^-)_Q \leqslant C$.

Consequently, we get $b_i^- \in L^\infty$ from the Lebesgue differential theorem and finish the proof of Theorem 1.2. □

## 4. Applications to Morrey space with variable exponents

Suppose that $\mathscr{P}$ is the set of Lebesgue measurable functions with $p(\cdot) : \mathbb{R}^n \to [1,\infty]$. For any $p(\cdot) \in \mathscr{P}$, we denote

$$p_+ = \sup_{x\in\mathbb{R}^n} p(x) \qquad \text{and} \quad p_- = \inf_{x\in\mathbb{R}^n} p(x).$$

Thus, it is easy to see $1 \leqslant p_- \leqslant p_+ \leqslant \infty$.

Now, we introduce the definition of Lebesgue space with variable exponent as follows.

DEFINITION 4.1. ([7, 9]) Let

$$\rho_p(f) = \int_{\mathbb{R}^n\setminus\mathbb{R}^n_\infty} |f(x)|^{p(x)}dx + \operatorname*{ess\,sup}_{x\in\mathbb{R}^n_\infty} |f(x)|$$

with

$$\mathbb{R}^n_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}.$$

Then, the Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is defined as

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f \in \mathscr{M}(\mathbb{R}^n) : \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\lambda > 0 : \rho_p\left(\frac{f}{\lambda}\right) \leqslant 1\right\} < \infty\right\}$$

with any Lebesgue measurable functions $f$.

Obviously, if we replace $X$ by $L^{p(\cdot)}(\mathbb{R}^n)$, $M^u_X(\mathbb{R}^n)$ becomes the Morrey space with variable exponent $M^u_{L^{p(\cdot)}}(\mathbb{R}^n)$.

For the studies of integral operators on $M^u_{L^{p(\cdot)}}(\mathbb{R}^n)$ with $u$ satisfies (2.3) or certain doubling conditions, one may see [11, 13, 14, 17, 28] et al. to find more details.

Next, we introduce the definitions of some classes which can be used in the study of function spaces with variable exponents.

DEFINITION 4.2. ([7, 9]) Let $p(\cdot):\mathbb{R}^n \to \mathbb{R}$. If there exists a constant $C>0$ and $p_\infty \in \mathbb{R}$, such that

$$|p(x)-p(y)| \leqslant \frac{C}{\log(e+1/|x-y|)}, \quad \forall x,y \in \mathbb{R}^n$$

and

$$|p(x)-p_\infty| \leqslant \frac{C}{\log(e+|x|)}, \quad \forall x \in \mathbb{R}^n$$

and we say that $p(\cdot)$ belongs to the globally log-Hölder continuous.

Moreover, for $\frac{1}{p(\cdot)}$ is globally log-Hölder continuous, we denote

$$\mathscr{P}_{\log} = \{p(\cdot) \in \mathscr{P}\}.$$

It is easy to see that $p(\cdot) \in \mathscr{P}_{\log}$ is equivalent to $p'(\cdot) \in \mathscr{P}_{\log}$ (see [7, 9]).

From [9], we have

LEMMA 4.3. ([9, Remark 4.1.5]) *If $p_+ < \infty$, then $p(\cdot) \in \mathscr{P}_{log}$ is equivalent to $p(\cdot)$ is globally log-Hölder continuous.*

LEMMA 4.4. ([9, Theorem 4.3.8, Corollary 4.4.12]) *Let $p(\cdot):\mathbb{R}^n \to [1,\infty]$. If $p \in \mathscr{P}_{log}$ and $p_- > 1$, then $L^{p(\cdot)} \in \mathbb{M}$.*

The boundedness of $M_b$ on $L^{p(\cdot)}(\mathbb{R}^n)$ can be found in [30].

LEMMA 4.5. ([30]) *Let $p(\cdot):\mathbb{R}^n \to [1,\infty]$. If $p(\cdot) \in \mathscr{P}_{log}$ and $p_- > 1$, then $M_b$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself with $b \in BMO$.*

Thus, we obtain

LEMMA 4.6. *Suppose that $p(\cdot), p_i(\cdot):\mathbb{R}^n \to [1,\infty]$ with $p_i(\cdot), p(\cdot) \in \mathscr{P}_{log}$, and $(p_i)_-, p_- > 1$. Then $\mathscr{M}^{b_i}(\vec{f})(x)$ is bounded from $L^{p_1(\cdot)}(\mathbb{R}^n) \times \cdots \times L^{p_m(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ with $\frac{1}{p(\cdot)} = \sum_{i=1}^{m} \frac{1}{p_i(\cdot)}$.*

*Proof.* Without loss of generality, it suffices to consider with $\mathscr{M}^{b_1}(\vec{f})(x)$.

As $\mathscr{M}^{b_1}(\vec{f})(x) \leqslant M_{b_1}(f_1)(x)M(f_2)(x)\cdots M(f_m)(x)$, then using the Hölder inequality on $L^{p(\cdot)}(\mathbb{R}^n)$ space(see [7, 9]) and Lemma 4.5, we have

$$\begin{aligned}\|\mathscr{M}^{b_1}(\vec{f})\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leqslant C\|M_{b_1}(f_1)\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|M(f_2)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\cdots\|M(f_m)\|_{L^{p_m(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C\prod_{i=1}^{m}\|f_i\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}\end{aligned}$$

with $p_i(\cdot), p(\cdot) \in \mathscr{P}_{\log}$ and $(p_i)_-, p_- > 1$.

Next, we give the norm of $\chi_B$ on $L^{p(\cdot)}(\mathbb{R}^n)$.

Suppose that $p(\cdot) \in \mathscr{P}_{\log}$ with $p_- > 1$. For any ball $B \subset \mathbb{R}^n$, we define the exponents $p_B$ and $p'_B$ by

$$\frac{1}{p_B} = \frac{1}{|B|}\int_B \frac{1}{p(x)}dx, \qquad \frac{1}{p'_B} = \frac{1}{|B|}\int_B \frac{1}{p'(x)}dx. \tag{4.1}$$

From [6, Proposition 4.66], we have

$$\|\chi_B\|_{L^{p(\cdot)}} \approx |B|^{\frac{1}{p_B}} \quad \text{and} \quad \|\chi_B\|_{L^{p'(\cdot)}} \approx |B|^{\frac{1}{p'_B}}. \tag{4.2}$$

Thus, for $p_i(\cdot) \in \mathscr{P}_{\log}$ and $(p_i)_- > 1$, there is

$$\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \prod_{i=1}^{m} \|\chi_B\|_{L^{p_i(\cdot)}(\mathbb{R}^n)} \tag{4.3}$$

with $\frac{1}{p(\cdot)} = \sum_{i=1}^{m} \frac{1}{p_i(\cdot)}$.

As it is easy to see $p_- > 1$ implies $p'_- > 1$. $\square$

Now, using Theorem 1.2, Lemmas 4.3–4.6 and the fact $p(\cdot) \in \mathscr{P}_{\log}$ is equivalent to $p'(\cdot) \in \mathscr{P}_{\log}$, we obtain

THEOREM 4.7. *Let $b_i$ be a real valued, locally integrable function in $\mathbb{R}^n$ with $i = 1,\cdots,m$ and $u_i$ satisfies the $\mathbb{W}^1_{X_i}$ condition. Denote $p(\cdot)$ pointwise by $\frac{1}{p(\cdot)} = \sum_{i=1}^{m} \frac{1}{p_i(\cdot)}$ with $p_i(\cdot), p(\cdot) \in \mathscr{P}_{log}$ and $(p_i)_-, p_- > 1$. Then, the following conditions are equivalent.*

*(I) The commutator $[\vec{b},\mathscr{M}]$ is bounded from $M^{u_1}_{L^{p_1(\cdot)}}(\mathbb{R}^n) \times \cdots M^{u_m}_{L^{p_m(\cdot)}}(\mathbb{R}^n)$ to $M^{u}_{L^{p_m(\cdot)}}(\mathbb{R}^n)$ with $u = u_1 \cdots u_m$.*

*(II) $b_i$ is in BMO and $b_i^-$ belongs to $L^\infty$.*

*(III) Define $M_Q(b_i)(x) = \sup_{Q_0 \ni x, Q_0 \subset Q} \frac{1}{|Q_0|} \int_{Q_0} |b_i(t)|dt$, for any $i = 1,2,\ldots,m$, there is*

$$\sup_Q \frac{\|(M_Q(b_i) - b_i)\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\prod_{i=1}^{m} \|\chi_Q\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}} < \infty.$$

REMARK 4.8. Obviously, Theorem 4.7 improves the results of [35, Theorem 1.2]. Moreover, as far as we know, Theorem 4.7 seems to be a new result even in the one-linear case.

## 5. Multilinear maximal function and its commutator on Morrey-Lorentz spaces

The Lorentz space $\mathscr{L}^{p,q}$, which is defined as

$$\mathscr{L}^{p,q} = \left\{ f \in \mathscr{L}^{p,q} : \|f\|_{\mathscr{L}^{p,q}} := \left( \int_0^\infty \left[ t^{\frac{1}{p}} f^*(t) \right]^q \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

with $0 < p,q < \infty$ and $f$ is a measurable function on $\mathbb{R}^n$. Moreover, $f^*$ is the rearrangement function of $f$, that is

$$f^*(t) := \inf\{y \geqslant 0 : m(f,y) \leqslant t\}$$

with

$$m(f,y) := |\{x \in \mathbb{R}^n : |f(x)| > y, y \geqslant 0\}|.$$

Obviously, for the case $p = q$, $\mathscr{L}^{p,p}$ becomes the classical $L^p$ spaces. Moreover, $\mathscr{L}^{p,q}$ is the Banach function space with $1 < p < \infty$ and $1 \leqslant q < \infty$.

The space $\mathscr{L}^{p,q}$ was first introduced by Lorentz [21] and was studied by many authors. One may see [5, 8, 22] et. al. to find more details.

In [26], Ragusa introduced the Morrey-Lorentz space $\mathscr{R}^{p,q,\lambda}$ on $\mathbb{R}^n$ as follows.

$$\mathscr{R}^{p,q,\lambda} = \left\{ f \in \mathscr{R}^{p,q,\lambda} : \|f\|_{\mathscr{R}^{p,q,\lambda}} := \sup_{x\in\mathbb{R}^n, t>0} \frac{1}{t^{\frac{\lambda}{q}}} \|f\chi_B\|_{\mathscr{L}^{p,q}} \right\}$$

with $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $0 \leqslant \lambda < n$ and $B = B(x,t)$. Obviously, $\mathscr{R}^{p,q,\lambda}$ becomes $L^{p,\lambda}$ if we choose $p = q$.

For the studies of integral operators on $\mathscr{R}^{p,q,\lambda}$, one may see the paper [2, 26, 31] et. al. to find more details.

Next, we will give an application of Theorem 1.2 under the Morrey-Lorentz setting. To see this, we give the following arguments and estimates.

Suppose that $\phi \in C^\infty(\mathbb{R}^n)$ is a non-negtive, smooth and rapidly decreasing function. Moreover, we assume that $\phi$ satisfies the following condition.

$$|\phi'(t)| \leqslant Ct^{-1}, \qquad \chi_{[0,1]}(t) \leqslant \phi(t) \leqslant \chi_{[0,2]}(t).$$

For any $\varepsilon > 0$, we denote

$$T_b^*(f)(x) = \sup_{\varepsilon>0} \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} |b(x) - b(y)| \phi\left(\frac{|x-y|}{\varepsilon}\right) f(y)dy$$

and

$$T^*(f)(x) = \sup_{\varepsilon>0} \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \phi\left(\frac{|x-y|}{\varepsilon}\right) f(y)dy$$

From [33, Lemma 3.2], we know that

$$M_b \sim T_b^* \text{ and } M \sim T^*. \tag{5.1}$$

Moreover, for $b \in \mathrm{BMO}$, by the similar estimates of [33, Lemma 3.3] and [25, Lemma 3.1], we can easily obtain

$$M_\delta^\sharp(T_b^*(f))(x) \leqslant C\|b\|_{\mathrm{BMO}} \left(M_\gamma(M(f))(x) + M^2(f)(x)\right) \tag{5.2}$$

for all bounded $f$ with compact support and $0 < \delta < \gamma$. Here $M^\sharp(f)(x)$ denotes the usual sharp maximal function, $M_\gamma(f)(x) = M(|f|^\gamma)(x)^{\frac{1}{\gamma}}$ and $M^2 = M \circ M$.

Then, we assume there exist a series of numbers $q_i, p_i$ satisfying $1 \leqslant q_i < p_i < \infty$. Moreover, denote $\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i}$ and $\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}$ with $1 \leqslant q < p < \infty$.

From [4, Chapter 3, Theorem 5.7 and Chapter 4, Theorem 4.6], we know that the Hardy-Littlewood maximal function is bounded on $\mathscr{L}^{p_i,q_i}$ with $1 \leqslant q_i < p_i < \infty$. If we choose $\gamma < q_i$ with $i = 1,2\cdots,m$, there is

$$\|M_\gamma(M(f))\|_{\mathscr{L}^{p_i,q_i}} = \|M(M(f)^\gamma)\|^{\frac{1}{\gamma}}_{\mathscr{L}^{\frac{p_i}{\gamma},\frac{q_i}{\gamma}}} \leqslant C\|M(f)^\gamma\|^{\frac{1}{\gamma}}_{\mathscr{L}^{\frac{p_i}{\gamma},\frac{q_i}{\gamma}}} \leqslant C\|f\|_{\mathscr{L}^{p_i,q_i}}.$$

Thus, using the Fefferman-Stein inequality on Lorentz space (see [5, Lemma 2.6] for the unweighted case with $u = 1$ and $\omega(t) = t^{\frac{p}{q}} - 1$) (5.2) and the assumption $f$ is bounded with compact support, we obtain

$$\begin{aligned}\|T_b^*(f)\|_{\mathscr{L}^{p_i,q_i}} &\leqslant \|M_\delta(T_b^*(f))\|_{\mathscr{L}^{p_i,q_i}} \leqslant \|M_\delta^\sharp(T_b^*(f))\|_{\mathscr{L}^{p_i,q_i}}\\ &\leqslant C\|b\|_{\mathrm{BMO}}\left(\|M_\gamma(M(f))\|_{\mathscr{L}^{p_i,q_i}} + \|M^2(f)\|_{\mathscr{L}^{p_i,q_i}}\right) \leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{\mathscr{L}^{p_i,q_i}}\end{aligned}$$

for all bounded $f$ with compact support.

Then, by the similar density arguments and estimates as in [5, p. 8989], we know that $T_b^*$ is bounded on $\mathscr{L}^{p_i,q_i}$ with all $f \in \mathscr{L}^{p_i,q_i}$. Thus, we get that $M_b$ is also bounded on $\mathscr{L}^{p_i,q_i}$ from (5.1).

Using the Hölder inequality on the Lorentz spaces (see [8, Proposition 2.11]) and [4, Chapter 3, Theorem 5.7 and Chapter 4, Theorem 4.6] again, we know that there exists a positive constant $C$ depending on $p, p_1, \cdots, p_m$, such that

$$\begin{aligned}\|\mathscr{M}^{b_1}(f)\|_{\mathscr{L}^{p,q}} &\leqslant \|M_{b_1}(f_1)M(f_2)\cdots M(f_m)\|_{\mathscr{L}^{p,q}}\\ &\leqslant \|M_{b_1}(f_1)\|_{\mathscr{L}^{p_1,q_1}}\prod_{i=2}^m\|M(f_i)\|_{\mathscr{L}^{p_i,q_i}} \leqslant C\|b_1\|_{\mathrm{BMO}}\prod_{i=1}^m\|f_i\|_{\mathscr{L}^{p_i,q_i}}.\end{aligned}$$

Similarly, there is

$$\|\mathscr{M}(f)\|_{\mathscr{L}^{p,q}} \leqslant C\prod_{i=1}^m\|f_i\|_{\mathscr{L}^{p_i,q_i}}$$

with $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$ and $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i}$.

Moreover, it is easy to see that for any ball $B = B(x,t)$, there is $\|\chi_B\|_{\mathscr{L}^{p,q}} = t^{\frac{n}{p}}$. Thus, it is easy to see

$$\|\chi_B\|_{\mathscr{L}^{p,q}} \sim \prod_{i=1}^m\|\chi_B\|_{\mathscr{L}^{p_i,q_i}} \tag{5.3}$$

with $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$.

From now on, we assume that $\frac{\lambda}{n} < \frac{q_i}{p_i}$. For any $x \in \mathbb{R}^n$ and $t > 0$, we denote $u_i(x,t) = t^{\frac{\lambda}{q_i}}$ with $q_i \geqslant 1$. Then, we obtain $u_i(x_1,t_1) \leqslant u_i(x_2,t_2)$ for $t_1 \leqslant t_2$ and $\forall x_1, x_2 \in \mathbb{R}^n$.

Thus, we have

$$\frac{\|\chi_{B(x_1,t_1)}\|_{\mathscr{L}^{p_i,q_i}}}{t_1^{\frac{\lambda}{q_i}}} = t_1^{\frac{n}{p_i}-\frac{\lambda}{q_i}} \leqslant t_2^{\frac{n}{p_i}-\frac{\lambda}{q_i}} = \frac{\|\chi_{B(x_2,t_2)}\|_{\mathscr{L}^{p_i,q_i}}}{t_2^{\frac{\lambda}{q_i}}}$$

with $u_i(x_1,t_1) \leqslant u_i(x_2,t_2)$ and $\forall x_1,x_2 \in \mathbb{R}^n$, which implies $u_i(x,t)$ satisfies (2.2).

Moreover, there is

$$\begin{aligned}\sum_{j=0}^{\infty}(j+1)\frac{\|\chi_{B(x,t)}\|_{\mathscr{L}^{p_i,q_i}}}{\|\chi_{B(x,2^{j+1}t)}\|_{\mathscr{L}^{p_i,q_i}}}u_i(x,2^{j+1}t) &= \sum_{j=0}^{\infty}(j+1)\frac{t^{\frac{n}{p_i}}}{(2^{j+1}t)^{\frac{n}{p_i}}}(2^{j+1}t)^{\frac{\lambda}{q_i}} \\ &= \sum_{j=0}^{\infty}(j+1)2^{(j+1)(\frac{\lambda}{q_i}-\frac{n}{p_i})}t^{\frac{\lambda}{q_i}} \\ &\leqslant Ct^{\frac{\lambda}{q_i}} = Cu_i(x,t).\end{aligned}$$

Consequently, we obtain $u_i(x,t) = t^{\frac{\lambda}{q_i}} \in \mathbb{W}^1_{\mathscr{L}^{p_i,q_i}}$.

Finally, from [4, Corollary 4.8], we know that the dual space of $\mathscr{L}^{p,q}$ is $\mathscr{L}^{p',q'}$ with the condition $1 \leqslant q < p < \infty$. Thus, using [4, Chapter 3, Theorem 5.7 and Chapter 4, Theorem 4.6] again, we obtain that the Hardy-Littlewood maximal function is also bounded on $\mathscr{L}^{p',q'}$, which implies $\mathscr{L}^{p,q} \in \mathbb{M}'$.

Combing the above arguments and estimates, we have the following conclusions.

REMARK 5.1. Suppose that there exists a series of numbers $q_i, p_i$ satisfying $1 \leqslant q_i < p_i < \infty$. Moreover, denote $\frac{1}{q} = \sum_{i=1}^{m}\frac{1}{q_i}$ and $\frac{1}{p} = \sum_{i=1}^{m}\frac{1}{p_i}$ with $1 \leqslant q < p < \infty$. If $0 \leqslant \lambda < n$ and $\frac{\lambda}{n} < \frac{q_i}{p_i}$, then we get

(i) $\mathscr{L}^{p,q} \in \mathbb{M}'$, $\mathscr{L}^{p_i,q_i} \in \mathbb{M}$, $u_i(x,t) = t^{\frac{\lambda}{q_i}} \in \mathbb{W}^1_{\mathscr{L}^{p_i,q_i}}$

(ii) $\|\chi_B\|_{\mathscr{L}^{p,q}} \sim \prod_{i=1}^{m}\|\chi_B\|_{\mathscr{L}^{p_i,q_i}}$.

(iii) Both $\mathscr{M}(\vec{f})(x)$ and $\mathscr{M}^{b_i}(\vec{f})(x)$ are bounded from $\mathscr{L}^{p_1,q_1} \times \cdots \times \mathscr{L}^{p_m,q_m}$ to $\mathscr{L}^{p,q}$ with $b_i \in \mathrm{BMO}$.

Thus, we have the following results according to Theorem 1.2.

THEOREM 5.2. *Let $b_i$ be a real valued, locally integrable function in $\mathbb{R}^n$ with $i = 1,\cdots,m$. Suppose that there exists a series of numbers $q_i, p_i$ satisfying $1 \leqslant q_i < p_i < \infty$. Moreover, denote $\frac{1}{q} = \sum_{i=1}^{m}\frac{1}{q_i}$ and $\frac{1}{p} = \sum_{i=1}^{m}\frac{1}{p_i}$ with $1 \leqslant q < p < \infty$. If $0 \leqslant \lambda < n$ and $\frac{\lambda}{n} < \frac{q_i}{p_i}$, then the following three statements are equivalent.*

*(I) The commutator $[\vec{b},\mathscr{M}]$ is bounded from $\mathscr{R}^{p_1,q_1,\lambda} \times \cdots \times \mathscr{R}^{p_m,q_m,\lambda}$ to $\mathscr{R}^{p,q,\lambda}$*

*(II) $b_i$ is in BMO and $b_i^-$ belongs to $L^\infty$.*

*(III) Define $M_Q(b_i)(x) = \sup_{Q_0 \ni x, Q_0 \subset Q}\frac{1}{|Q_0|}\int_{Q_0}|b_i(t)|dt$, for any $i = 1,2,\ldots,m$, there is*

$$\sup_{Q}\frac{\|(M_Q(b_i)-b_i)\chi_Q\|_{\mathscr{L}^{p,q}}}{\prod_{i=1}^{m}\|\chi_Q\|_{\mathscr{L}^{p_i,q_i}}} < \infty.$$

REMARK 5.3. As far as we know, Theorem 5.2 also seems to be a new result even in the one-linear case.

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## REFERENCES

[1] M. AGCAYAZI, A. GOGATISHVILI, K. KOCA, *A note on maximal commutators and commutators of maximal functions*, J. Math. Soc. Japan **67**, 581–593 (2015).

[2] C. AYKOL, V. S. GULIYEV, A. SERBETCI, *Boundedness of the maximal operator in the local Morrey-Lorentz spaces*, J. Inequal. Appl., **2013**, 346 (2013).

[3] J. BASTERO, M. MILMAN, F. J. RUIZ, *Commutators for the maximal and sharp functions*, Proc. Amer. Math. Soc. **128**, 3329–3334 (2000).

[4] C. BENNETT, R. SHARPLEY, *Interpolations of Operators*, Academic Press, Cambridge (1988).

[5] M. J. CARRO, H. LI, J. SORIA, Q. SUN, *Calderón-Zygmund Operators and Commutators on Weighted Lorentz Spaces*, J. Geom. Anal. **31**: 8979–8990 (2021).

[6] L. CHAFFEE, D. CRUZ-URIBE, *Necessary conditions for the boundedness of linear and bilinear commutators on Banach function spaces*, Math. Inequal. Appl. **21**, 1–16 (2018).

[7] D. CRUZ-URIBE, A. FIORENZA, *Variable Lebesgue Spaces*, Birkhuser, Basel (2013).

[8] N. A. DAO, S. G. KRANTZ, *Lorentz boundedness and compactness characterization of integral commutators on spaces of homogeneous type*, Nonlinear Anal. **203**, 112162 (2021).

[9] L. DIENING, P. HARJULEHTO, P. HÄSTÖ AND M. RUŽIČKA, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Math., Springer (2011).

[10] L. GRAFAKOS, R. TORRES, *Multilinear Calderón-Zygmund theory*, Adv. Math. **165**, 124–164 (2002).

[11] K.-P. HO, *Vector-valued singular integral operators on Morrey type spaces and variable Triebel-Lizorkin-Morrey spaces*, Ann. Acad. Sci. Fenn. Math. **37**, 375–406 (2012).

[12] K.-P. HO, *Atomic decomposition of Hardy spaces and characterization of BMO via Banach function spaces*, Anal. Math. **38**, 173–185 (2012).

[13] K.-P. HO, *Vector-valued operators with singular kernel and Triebel-Lizorkin-block sopaces with variable exponents*, Kyoto J. Math. **56**, 97–124 (2016).

[14] K.-P. HO, *Fractional integral operators with homogeneous kernels on Morrey type spaces with variable exponents*, J. Math. Soc. Japan **69**, 1059–1077 (2017).

[15] K.-P. HO, *Singular integral operators with rough kernels on Morrey type spaces*, Studia Math. **244**, 217–243 (2019).

[16] K.-P. HO, *Weak type estimates of singular integral operators on Morrey-Banach spaces*, Integr. Equ. Oper. Theory **91**, 18 pp. (2019).

[17] K.-P. HO, *Weak type estimates of the fractional integral operators on Morrey spaces with variable exponents*, Acta Appl. Math. **159**, 1–10 (2019).

[18] K.-P. HO, *Definability of singular integral operators on Morrey-Banach spaces*, J. Math. Soc. Japan **72**, 155–170 (2020).

[19] K.-P. HO, *Nonlinear commutators on Morrey-Banach spaces*, J. Pseudo-Differ. Oper. Appl. **12**, 12 pp (2021).

[20] A. K. LERNER, S. OMBROSI, C. PÉREZ, R. TORRES, R. TRUJILLO-GONZALES, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory*, Adv. Math. **220**, 1222–1264 (2009).

[21] G. G. LORENTZ, *Some new functional spaces*, Ann. Math. **51**, 37–55 (1950).

[22] G. G. LORENTZ, *On the theory of space*, Pac. J. Math. **1**, 411–429 (1951).

[23] C. B. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, 126–166 (1938).

[24] B. MUCKENHOUPT, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165**, 207–226 (1972).

[25] C. PÉREZ, *Endpoint estimates for commutators of singular integral operators*, J. Funct. Anal. **128**, 163–185 (1995).

[26] M. A. RAGUSA, *Embeddings for Morrey-Lorentz Spaces*, J. Optim. Theory Appl. **154**, 491–499 (2012).

[27] H. TRIEBEL, *Hybrid Function Spaces, Heat and Navier-Stokes Equations*, Switzerland, EMS Tracts in Mathematics (2015).

[28] W. WANG, J. XU, *Multilinear Calderon-Zygmund operators and their commutators with BMO functions in variable exponent Morrey spaces*, Front. Math. China. **12**, 1235–1246 (2017).

[29] C. XIE, *Some estimates of commutators*, Real. Anal. Exch. **36**, 405–416 (2010/2011).

[30] J. XU, *The boundedness of multilinear commutators of singular integrals on Lebesgue spaces with variable exponent*, Czechoslovak Math. J. **57**, 13–27 (2007).

[31] T.-L. YEE, K.-P. HO, *Fractional integral operators with homogeneous kernels on generalized Lorentz-Morrey spaces*, J. Math. Inequal., **15**, 17–30 (2021).

[32] X. YU, P. ZHANG, H. LI, *Some estimates for the commutators of multilinear maximal function on Morrey type space*, Open Math. **19**, 515–530 (2021).

[33] P. ZHANG, *Multiple Weighted Estimates for Commutators of Multilinear Maximal Function*, Acta Math. Sinica **31**, 973–994 (2015).

[34] P. ZHANG, *Commutators of Multi-sublinear Maximal Functions with Lipschitz Functions*, Results Math. **74**, 21 pp (2019).

[35] P. ZHANG, J. WU, *Hardy-Littlewood maximal operator on* $L^{p(x)}(\mathbb{R})$, Math. Inequal. Appl. **7**, 255–265 (2014).

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