Journal of
Mathematical
Inequalities
Volume 16, Number 3 (2022), 923–945 doi:10.7153/jmi-2022-16-63

A METHOD TO PROVE INEQUALITIES AND ITS APPLICATIONS

CHAO-PING CHEN* AND BRANKO MALEŠEVIĆ

(*Communicated by T. Burić*)

Abstract. In this paper, we provide a method to prove an inequality. As applications of the method developed here, we establish Wilker and Huygens type inequalities for Gauss lemniscate functions, and present sharp inequalities between the inverse hyperbolic tangent and inverse sine functions. We also present sharp inequalities for trigonometric functions.

1. A method to prove inequalities

Let the functions f and g be both strictly increasing on $[a,b]$, and let

$$F(x) = f(x) - g(x), \qquad x \in [a,b].$$

In this paper, we first provide a method to prove

$$F(x) > 0, \qquad x \in [a,b].$$

The details of this method are given below.

We divide the interval $[a,b]$ into n subintervals

$$[a,b] = [x_0,x_1] \cup [x_1,x_2] \cup \cdots \cup [x_{n-1},x_n],$$

where $x_0 = a$ and $x_n = b$. The choices of x_i $(i = 0,1,2,\ldots,n)$ are as follows:

Solving the equation $f(x_0) = g(x)$ yields

$$x_1 = g^{-1}\big(f(x_0)\big),$$

where g^{-1} is the inverse function of g. Since the functions f and g are both strictly increasing on $[x_0,x_1]$, we have

$$\min_{x\in[x_0,x_1]} f(x) = f(x_0) = g(x_1) = \max_{x\in[x_0,x_1]} g(x),$$

Mathematics subject classification (2020): 26D05.
Keywords and phrases: Inequalities, lemniscate functions, trigonometric functions, inverse trigonometric functions, inverse hyperbolic functions.

* Corresponding author.

Paper JMI-16-63

and then, we see that

$$f(x) > g(x), \qquad x \in [x_0, x_1].$$

Solving the equation $f(x_1) = g(x)$ yields

$$x_2 = g^{-1}\big(f(x_1)\big),$$

and we have

$$f(x) > g(x), \qquad x \in [x_1, x_2].$$

Continuing the above process, we get

$$x_{n-1} = g^{-1}\big(f(x_{n-2})\big),$$

and we have

$$f(x) > g(x), \qquad x \in [x_{n-2}, x_{n-1}].$$

Finally, we show by elementary calculation that

$$f(x_{n-1}) \geqslant g(x_n),$$

which implies

$$f(x) > g(x), \qquad x \in [x_{n-1}, x_n].$$

Our method shows $f(x) > g(x)$ on every subinterval. This derives $F(x) = f(x) - g(x) > 0$ for all $x \in [a,b]$.

REMARK 1.1. Let the functions f and g be both strictly increasing (decreasing) on $[a,b]$. In order to show

$$F(x) = f(x) - g(x) > 0, \qquad x \in [a,b],$$

we frequently (for convenience) divide the interval $[a,b]$ into n equal parts

$$[a,b] = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-1}, x_n],$$

where

$$x_i = a + \frac{b-a}{n} i, \qquad i = 0, 1, 2, \ldots, n.$$

And then, we show by elementary calculation that

$$f(x_i) > g(x_{i+1}) \quad \big(f(x_{i+1}) > g(x_i)\big), \qquad i = 0, 1, 2, \ldots, n-1.$$

This means that $f(x) > g(x)$ holds on every subinterval. We then obtain $F(x) = f(x) - g(x) > 0$ for all $x \in [a,b]$.

As applications of the method developed here, we establish Wilker and Huygens type inequalities for Gauss lemniscate functions (Section 2). We present sharp inequalities between the inverse hyperbolic tangent and inverse sine functions (Section 3). Also, we present sharp inequalities for trigonometric functions (Section 4).

The numerical values given have been calculated using the computer program MAPLE 11.

2. Wilker and Huygens type inequalities for Gauss lemniscate functions

Wilker [48] proposed, and then Sumner et al. [46] proved, for $0 < x < \pi/2$,

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \tag{2.1}$$

and

$$2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45} x^3 \tan x, \tag{2.2}$$

where the constants $(2/\pi)^4$ and $8/45$ are the best possible.

A related inequality that is of interest to us is Huygens' inequality [20], which asserts that

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3, \qquad 0 < |x| < \frac{\pi}{2}. \tag{2.3}$$

Chen and Cheung [11] developed (2.3) to produce a double inequality

$$3 + \frac{3}{20} x^3 \tan x < 2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} < 3 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x \tag{2.4}$$

for $0 < |x| < \frac{\pi}{2}$, where the constants $\frac{3}{20}$ and $\left(\frac{2}{\pi}\right)^4$ are the best possible.

The Wilker and Huygens type inequalities (2.1), (2.2), (2.3) and (2.4) have attracted much interest of many mathematicians and have motivated a large number of research papers involving different proofs, various generalizations and improvements (cf. [2, 7, 10, 11, 14, 15, 16, 19, 24, 27, 28, 29, 32, 33, 34, 35, 42, 44, 49, 50, 51, 52, 53, 58, 59, 60, 61, 63, 64, 62] and the references cited therein).

The hyperbolic versions of the Wilker and Huygens type inequalities were established in [42, 51, 61]. The Wilker and Huygens type inequalities for inverse trigonometric functions were presented in [7, 24, 58]. The Wilker and Huygens type inequalities have also been established for the lemniscate functions and Jacobian elliptic and theta functions. For more details see [8, 9, 36] and [37], respectively.

The lemniscate, also called the lemniscate of Bernoulli, is the locus of points (x, y) in the plane satisfying the equation $(x^2 + y^2)^2 = x^2 + y^2$. In polar coordinates (r, θ), the equation becomes $r^2 = \cos(2\theta)$ and its arc length is given by the function

$$\operatorname{arcsl} x = \int_0^x \frac{1}{\sqrt{1 - t^4}} \mathrm{d}t, \qquad |x| \leqslant 1, \tag{2.5}$$

where arcsl is called the arc lemniscate sine function studied by Gauss in 1797-1798. Another lemniscate function investigated by Gauss is the hyperbolic arc lemniscate sine function, defined as

$$\operatorname{arcslh} x = \int_0^x \frac{1}{\sqrt{1 + t^4}} \mathrm{d}t, \qquad x \in \mathbb{R}. \tag{2.6}$$

The functions (2.5) and (2.6) can be found (see [45, Ch. 1], [5, p. 259] and [6, 8, 9, 18, 36, 38, 39, 40, 41]).

Another pair of lemniscate functions, the arc lemniscate tangent arctl and the hyperbolic arc lemniscate tangent arctlh, have been introduced in [38, (3.1)–(3.2)]. Therein it has been proven that

$$\operatorname{arctl} x = \operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1+x^4}}\right), \qquad x \in \mathbb{R} \tag{2.7}$$

and

$$\operatorname{arctlh} x = \operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1-x^4}}\right), \qquad |x| < 1 \tag{2.8}$$

(see [38, Prop. 3.1]).

Recently, numerous inequalities have been given for the lemniscate functions [8, 9, 18, 23, 40, 47]. For example, Neuman [40] proved the following inequalities:

$$\left(\frac{5}{3+2(1-x^4)^{1/2}}\right)^{1/2} < \frac{\operatorname{arcsl} x}{x} < (1-x^4)^{-1/10} \tag{2.9}$$

and

$$\left(\frac{5}{3+2(1+x^4)^{1/2}}\right)^{1/2} < \frac{\operatorname{arcslh} x}{x} < (1+x^4)^{-1/10} \tag{2.10}$$

for $0 < |x| < 1$.

In analogy with (2.2) and (2.4), in this section we establish Wilker and Huygens type inequalities for Gauss lemniscate functions.

2.1. Wilker type inequalities

For $0 \leqslant x \leqslant 1$, let

$$W_1(x) = \frac{\left(\frac{\operatorname{arcsl} x}{x}\right)^2 + \frac{\operatorname{arctl} x}{x} - 2}{x^3 \operatorname{arctl} x}.$$

The computer program MAPLE 11 suggests that the function $W_1(x)$ is strictly increasing on $[0,1]$, and

$$W_1(0) = \lim_{x\to 0} W_1(x) = \frac{1}{20} \quad \text{and} \quad W_1(1) = 0.6860098\ldots.$$

This fact motivated us to establish Theorem 2.1.

THEOREM 2.1. *For $0 < x < 1$, we have*

$$2 + \frac{1}{20}x^3 \operatorname{arctl} x < \left(\frac{\operatorname{arcsl} x}{x}\right)^2 + \frac{\operatorname{arctl} x}{x} < 2 + \frac{687}{1000}x^3 \operatorname{arctl} x. \tag{2.11}$$

Proof. The left-hand side of (2.11) has been proved in [8]. Here, we only prove the right-hand side of (2.11). It suffices to show that for $0<x<1$,

$$g(x)<0,$$

where

$$g(x)=\left(\frac{\operatorname{arcsl}x}{x}\right)^2+\left(\frac{1}{x}-\frac{687}{1000}x^3\right)\operatorname{arctl}x-2. \tag{2.12}$$

We consider two cases to prove $g(x)<0$ for $0<x<1$.

Case 1. $0<x<0.9$.

It follows from [9] that

$$x-\frac{3}{20}x^5<\operatorname{arctl}x<x, \qquad 0<x<1. \tag{2.13}$$

By the right-hand sides of (2.9) and (2.13), we have

$$g(x)<\left(\frac{1}{1-x^4}\right)^{1/5}+\left(\frac{1}{x}-\frac{687}{1000}x^3\right)x-2=-\left\{\left(1+\frac{687}{1000}x^4\right)-\left(\frac{1}{1-x^4}\right)^{1/5}\right\}.$$

Elementary calculations show that, for $0<x<0.9$,

$$\left(1+\frac{687}{1000}x^4\right)^5-\frac{1}{1-x^4}>1+\frac{687}{200}x^4-\frac{1}{1-x^4}=\frac{x^4(487-687x^4)}{200(1-x^4)}>0.$$

We then obtain $g(x)<0$ for $0<x<0.9$.

Case 2. $0.9\leqslant x<1$.

Write (2.12) as

$$-x^2g(x)=f_1(x)-g_1(x),$$

where

$$f_1(x)=2x^2+\frac{687}{1000}x^5\operatorname{arctl}x \quad \text{and} \quad g_1(x)=(\operatorname{arcsl}x)^2+x\operatorname{arctl}x$$

are both strictly increasing. We divide the interval $[0.9,1]$ into 1000 subintervals:

$$[0.9,1]=\bigcup_{k=0}^{999}\left[0.9+\frac{k}{10000},0.9+\frac{k+1}{10000}\right] \quad \text{for} \quad k=0,1,2,\ldots,999.$$

By direct computation we get

$$f_1\left(0.9+\frac{0}{10000}\right)-g_1\left(0.9+\frac{0+1}{10000}\right)=0.15181132183\ldots>0,$$

$$f_1\left(0.9+\frac{1}{10000}\right)-g_1\left(0.9+\frac{1+1}{10000}\right)=0.15198131579\ldots>0,$$

$$\vdots$$

$$f_1\left(0.9+\frac{998}{10000}\right)-g_1\left(0.9+\frac{998+1}{10000}\right)=0.02557091989\ldots>0,$$

$$f_1\left(0.9+\frac{999}{10000}\right)-g_1\left(0.9+\frac{999+1}{10000}\right)=0.00013841386\ldots>0.$$

We then obtain

$$f_1\left(0.9+\frac{k}{10000}\right)-g_1\left(0.9+\frac{k+1}{10000}\right)>0 \quad \text{for} \quad k=0,1,2,\ldots,999.$$

Hence,

$$-x^2g(x)>0 \text{ for } x\in\left[0.9+\frac{k}{10000},0.9+\frac{k+1}{10000}\right] \text{ and } k=0,1,2,\ldots,999.$$

This implies that $g(x)$ is negative for $0.9\leqslant x<1$. Hence, $g(x)<0$ holds for $0<x<1$. The proof of Theorem 2.1 is complete. $\square$

For $0\leqslant x\leqslant 1$, let

$$W_2(x)=\frac{2-\left(\frac{x}{\operatorname{arcsl}x}\right)^2-\frac{x}{\operatorname{arctl}x}}{x^3\operatorname{arctl}x}.$$

By using the computer program MAPLE 11, we find that the function $W_2(x)$ is strictly increasing on $[0,1]$, and

$$W_2(0)=\lim_{x\to 0}W_2(x)=\frac{1}{20} \quad \text{and} \quad W_2(1)=0.3367687\ldots.$$

This fact motivated us to establish Theorem 2.2.

THEOREM 2.2. *For $0<x<1$, we have*

$$2-\frac{337}{1000}x^3\operatorname{arctl}x<\left(\frac{x}{\operatorname{arcsl}x}\right)^2+\frac{x}{\operatorname{arctl}x}<2-\frac{1}{20}x^3\operatorname{arctl}x. \tag{2.14}$$

Proof. The left-hand side of (2.14) is obtained by considering the function $f(x)$ defined, for $0<x<1$, by

$$f(x)=\left(\frac{x}{\operatorname{arcsl}x}\right)^2+\frac{x}{\operatorname{arctl}x}-2+\frac{337}{1000}x^3\operatorname{arctl}x. \tag{2.15}$$

We consider two cases to prove $f(x) > 0$ for $0 < x < 1$.

Case 1. $0 < x < 0.9$.

By the right-hand side of (2.9) and (2.13), we have

$$f(x) > \left(1-x^4\right)^{1/5} - 1 + \frac{337}{1000}x^3\left(x - \frac{3}{20}x^5\right)$$
$$= \left(1-x^4\right)^{1/5} - \left(1 - \frac{337}{1000}x^4 + \frac{1011}{20000}x^8\right).$$

Elementary calculations show that, for $0 < x < 0.9$,

$$\begin{aligned}
&\left(1-x^4\right) - \left(1 - \frac{337}{1000}x^4 + \frac{1011}{20000}x^8\right)^5 \\
&= x^4\left\{\frac{137}{200} - \frac{34711}{25000}x^4 + \frac{72343453}{100000000}x^8 - \frac{52454000461}{200000000000}x^{12}\right\} \\
&\quad + x^{20}\left\{\frac{68874459843457}{1000000000000000} - \frac{53030994466071}{4000000000000000}x^4\right\} \\
&\quad + x^{28}\left\{\frac{73943762523813}{40000000000000000} - \frac{35869109293341}{200000000000000000}x^4\right\} \\
&\quad + x^{36}\left\{\frac{352074461122017}{32000000000000000000} - \frac{1056223383366051}{320000000000000000000}x^4\right\} > 0,
\end{aligned}$$

since each of the terms in the braces is positive for $0 < x < 0.9$. We then obtain $f(x) > 0$ for $0 < x < 0.9$.

Case 2. $0.9 \leqslant x < 1$.

Write (2.15) as

$$f(x) = f_2(x) - g_2(x),$$

where

$$f_2(x) = \frac{x}{\operatorname{arctl} x} + \frac{337}{1000}x^3 \operatorname{arctl} x \quad \text{and} \quad g_2(x) = 2 - \left(\frac{x}{\operatorname{arcsl} x}\right)^2$$

are both strictly increasing. The proofs of the monotonicity properties for $f_1(x)$ and $f_2(x)$ are easy, we here omit the proofs.

We divide the interval $[0.9, 1]$ into 1000 subintervals:

$$[0.9, 1] = \bigcup_{k=0}^{999}\left[0.9 + \frac{k}{10000}, 0.9 + \frac{k+1}{10000}\right] \quad \text{for} \quad k = 0, 1, 2, \ldots, 999.$$

By direct computation we get

$$f_2\left(0.9 + \frac{0}{10000}\right) - g_2\left(0.9 + \frac{0+1}{10000}\right) = 0.11846213787\ldots > 0,$$

$$f_2\left(0.9+\frac{1}{10000}\right)-g_2\left(0.9+\frac{1+1}{10000}\right)=0.11847537183\ldots>0,$$

$$\vdots$$

$$f_2\left(0.9+\frac{998}{10000}\right)-g_2\left(0.9+\frac{998+1}{10000}\right)=0.00877117141\ldots>0,$$

$$f_2\left(0.9+\frac{999}{10000}\right)-g_2\left(0.9+\frac{999+1}{10000}\right)=0.00005905035\ldots>0.$$

Hence,

$$f(x)>0 \quad \text{for} \quad x\in\left[0.9+\frac{k}{10000},0.9+\frac{k+1}{10000}\right] \quad \text{and} \quad k=0,1,2,\ldots,999.$$

This implies that $f(x)$ is positive for $0.9\leqslant x<1$. Hence, $f(x)>0$ holds for $0<x<1$.

The right-hand side of (2.14) is obtained by considering the function $F(x)$ defined, for $0<x<1$, by

$$F(x)=\left(\frac{x}{\operatorname{arcsl}x}\right)^2+\frac{x}{\operatorname{arctl}x}-2+\frac{1}{20}x^3\operatorname{arctl}x.$$

By the left-hand side of (2.9) and (2.13), we have, for $0<x<1$,

$$\begin{aligned}F(x)&<\frac{3+2(1-x^4)^{1/2}}{5}+\frac{x}{x-\frac{3}{20}x^5}-2+\frac{1}{20}x^4\\&=-\frac{160-104x^4+3x^8-(160-24x^4)\sqrt{1-x^4}}{20(20-3x^4)}.\end{aligned}$$

Elementary calculations show that, for $0<x<1$,

$$\left(160-104x^4+3x^8\right)^2-\left((160-24x^4)\sqrt{1-x^4}\right)^2=x^8(3520-48x^4+9x^8)>0.$$

We then obtain $F(x)<0$ for $0<x<1$. The proof of Theorem 2.2 is complete. □

2.2. Huygens type inequalities

For $0\leqslant x\leqslant 1$, let

$$H_1(x)=\frac{\frac{2\operatorname{arcsl}x}{x}+\frac{\operatorname{arctl}x}{x}-3}{x^3\operatorname{arctl}x}.$$

By using the computer program MAPLE 11, we find that the function $H_1(x)$ is strictly increasing on $[0,1]$, and

$$H_1(0)=\lim_{x\to 0}H_1(x)=\frac{1}{20} \quad \text{and} \quad H_1(1)=0.57799173\ldots.$$

This fact motivated us to establish Theorem 2.3.

THEOREM 2.3. *For* $0 < x < 1$, *we have*

$$3+\frac{1}{20}x^3\operatorname{arctl}x < 2\left(\frac{\operatorname{arcsl}x}{x}\right)+\frac{\operatorname{arctl}x}{x} < 3+\frac{578}{1000}x^3\operatorname{arctl}x. \tag{2.16}$$

Proof. By the left-hand sides of (2.9) and (2.13), we have, for $0 < x < 1$,

$$\begin{aligned}
&2\left(\frac{\operatorname{arcsl}x}{x}\right)+\frac{\operatorname{arctl}x}{x}-3-\frac{1}{20}x^3\operatorname{arctl}x\\
&=2\left(\frac{\operatorname{arcsl}x}{x}\right)+\left(\frac{1}{x}-\frac{1}{20}x^3\right)\operatorname{arctl}x-3\\
&>2\left(\frac{5}{3+2(1-x^4)^{1/2}}\right)^{1/2}+\left(\frac{1}{x}-\frac{1}{20}x^3\right)\left(x-\frac{3}{20}x^5\right)-3\\
&=2\left(\frac{5}{3+2(1-x^4)^{1/2}}\right)^{1/2}-\left(2+\frac{1}{5}x^4\right)+\frac{3}{400}x^8.
\end{aligned}$$

It is easy to prove that

$$2\left(\frac{5}{3+2(1-x^4)^{1/2}}\right)^{1/2} > 2+\frac{1}{5}x^4 \quad \text{for} \quad 0<x<1$$

(we here omit the proof). We then see that the left-hand side of (2.16) is valid for $0 < x < 1$.

The right-hand side of (2.16) is obtained by considering the function $G(x)$ defined, for $0 < x < 1$, by

$$G(x)=3-2\left(\frac{\operatorname{arcsl}x}{x}\right)-\left(\frac{1}{x}-\frac{578}{1000}x^3\right)\operatorname{arctl}x. \tag{2.17}$$

We consider two cases to prove $G(x) > 0$ for $0 < x < 1$.

Case 1. $0 < x < 0.9$.

By the right-hand sides of (2.9) and (2.13), we have

$$G(x)>3-2\left(\frac{1}{(1-x^4)^{1/10}}\right)-\left(\frac{1}{x}-\frac{578}{1000}x^3\right)x=2\left\{1+\frac{289}{1000}x^4-\frac{1}{(1-x^4)^{1/10}}\right\}.$$

Elementary calculations show that, for $0 < x < 0.9$,

$$\begin{aligned}
\left(1+\frac{289}{1000}x^4\right)^{10}-\frac{1}{1-x^4} &> 1+\frac{289}{100}x^4+\frac{751689}{200000}x^8-\frac{1}{1-x^4}\\
&=\frac{x^4(378000+173689x^4-751689x^8)}{200000(1-x^4)}>0.
\end{aligned}$$

We then obtain $G(x) > 0$ for $0 < x < 0.9$.

Case 2. $0.9 \leqslant x < 1$.
Write (2.17) as

$$xG(x) = f_3(x) - g_3(x),$$

where

$$f_3(x) = 3x + \frac{578}{1000}x^4 \operatorname{arctl} x. \quad \text{and} \quad g_3(x) = 2\operatorname{arcsl} x + \operatorname{arctl} x$$

are both strictly increasing. We divide the interval $[0.9, 1]$ into 100000 subintervals:

$$[0.9, 1] = \bigcup_{k=0}^{99999} \left[0.9 + \frac{k}{1000000}, 0.9 + \frac{k+1}{1000000}\right] \quad \text{for} \quad k = 0, 1, 2, \ldots, 99999.$$

By direct computation we get

$$f_3\left(0.9 + \frac{0}{1000000}\right) - g_3\left(0.9 + \frac{0+1}{1000000}\right) = 0.210401355429\ldots > 0,$$

$$f_3\left(0.9 + \frac{1}{1000000}\right) - g_3\left(0.9 + \frac{1+1}{1000000}\right) = 0.210401920798\ldots > 0,$$

$$\vdots$$

$$f_3\left(0.9 + \frac{99998}{1000000}\right) - g_3\left(0.9 + \frac{99998+1}{1000000}\right) = 0.001997169404\ldots > 0.$$

$$f_3\left(0.9 + \frac{99999}{1000000}\right) - g_3\left(0.9 + \frac{99999+1}{1000000}\right) = 0.000001989345\ldots > 0.$$

We then obtain

$$f_3\left(0.9 + \frac{k}{1000000}\right) - g_3\left(0.9 + \frac{k+1}{1000000}\right) > 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, 99999.$$

Hence,

$$xG(x) > 0 \text{ for } x \in \left[0.9 + \frac{k}{1000000}, 0.9 + \frac{k+1}{1000000}\right] \text{ and } k = 0, 1, 2, \ldots, 99999.$$

This implies that $G(x)$ is positive for $0.9 \leqslant x < 1$. Hence, $G(x) > 0$ holds for $0 < x < 1$. The proof of Theorem 2.3 is complete. □

For $0 \leqslant x \leqslant 1$, let

$$H_2(x) = \frac{3 - 2\left(\frac{x}{\operatorname{arcsl} x}\right) - \frac{x}{\operatorname{arctl} x}}{x^3 \operatorname{arctl} x}.$$

By using the computer program MAPLE 11, we find that the function $H_2(x)$ is strictly increasing on $[0,1]$, and

$$H_2(0) = \lim_{x\to 0} H_2(x) = \frac{1}{20} \quad \text{and} \quad H_2(1) = 0.399613\ldots.$$

This fact motivated us to establish Theorem 2.4.

THEOREM 2.4. *For* $0 < x < 1$, *we have*

$$3 - \frac{2}{5}x^3 \operatorname{arctl} x < 2\left(\frac{x}{\operatorname{arcsl} x}\right) + \frac{x}{\operatorname{arctl} x} < 3 - \frac{1}{20}x^3 \operatorname{arctl} x. \tag{2.18}$$

Proof. The left-hand side of (2.18) is obtained by considering the function $J(x)$ defined, for $0 < x < 1$, by

$$J(x) = 2\left(\frac{x}{\operatorname{arcsl} x}\right) + \frac{x}{\operatorname{arctl} x} - 3 + \frac{2}{5}x^3 \operatorname{arctl} x. \tag{2.19}$$

We consider two cases to prove $J(x) > 0$ for $0 < x < 1$.

Case 1. $0 < x < 0.9$.

By the right-hand side of (2.9) and (2.13), we have

$$\begin{aligned} J(x) &> 2(1-x^4)^{1/10} - 2 + \frac{2}{5}x^3\left(x - \frac{3}{20}x^5\right) \\ &= 2\left\{(1-x^4)^{1/10} - \left(1 - \frac{1}{5}x^4 + \frac{3}{100}x^8\right)\right\}. \end{aligned}$$

Elementary calculations show that, for $0 < x < 0.9$,

$$\begin{aligned} &(1-x^4) - \left(1 - \frac{1}{5}x^4 + \frac{3}{100}x^8\right)^{10} \\ &\quad = x^4\left\{1 - \frac{21}{10}x^4 + \frac{3}{2}x^8 - \frac{1617}{2000}x^{12}\right\} + x^{20}\left\{\frac{2169}{6250} - \frac{3063}{25000}x^4\right\} \\ &\qquad + x^{28}\left\{\frac{4539}{125000} - \frac{91557}{10000000}x^4\right\} + x^{36}\left\{\frac{49511}{25000000} - \frac{184607}{500000000}x^4\right\} \\ &\qquad + \cdots + x^{76}\left\{\frac{19683}{5000000000000000000} - \frac{59049}{100000000000000000000}x^4\right\} > 0, \end{aligned}$$

since each of the terms in the braces is positive for $0 < x < 0.9$. We then obtain $J(x) > 0$ for $0 < x < 0.9$.

Case 2. $0.9 \leqslant x < 1$.

Write (2.19) as

$$J(x) = f_4(x) - g_4(x),$$

where

$$f_4(x) = \frac{x}{\operatorname{arctl} x} + \frac{2}{5}x^3 \operatorname{arctl} x \quad \text{and} \quad g_4(x) = 3 - \frac{2x}{\operatorname{arcsl} x}$$

are both strictly increasing. We divide the interval $[0.9, 1]$ into 1000 subintervals:

$$[0.9, 1] = \bigcup_{k=0}^{999} \left[0.9 + \frac{k}{10000}, 0.9 + \frac{k+1}{10000}\right] \quad \text{for} \quad k = 0, 1, 2, \ldots, 999.$$

By direct computation we get

$$f_4\left(0.9 + \frac{0}{10000}\right) - g_4\left(0.9 + \frac{0+1}{10000}\right) = 0.148928720936\ldots > 0,$$

$$f_4\left(0.9 + \frac{1}{10000}\right) - g_4\left(0.9 + \frac{1+1}{10000}\right) = 0.148947921862\ldots > 0,$$

$$\vdots$$

$$f_4\left(0.9 + \frac{998}{10000}\right) - g_4\left(0.9 + \frac{998+1}{10000}\right) = 0.011580368463\ldots > 0,$$

$$f_4\left(0.9 + \frac{999}{10000}\right) - g_4\left(0.9 + \frac{999+1}{10000}\right) = 0.000177032765\ldots > 0.$$

Hence,

$$J(x) > 0 \quad \text{for} \quad x \in \left[0.9 + \frac{k}{10000}, 0.9 + \frac{k+1}{10000}\right] \quad \text{and} \quad k = 0, 1, 2, \ldots, 999.$$

This implies that $J(x)$ is positive for $0.9 \leqslant x < 1$. Hence, $J(x) > 0$ holds for $0 < x < 1$.

By the left-hand side of (2.9) and (2.13), we have, for $0 < x < 1$,

$$\begin{aligned}
&3 - \frac{1}{20}x^3 \operatorname{arctl} x - 2\left(\frac{x}{\operatorname{arcsl} x}\right) - \frac{x}{\operatorname{arctl} x} \\
&\quad > 3 - \frac{1}{20}x^4 - 2\left(\frac{3 + 2\sqrt{1 - x^4}}{5}\right)^{1/2} - \frac{x}{x - \frac{3}{20}x^5} \\
&\quad = \frac{800 - 200x^4 + 3x^8}{20(20 - 3x^4)} - 2\left(\frac{3 + 2\sqrt{1 - x^4}}{5}\right)^{1/2} > 0.
\end{aligned}$$

The proof of the last inequality is easy, we here omit the proof. Hence, the right-hand side of (2.18) is valid for $0 < x < 1$. The proof of Theorem 2.4 is complete. $\square$

3. Inequalities between the inverse hyperbolic tangent and inverse sine functions

It is known in the literature that

$$(\arctan x)^2 \leqslant \frac{x\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \tag{3.1}$$

for $x \in \mathbb{R}$. Masjed-Jamei [25] first established the inequality (3.1) for $|x| < 1$, and then Zhu and Malešević [65, Theorem 1.1] proved the inequality (3.1) for $x \in \mathbb{R}$. Inequality (3.1) gives the upper bound for the square of the inverse tangent function $\arctan x$ by the inverse hyperbolic sine function $\operatorname{arcsinh} x = \ln(x+\sqrt{1+x^2})$.

Zhu and Malešević [66] obtained a general result on the natural approximation of the function $(\arctan x)^2 - (x \operatorname{arcsinh} x)/\sqrt{1+x^2}$, and proved a conjecture raised by Zhu and Malešević [65]. Recently, Chen and Malešević [13] developed (3.1) to produce a double inequality and proved that for $x > 0$,

$$\frac{x \operatorname{arcsinh} x}{\sqrt{1+x^2+\alpha x^4}} < (\arctan x)^2 < \frac{x \operatorname{arcsinh} x}{\sqrt{1+x^2+\beta x^4}}, \tag{3.2}$$

with the best possible constants

$$\alpha = \frac{2}{45} \quad \text{and} \quad \beta = 0. \tag{3.3}$$

For $0 < x < 1$, the following inequality were proved in [13, 65]:

$$\frac{x \arcsin x}{1-\frac{1}{2}x^2} < (\operatorname{arctanh} x)^2 < \frac{x \arcsin x}{\sqrt{1-x^2}}, \tag{3.4}$$

which is an analogue of (3.1). Inequality (3.4) gives the upper and lower bounds for the square of inverse hyperbolic tangent function $\operatorname{arctanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}$ by inverse sine function $\arcsin x$.

By using the computer program MAPLE 11, Chen and Malešević [13] conjectured that for $0 < x < 1$,

$$\frac{x \arcsin x}{\left(1-\frac{41}{45}x^2\right)^{\frac{45}{82}}} < (\operatorname{arctanh} x)^2, \tag{3.5}$$

and the authors pointed out that the lower bound in (3.5) is better than the lower bound in (3.4).

In this section, we prove the conjecture (3.5).

THEOREM 3.1. *For $0 < x < 1$, the inequality (3.5) holds, and the power number $\frac{45}{82}$ in (3.5) is the best.*

Proof. We consider two cases to prove the inequality (3.5).

Case 1. $0<x<0.9$.
Let

$$F(x)=\left(1-\frac{41}{45}x^2\right)^{\frac{45}{82}}\frac{(\operatorname{arctanh}x)^2}{x}-\arcsin x.$$

Differentiation yields

$$F'(x)=\left(1-\frac{41}{45}x^2\right)^{\frac{45}{82}}G(x),$$

where

$$G(x)=\frac{2\operatorname{arctanh}x}{x(1-x^2)}-\frac{(4x^2+45)(\operatorname{arctanh}x)^2}{x^2(45-41x^2)}-\frac{1}{\left(1-\frac{41}{45}x^2\right)^{\frac{45}{82}}\sqrt{1-x^2}}.$$

From the continued fraction [17, p. 216, Eq. (11.6.8)]

$$\operatorname{arctanh}x=\cfrac{\frac{x}{1-x^2}}{1+\cfrac{\frac{\frac{1\cdot 2}{1\cdot 3}x^2}{1-x^2}}{1+\cfrac{\frac{\frac{1\cdot 2}{3\cdot 5}x^2}{1-x^2}}{1+\cfrac{\frac{\frac{3\cdot 4}{5\cdot 7}x^2}{1-x^2}}{1+\cfrac{\frac{\frac{3\cdot 4}{7\cdot 9}x^2}{1-x^2}}{1+\ddots}}}}},$$

we find, for $0<x<1$,

$$\cfrac{\frac{x}{1-x^2}}{1+\cfrac{\frac{\frac{1\cdot 2}{1\cdot 3}x^2}{1-x^2}}{1+\cfrac{\frac{\frac{1\cdot 2}{3\cdot 5}x^2}{1-x^2}}{1+\cfrac{\frac{\frac{3\cdot 4}{5\cdot 7}x^2}{1-x^2}}{1}}}}<\operatorname{arctanh}x<\cfrac{\frac{x}{1-x^2}}{1+\cfrac{\frac{\frac{1\cdot 2}{1\cdot 3}x^2}{1-x^2}}{1+\cfrac{\frac{\frac{1\cdot 2}{3\cdot 5}x^2}{1-x^2}}{1+\cfrac{\frac{\frac{3\cdot 4}{5\cdot 7}x^2}{1-x^2}}{1+\cfrac{\frac{\frac{3\cdot 4}{7\cdot 9}x^2}{1-x^2}}{1}}}}},$$

which can be written for $0<x<1$ as

$$\frac{5x(21-11x^2)}{3(35-30x^2+3x^4)}<\operatorname{arctanh}x<\frac{x(315-420x^2+113x^4)}{15(1-x^2)(21-14x^2+x^4)}. \tag{3.6}$$

Using (3.6), we obtain

$$G(x) > \frac{2}{x(1-x^2)}\left(\frac{5x(21-11x^2)}{3(35-30x^2+3x^4)}\right) - \frac{4x^2+45}{x^2(45-41x^2)}\left(\frac{x(315-420x^2+113x^4)}{15(1-x^2)(21-14x^2+x^4)}\right)^2 - \frac{1}{\left(1-\frac{41}{45}x^2\right)^{\frac{45}{82}}\sqrt{1-x^2}} = I(x)-J(x),$$

where

$$I(x) = \frac{I_1(x)}{225(35-30x^2+3x^4)(1-x^2)^2(45-41x^2)(21-14x^2+x^4)^2} \tag{3.7}$$

with

$$I_1(x) = 156279375 - 640993500x^2 + 1069480125x^4 - 924767550x^6 + 434967225x^8 - 106273040x^{10} + 11773755x^{12} - 491478x^{14},$$

and

$$J(x) = \frac{1}{\left(1-\frac{41}{45}x^2\right)^{\frac{45}{82}}\sqrt{1-x^2}}. \tag{3.8}$$

We now prove that

$$I(x) > J(x) \quad \text{for} \quad 0 < x < 0.9. \tag{3.9}$$

For $0 < x < 0.9$, let

$$P(x) = \ln I(x) - \ln J(x).$$

Differentiation yields

$$P'(x) = \frac{4x^5 P_{20}(x)}{(45-41x^2)(21-14x^2+x^4)(1-x^2)(35-30x^2+3x^4)P_{14}(x)},$$

where

$$P_{20}(x) = 273124254375 - 1473529286250x^2 + 3396006880875x^4 - 4369916716200x^6 + 3448278347150x^8 - 1729511498540x^{10} + 554979220590x^{12} - 112293885192x^{14} + 13783416459x^{16} - 954301626x^{18} + 28751463x^{20}$$

and

$$P_{14}(x) = 156279375 - 640993500x^2 + 1069480125x^4 - 924767550x^6 + 434967225x^8 - 106273040x^{10} + 11773755x^{12} - 491478x^{14}.$$

Noting that $P_{20}(x) > 0$ and $P_{14}(x) > 0$ hold for $0 < x < 0.9$, we obtain $P'(x) > 0$ for $0 < x < 0.9$. Hence, $P(x)$ is strictly increasing on $(0.0.9)$, and we have

$$P(x) > \lim_{u \to 0} P(u) = 0 \quad \text{for} \quad 0 < x < 0.9.$$

This means that the inequality (3.9) is valid. We then obtain $G(x) > 0$ and $F'(x) > 0$ for $0 < x < 0.9$. Hence, $F(x)$ is strictly increasing on $(0.0.9)$, and we have

$$F(x) > \lim_{u \to 0} F(u) = 0 \quad \text{for} \quad 0 < x < 0.9.$$

This means that the inequality (3.5) is valid for $0 < x < 0.9$.

Case 2. $0.9 \leqslant x < 1$.

Let

$$g(x) = g_1(x) + g_2(x),$$

where

$$g_1(x) = -\frac{x \arcsin x}{\left(1 - \frac{41}{45}x^2\right)^{\frac{45}{82}}} \quad \text{and} \quad g_2(x) = (\operatorname{arctanh} x)^2.$$

Let $0.9 \leqslant r < x < s \leqslant 1$. Since $g_1(x)$ is strictly decreasing and $g_2(x)$ is strictly increasing, we obtain

$$g(x) > g_1(s) + g_2(r) =: \sigma(r, s).$$

We divide the interval $[0.9, 1]$ into 100 subintervals:

$$[0.9, 1] = \bigcup_{k=0}^{99} \left[0.9 + \frac{k}{1000}, 0.9 + \frac{k+1}{1000}\right].$$

By direct computation we get

$$g_2\left(0.9 + \frac{k}{1000}\right) > -g_1\left(0.9 + \frac{k+1}{1000}\right) \quad \text{for} \quad k = 0, 1, 2, \ldots, 99;$$

ie.

$$\sigma\left(0.9 + \frac{k}{1000}, 0.9 + \frac{k+1}{1000}\right) > 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, 99.$$

Hence,

$$g(x) > 0 \quad \text{for} \quad x \in \left[0.9 + \frac{k}{1000}, 0.9 + \frac{k+1}{1000}\right] \quad \text{and} \quad k = 0, 1, 2, \ldots, 99.$$

This implies that $g(x)$ is positive for $0.9 \leqslant x < 1$. Hence, the inequality (3.5) holds for $0.9 \leqslant x < 1$.

If we write (3.5) as

$$\frac{\ln\left(\frac{x\arcsin x}{(\operatorname{arctanh} x)^2}\right)}{\ln(1-\frac{41}{45}x^2)} > \frac{45}{82},$$

we find

$$\lim_{x\to 0^+} \frac{\ln\left(\frac{x\arcsin x}{(\operatorname{arctanh} x)^2}\right)}{\ln(1-\frac{41}{45}x^2)} = \frac{45}{82}.$$

Hence, the inequality (3.5) holds for $0<x<1$, and the power number $\frac{45}{82}$ in (3.5) is the best. The proof of Theorem 3.1 is complete. □

4. Inequalities for trigonometric functions

It is known in the literature that

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2+\cos x}{3} \tag{4.1}$$

for $0<|x|<\pi/2$. The left-hand side inequality was obtained by Adamović and Mitrinović (see [26, p. 238]), while the right-hand side inequality was first mentioned by the German philosopher and theologian Nicolaus de Cusa (1401-1464), by a geometrical method. Huygens [20] gave a rigorous proof of the right-hand side inequality, and then used it to estimate the number π. The right-hand side inequality is now known as Cusa's inequality (see [28, 42, 62]).

The inequalities (4.1) have attracted much interest of many mathematicians and have motivated a large number of research papers; see, for example, [3, 4, 12, 16, 21, 28, 30, 31, 42, 43, 50, 54, 55, 56, 57, 62] and the references cited therein.

By using inequalities involving Schwab-Borchardt mean, Neuman [31] presented the following inequality chain:

$$\begin{aligned}(\cos x)^{1/3} &< \left(\cos x\frac{\sin x}{x}\right)^{1/4} < \left(\frac{\sin x}{\operatorname{arctanh}(\sin x)}\right)^{1/2} < \left(\frac{\cos x+(\sin x)/x}{2}\right)^{1/2}\\ &< \left(\frac{1+2\cos x}{3}\right)^{1/2} < \left(\frac{1+\cos x}{2}\right)^{2/3} < \frac{\sin x}{x}, \qquad 0<x<\frac{\pi}{2},\end{aligned} \tag{4.2}$$

which improves the first inequality in (4.1). Yang [55] proved that for $0<x<\pi/2$,

$$\frac{\sin x}{x} < \left(\frac{2}{3}\cos\frac{x}{2}+\frac{1}{3}\right)^2 < \cos^3\frac{x}{3} < \frac{2+\cos x}{3}, \tag{4.3}$$

which improves the second inequality in (4.1).

Motivated by (4.1), in this section we establish sharp inequalities for trigonometric functions. By using the obtained results, we present inequality chain and improve the double inequality (4.1).

The following elementary power series expansions are useful in our investigation.

$$\cot x=\frac{1}{x}-\sum_{n=1}^{\infty}\frac{2^{2n}|B_{2n}|}{(2n)!}x^{2n-1},\quad 0<|x|<\pi, \tag{4.4}$$

where B_n $(n=0,1,2,\ldots)$ are the Bernoulli numbers defined by

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!},\qquad |z|<2\pi.$$

The first inequality in (4.1) is equivalent to

$$\frac{x}{\tan x}<\left(\frac{\sin x}{x}\right)^2,\qquad 0<x<\frac{\pi}{2}. \tag{4.5}$$

By using power series expansions for $\cos x$ and $\cot x$, we have

$$\begin{aligned}\left(\frac{\sin x}{x}\right)^2-\frac{x}{\tan x}&=\frac{1-\cos(2x)}{2x^2}-x\cot x\\&=\sum_{n=2}^{\infty}\left(\frac{(-1)^n2^{2n+1}}{(2n+2)!}+\frac{2^{2n}|B_{2n}|}{(2n)!}\right)x^{2n}\\&=\frac{1}{15}x^4-\frac{1}{945}x^6+\frac{1}{2835}x^8+\frac{8}{467775}x^{10}+\cdots.\end{aligned} \tag{4.6}$$

This fact motivated us to establish Theorem 4.1.

THEOREM 4.1. (i) *For* $0<x<\pi/2$,

$$\frac{1}{15}x^3\sin x<\left(\frac{\sin x}{x}\right)^2-\frac{x}{\tan x}<\left(\frac{2}{\pi}\right)^5x^3\sin x, \tag{4.7}$$

where the constants $1/15$ *and* $(2/\pi)^5$ *are the best possible.*

(ii) *For* $0<x<\pi/2$,

$$\frac{1}{15}x^4-\frac{1}{945}x^5\sin x<\left(\frac{\sin x}{x}\right)^2-\frac{x}{\tan x}<\frac{1}{15}x^4-\frac{2\pi^6-1920}{15\pi^7}x^5\sin x, \tag{4.8}$$

where the $1/945$ *and* $(2\pi^6-1920)/(15\pi^7)$ *are the best possible.*

Proof. We only prove the inequality (4.7). The proof of (4.8) is analogous. The inequality (4.7) is obtained by considering the function $f(x)$ defined by

$$f(x)=\frac{\left(\frac{\sin x}{x}\right)^2-\frac{x}{\tan x}}{x^3\sin x},\qquad 0<x<\frac{\pi}{2}.$$

Differentiating $f(x)$ and applying power series expansions for $\sin x$ and $\cos x$, we find

$$\begin{aligned}
x^6\sin^3 x f'(x) &= (2x^3+x)\sin x\cos x - x\sin x\cos^3 x + (x^4+10)\cos^2 x \\
&\quad - 5\cos^4 x + x^4 - 5 \\
&= \left(x^3+\frac{1}{2}x\right)\sin(2x) - x\left(\frac{\sin(4x)+2\sin(2x)}{8}\right) + (x^4+10)\left(\frac{1+\cos(2x)}{2}\right) \\
&\quad - 5\left(\frac{\cos(4x)+4\cos(2x)+3}{8}\right) + x^4 - 5 \\
&= \left(x^3+\frac{1}{4}x\right)\sin(2x) - \frac{1}{8}x\sin(4x) - \frac{5}{8}\cos(4x) + \frac{x^4+5}{2}\cos(2x) + \frac{3}{2}x^4 - \frac{15}{8} \\
&= \frac{19}{945}x^{10} - \frac{59}{14175}x^{12} + \frac{4}{10395}x^{14} - \frac{4271}{212837625}x^{16} + \frac{151}{273648375}x^{18} \\
&\quad + \frac{1}{516891375}x^{20} - \frac{1448}{1443677610375}x^{22} + \sum_{n=12}^{\infty}(-1)^n v_n(x), \qquad (4.9)
\end{aligned}$$

where

$$v_n(x) = \frac{(40-2n-2n^2-8n^3+8n^4)4^n + (n-10)16^n}{16\cdot(2n)!}x^{2n}.$$

Elementary calculations show that for $0 < x < \pi/2$ and $n \geqslant 12$,

$$\begin{aligned}
\frac{v_{n+1}(x)}{v_n(x)} &= \frac{x^2\Big((144+8n+88n^2+96n^3+32n^4)4^n + (16n-144)16^n\Big)}{2(2n+1)(n+1)\Big((40-2n-2n^2-8n^3+8n^4)4^n+(n-10)16^n\Big)} \\
&< \left(\frac{(\pi/2)^2}{n+1}\right)\frac{(144+8n+88n^2+96n^3+32n^4)4^n + (16n-144)16^n}{2(2n+1)\Big((40-2n-2n^2-8n^3+8n^4)4^n+(n-10)16^n\Big)} \\
&< \frac{(144+8n+88n^2+96n^3+32n^4)4^n + (16n-144)16^n}{2(2n+1)\Big((40-2n-2n^2-8n^3+8n^4)4^n+(n-10)16^n\Big)}
\end{aligned}$$

and

$$\begin{aligned}
&2(2n+1)\Big((40-2n-2n^2-8n^3+8n^4)4^n+(n-10)16^n\Big) \\
&\qquad - \Big((144+8n+88n^2+96n^3+32n^4)4^n + (16n-144)16^n\Big) \\
&= (32n^5-48n^4-120n^3-100n^2+148n-64)4^n + (4n^2-54n+124)16^n > 0.
\end{aligned}$$

We then obtain that for $0 < x < \pi/2$ and $n \geqslant 12$,

$$\frac{v_{n+1}(x)}{v_n(x)} < 1.$$

Hence, for every $x\in(0,\pi/2)$, the sequence $n\longmapsto v_n(x)$ is strictly decreasing for $n\geqslant 12$. Therefore, we obtain from (4.9) that for $0<x<\pi/2$,

$$x^6\sin^3 x f'(x)>x^{10}\left(\frac{19}{945}-\frac{59}{14175}x^2\right)+x^{14}\left(\frac{4}{10395}-\frac{4271}{212837625}x^2\right)+x^{18}\left(\frac{151}{273648375}+\frac{1}{516891375}x^2-\frac{1448}{1443677610375}x^4\right)>0,$$

which implies $f'(x)>0$ for $0<x<\pi/2$. So, the function $f(x)$ is strictly increasing for $0<x<\pi/2$, and we have

$$\frac{1}{15}=\lim_{t\to 0^+}f(t)<f(x)=\frac{\left(\frac{\sin x}{x}\right)^2-\frac{x}{\tan x}}{x^3\sin x}<\lim_{t\to(\pi/2)^-}f(t)=\left(\frac{2}{\pi}\right)^5$$

for all $0<x<\pi/2$, with the constants $1/15$ and $(2/\pi)^5$ being possible. The proof is complete. □

REMARK 4.1. The inequalities (4.8) are sharper than the inequalities (4.7). Noting that

$$1>\left(1-\frac{1}{15}\frac{x^5}{\sin x}\right)^{1/3}>\left(1-\frac{\pi^5}{480}\right)^{1/3}=0.71299468\ldots,\quad 0<x<\frac{\pi}{2},$$

from (4.1) and (4.7) we obtain the following inequality chain:

$$\left(1-\left(\frac{2}{\pi}\right)^5\frac{x^5}{\sin x}\right)^{1/3}\frac{\sin x}{x}<(\cos x)^{1/3}<\left(1-\frac{1}{15}\frac{x^5}{\sin x}\right)^{1/3}\frac{\sin x}{x}<\frac{\sin x}{x}<\frac{2+\cos x}{3}\tag{4.10}$$

for $0<x<\pi/2$.

Acknowledgements. The authors are grateful to anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

The first author supported by the Key Science Research Project in the Universities of the Henan Province (20B110007) and by the Fundamental Research Funds for the Universities of the Henan Province (NSFRF210446).

The second author was supported in part by the Serbian Ministry of Education, Science and Technological Development, under project 451-03-68/2022-14/200103.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Dover, New York, 1965.

[2] A. BARICZ AND J. SÁNDOR, *Extensions of generalized Wilker inequality to Bessel functions*, J. Math. Inequal. **2** (2008), 397–406.

[3] G. BERCU, *Padé approximant related to remarkable inequalities involving trigonometric functions*, J. Inequal. Appl. **2016** (2016), 99.

[4] B. A. BHAYO AND J. SÁNDOR, *On Jordan's, Redheffer's and Wilker's inequality*, Math. Inequal. Appl. **19** (2016), 823–839.

[5] J. M. BORWEIN AND P. B. BORWEIN, *Pi and the AGM: a study in the analytic number theory and computational complexity*, John Wiley and Sons, New York, 1987.

[6] B. C. CARLSON, *Algorithms involving arithmetic and geometric means*, Amer. Math. Monthly **78** (1971), 496–505.

[7] C.-P. CHEN, *Sharp Wilker- and Huygens-type inequalities for inverse trigonometric and inverse hyperbolic functions*, Integral Transforms Spec. Funct. **23** (12) (2012), 865–873.

[8] C.-P. CHEN, *Wilker and Huygens type inequalities for the Lemniscate functions*, J. Math. Inequal. **6** (4) (2012), 673–684.

[9] C.-P. CHEN, *Wilker and Huygens type inequalities for the Lemniscate functions, II*, Math. Inequal. Appl. **16** (2) (2013), 577–586.

[10] C.-P. CHEN AND W.-S. CHEUNG, *Inequalities and solution to Oppenheim's problem*, Integral Transforms Spec. Funct. **23** (5) (2012), 325–336.

[11] C.-P. CHEN AND W.-S. CHEUNG, *Sharpness of Wilker and Huygens type inequalities*, J. Inequal. Appl. **2012** (2012), 72, https://link.springer.com/article/10.1186/1029-242X-2012-72.

[12] C.-P. CHEN AND W.-S. CHEUNG, *Sharp Cusa and Becker-Stark inequalities*, J. Inequal. Appl. **2011** (2011), 136.

[13] C.-P. CHEN AND B. MALEŠEVIĆ, *Inequalities related to certain inverse trigonometric and inverse hyperbolic functions*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. **114** (2020) RACSAM (2020), no. 2, Article number: 105, https://doi.org/10.1007/s13398-020-00836-0.

[14] C.-P. CHEN AND R. B. PARIS, *Series representations of the remainders in the expansions for certain trigonometric and hyperbolic functions with applications*, **2016** (2016) http://arxiv.org/abs/1601.03180.

[15] C.-P. CHEN AND R. B. PARIS, *On the Wilker and Huygens-type inequalities*, **2016** (2016) http://arxiv.org/abs/1602.00436.

[16] C.-P. CHEN AND J. SÁNDOR, *Inequality chains for Wilker, Huygens and Lazarević type inequalities*, J. Math. Inequal. **8** (1) (2014), 55–67.

[17] A. CUYT, V. B. PETERSEN, B. VERDONK, H. WAADELAND AND W. B. JONES, *Handbook of continued fractions for special functions*, Springer, New York, 2008.

[18] J. E. DENG AND C.-P. CHEN, *Sharp Shafer-Fink type inequalities for Gauss lemniscate functions*, J. Inequal. Appl. **2014** (2014), 35, http://www.journalofinequalitiesandapplications.com/content/2014/1/35.

[19] B.-N. GUO, B.-M. QIAO, F. QI AND W. LI, *On new proofs of Wilker inequalities involving trigonometric functions*, Math. Inequal. Appl. **6** (2003), 19–22.

[20] C. HUYGENS, *Oeuvres Completes 1888–1940*, Société Hollondaise des Science, Haga.

[21] R. KLÉN, M. VISURI AND M. VUORINEN, *On Jordan type inequalities for hyperbolic functions*, J. Inequal. Appl. **2010** (2010), Article ID 362548, 14 p.

[22] D. H. LEHMER, *Interesting series involving the central binomial coefficient*, Am. Math. Mon. **92** (1985), 449–457.

[23] J. LIU AND C.-P. CHEN, *Padé approximant related to inequalities for Gauss lemniscate functions*, J. Inequal. Appl. **2016** (2016), 320, https://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/s13660-016-1262-2.

[24] B. J. MALEŠEVIĆ, B. BANJAC AND I. JOVOVIĆ, *A proof of two conjectures of Chao-Ping Chen for inverse trigonometric functions*, J. Math. Inequal. **11** (1) (2017), 151–162.

[25] M. MASJED-JAMEI, *A main inequality for several special functions*, Comput. Math. Appl. **60** (2010), 1280–1289.

[26] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[27] C. MORTICI, *A subtly analysis of Wilker inequality*, Appl. Math. Comput. **23** (2014), 516–520.

[28] C. MORTICI, *The natural approach of Wilker-Cusa-Huygens inequalities*, Math. Inequal. Appl. **14** (2011), 535–541.

[29] M. NENEZIĆ, B. MALEŠEVIĆ AND C. MORTICI, *New approximations of some expressions involving trigonometric functions*, Appl. Math. Comput. **283** (2016), 299–315.

[30] E. NEUMAN, *Inequalities for the Schwab-Borchardt mean and their applications*, J. Math. Inequal. **5** (4) (2011), 601–609.

[31] E. NEUMAN, *Refinements and generalizations of certain inequalities involving trigonometric and hyperbolic functions*, Adv. Inequal. Appl. **1** (1) (2012), 1–11.

[32] E. NEUMAN, *One- and two-sided inequalities for Jacobian elliptic functions and related results*, Integral Transforms Spec. Funct. **21** (2010), 399–407.

[33] E. NEUMAN, *On Wilker and Huygens type inequalities*, Math. Inequal. Appl. **15** (2) (2012), 271–279.

[34] E. NEUMAN, *Wilker and Huygens-type inequalities for the generalized trigonometric and for the generalized hyperbolic functions*, Appl. Math. Comput. **230** (2014), 211–217.

[35] E. NEUMAN, *Inequalities for weighted sums of powers and their applications*, Math. Inequal. Appl. **24** (4) (2012), 995–1005.

[36] E. NEUMAN, *On lemniscate functions*, Integral Transforms Spec. Funct. **24** (3) (2013), 164–171.

[37] E. NEUMAN, *Wilker and Huygens type inequalities for Jacobian elliptic and theta functions*, Integral Transforms Spec. Funct. **25** (3) (2014), 240–248.

[38] E. NEUMAN, *On Gauss lemniscate functions and lemniscatic mean*, Math. Pannon. **18** (2007), 77–94

[39] E. NEUMAN, *Two-sided inequalities for the lemniscate functions*, J. Inequal. Spec. Funct. **1** (2010), 1–7.

[40] E. NEUMAN, *On Gauss lemniscate functions and lemniscatic mean II*, Math. Pannon. **23** (2012), 65–73.

[41] E. NEUMAN, *Inequalities for Jacobian elliptic functions and Gauss lemniscate functions*, Appl. Math. Comput. **218** (2012), 7774–7782.

[42] E. NEUMAN AND J. SÁNDOR, *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker and Huygens inequalities*, Math. Inequal. Appl. **13** (4) (2010), 715–723.

[43] Y. NISHIZAWA, *Sharp exponential approximate inequalities for trigonometric functions, Results*, Math. **71** (2017), 609–621.

[44] I. PINELIS, *L'Hospital rules of monotonicity and Wilker-Anglesio inequality*, Amer. Math. Monthly, **111** (2004), 905–909.

[45] C. L. SIEGEL, *Topics in Complex Function Theory*, Vol. 1. Wiley, New York, 1969.

[46] J. S. SUMNER, A. A. JAGERS, M. VOWE AND J. ANGLESIO, *Inequalities involving trigonometric functions*, Amer. Math. Monthly **98** (1991), 264–267.

[47] J. L. SUN AND C.-P. CHEN, *Shafer-type inequalities for inverse trigonometric functions and Gauss lemniscate functions*, J. Inequal. Appl. **2016** (2016), Article number 212, http://www.journalofinequalitiesandapplications.com/content/2016/1/212.

[48] J. B. WILKER, *Problem E 3306*, Amer. Math. Monthly **96** (1989), 55.

[49] S.-H. WU, *On extension and refinement of Wilker's inequality*, Rocky Mountain J. Math. **39** (2009), 683–687.

[50] S.-H. WU AND A. BÁRICZ, *Generalizations of Mitrinović, Adamović and Lazarević inequalities and their applications*, Publ. Math. Debrecen **75** (3–4) (2009), 447–458.

[51] S.-H. WU AND L. DEBNATH, *Wilker-type inequalities for hyperbolic functions*, Appl. Math. Lett. **25** (5) (2012), 837–842.

[52] S.-H. WU AND H. M. SRIVASTAVA, *A weighted and exponential generalization of Wilker's inequality and its applications*, Integral Transforms and Spec. Funct. **18** (2007), 529–535.

[53] S.-H. WU AND H. M. SRIVASTAVA, *A further refinement of Wilker's inequality*, Integral Transforms and Spec. Funct. **19** (2008), 757–765.

[54] Z.-H. YANG, *Renements of a two-sided inequality for trigonometric functions*, J. Math. Inequal. **7** (4) (2013), 601–615.

[55] Z.-H. YANG, *Three families of two-parameter means constructed by trigonometric functions*, J. Inequal. Appl. **2013** (2013), Article 541.
[56] Z.-H. YANG, *New sharp Jordan type inequalities and their applications*, Gulf J. Math. **2** (1) (2014), 1–10.
[57] Z.-H. YANG, Y.-M. CHU AND X.-H. ZHANG, *Sharp Cusa type inequalities with two parameters and their applications*, Appl. Math. Comput. **268** (2015), 1177–1198.
[58] B. ZHANG AND C.-P. CHEN, *Sharp Wilker and Huygens type inequalities for trigonometric and inverse trigonometric functions*, J. Math. Inequal. **14** (3) (2020), 673–684.
[59] L. ZHANG AND L. ZHU, *A new elementary proof of Wilker's inequalities*, Math. Inequal. Appl. **11** (2008), 149–151.
[60] L. ZHU, *A new simple proof of Wilker's inequality*, Math. Inequal. Appl. **8** (2005), 749–750.
[61] L. ZHU, *On Wilker-type inequalities*, Math. Inequal. Appl. **10** (4) (2007), 727–731.
[62] L. ZHU, *A source of inequalities for circular functions*, Comput. Math. Appl. **58** (10) (2009), 1998–2004.
[63] L. ZHU, *Some new Wilker-type inequalities for circular and hyperbolic functions*, Abstr. Appl. Anal. **2009** (2009), Article ID 485842,
http://downloads.hindawi.com/journals/aaa/2009/485842.pdf.
[64] L. ZHU, *Inequalities for Hyperbolic functions and their Applications*, J. Inequal. Appl. **2010** (2010), Article number 130821,
https://journalofinequalitiesandapplications.springeropen.com/articles/10.1155/2010/130821.
[65] L. ZHU AND B. MALEŠEVIĆ, *Inequalities between the inverse hyperbolic tangent and the inverse sine and the analogue for corresponding functions*, J. Inequal. Appl. **2019**, Article ID 93, 10 pages,
https://link.springer.com/article/10.1186/s13660-019-2046-2.
[66] L. ZHU AND B. MALEŠEVIĆ, *Natural approximation of Masjed-Jamei's inequality*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **114** (1) (2020), Article number: 25,
https://link.springer.com/article/10.1007%2Fs13398-019-00735-z.

(Received September 20, 2020)

Chao-Ping Chen
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454000, Henan Province, China
e-mail: chenchaoping@sohu.com

Branko Malešević
School of Electrical Engineering
University of Belgrade
Bulevar Kralja Aleksandra 73, 11000, Belgrade, Serbia
e-mail: branko.malesevic@etf.bg.ac.rs