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SPECTRAL PROPERTIES AND CHARACTERIZATION OF QUASI–n–HYPONORMAL OPERATORS

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Abstract. In this paper the class of quasi-n-hyponormal operators is introduced. The representation and characterization of the operators on a Hilbert space are established. Using these results, we obtain some spectral properties of the quasi-n-hyponormal operators. Finally we show that the class of n-hyponormal operators is properly contained in the class of quasi-n-hyponormal operators.

1. Introduction

Let $B(H)$ denote the algebra of all bounded linear operators on an infinite dimensional complex separable Hilbert space H. If $T \in B(H)$, we shall write $N(T)$ and $R(T)$ for the null space and the range space of T, respectively. Normal operator plays a crucial role in the development of operator theory and has been widely studied due to its fundamental importance in the theory of automatic continuity and harmonic analysis. As natural extension of normal operators, an operator T is said to be an n-normal operator [1] if $T^nT^* = T^*T^n$, where n is a positive integer. By Fuglede-Putnam theorem, it is easy to see that T is n-normal if and only if T^n is normal (see [1]), in particular for $n = 1$, a 1-normal operator is a normal operator. In [13] an operator $T \in B(H)$ is called nth root of hyponormal (abbrev. n-hyponormal), if T^n is hyponormal ($T^*T \geqslant TT^*$) for some positive integer n. Equivalently, T is an n-hyponormal operator if and only if $T^{*n}T^n \geqslant T^nT^{*n}$ for some positive integer n. Clearly, the classes of hyponormal and 1-hyponormal operators coincide. n-normal operators and n-hyponormal operators attract much attention and they have many interesting properties (see [3, 4, 5, 8, 12, 13, 17]).

DEFINITION 1.1. For a positive integer n, an operator $T \in B(H)$ is called quasi-n-hyponormal if

$$T^*(T^{*n}T^n - T^nT^{*n})T \geqslant 0.$$

In particular for $n = 1$, this operator is called quasi-hyponormal. It is clear that the class of n-normal operators $\subseteq$ the class of n-hyponormal operators $\subseteq$ the class of quasi-n-hyponormal operators.

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2. Main results

LEMMA 2.1. *If $T \in B(H)$ does not have a dense range, then the following statements are equivalent:*
(i) *T is a quasi-n-hyponormal operator;*
(ii) $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$ *on $H = \overline{R(T)} \oplus N(T^*)$, where $T_1^{*n}T_1^n - T_1^nT_1^{*n} \geqslant T_1^{n-1}T_2T_2^*T_1^{*(n-1)}$.*
Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. (i) $\Rightarrow$ (ii) Consider the matrix representation of T with respect to the decomposition $H = \overline{R(T)} \oplus N(T^*)$:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}.$$

Let P be the projection onto $\overline{R(T)}$. Since T is a quasi-n-hyponormal operator, we have

$$P(T^{*n}T^n - T^nT^{*n})P \geqslant 0.$$

Thus

$$T_1^{*n}T_1^n - T_1^nT_1^{*n} \geqslant T_1^{n-1}T_2T_2^*T_1^{*(n-1)}.$$

Since $\sigma(T_1) \cap \{0\}$ has no interior points, [11, Corollary 7] deduces that $\sigma(T) = \sigma(T_1) \cup \{0\}$.

(ii) $\Rightarrow$ (i) Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$ on $H = \overline{R(T)} \oplus N(T^*)$, where $T_1^{*n}T_1^n - T_1^nT_1^{*n} - T_1^{n-1}T_2T_2^*T_1^{*(n-1)} \geqslant 0$. Then

$$\begin{aligned}
&T^*(T^{*n}T^n - T^nT^{*n})T \\
&= \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}^* \\
&\quad \times \left(\begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}^{*n} \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}^{n} - \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}^{n} \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}^{*n} \right) \\
&\quad \times \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} D & T_1^{*n}T_1^{n-1}T_2 \\ T_2^*T_1^{*(n-1)}T_1^n & T_2^*T_1^{*(n-1)}T_1^{n-1}T_2 \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} T_1^*DT_1 & T_1^*DT_2 \\ T_2^*DT_1 & T_2^*DT_2 \end{pmatrix} \\
&= \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \\
&= T^*(D \oplus D)T,
\end{aligned}$$

where $D = T_1^{*n}T_1^n - T_1^nT_1^{*n} - T_1^{n-1}T_2T_2^*T_1^{*(n-1)}$. It follows that $T^*(T^{*n}T^n - T^nT^{*n})T \geqslant 0$. Therefore, T is a quasi-n-hyponormal operator. $\square$

LEMMA 2.2. *Suppose that $T \in B(H)$ is a quasi-n-hyponormal operator and $R(T)$ is dense. Then T is an n-hyponormal operator.*

Proof. The conclusion is evident by Definition 1.1. □

THEOREM 2.3. *An operator $T \in B(H)$ is quasi-n-hyponormal if and only if T has the matrix representation*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*),$$

where $T_1^{*n}T_1^n - T_1^nT_1^{*n} \geqslant T_1^{n-1}T_2T_2^*T_1^{*(n-1)}$. The space $\overline{R(T)}$ or $N(T^*)$ may be absent, that is, equals to $\{0\}$.

Proof. Clearly by Lemma 2.1 and Lemma 2.2. □

COROLLARY 2.4. [10] *An operator $T \in B(H)$ is quasi-hyponormal if and only if T has the matrix representation*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \oplus N(T^*),$$

where $T_1^*T_1 - T_1T_1^* \geqslant T_2T_2^*$. The space $\overline{R(T)}$ or $N(T^*)$ may be absent, that is, equals to $\{0\}$.

Proof. Clearly by Theorem 2.3. □

DEFINITION 2.5. [14] A bounded linear operator T on H is called scalar of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital morphism of topological algebra $\Phi : C_0^m(\mathbb{C}) \to B(H)$ such that $\Phi(z) = T$, where z stands for the identity function on $\mathbb{C}$, and $C_0^m(\mathbb{C})$ stands for the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order $m, 0 \leqslant m \leqslant \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to an invariant subspace.

LEMMA 2.6. [16, Theorem 2.4] *Suppose that $T \in B(H)$ is a quasi-n-hyponormal operator. Then T is subscalar of order $2n+2$.*

COROLLARY 2.7. *Suppose that $T \in B(H)$ is a quasi-nilpotent quasi-n-hyponormal operator. Then T is nilpotent.*

Proof. Since a quasi-nilpotent subscalar operator is nilpotent, it follows by Lemma 2.6 that T is nilpotent. □

DEFINITION 2.8. An operator $T \in B(H)$ is said to belong to the class $H(p)$ if there exists a natural number $p := p(\lambda)$ such that

$$H_0(\lambda I - T) = N(\lambda I - T)^p \text{ for all } \lambda \in \mathbb{C},$$

where $H_0(\lambda I - T) := \{x \in H : \lim_{n\to\infty} ||(\lambda I - T)^n x||^{\frac{1}{n}} = 0\}$.

LEMMA 2.9. [15] *Every subscalar operator $T \in B(H)$ is $H(p)$.*

COROLLARY 2.10. *Every quasi-n-hyponormal operator is $H(p)$.*

Proof. It follows by Lemma 2.6 and Lemma 2.9. □

LEMMA 2.11. [7] *$H(p)$ operators satisfy Weyl's theorem (i.e., $\sigma(T) - \omega(T) = \pi_{00}(T)$, where $\omega(T)$ is the Weyl spectrum of T and $\pi_{00}(T)$ is the set of all isolated points which are eigenvalues of T with finite multiplicities.)*

COROLLARY 2.12. *Every quasi-n-hyponormal operator satisfies Weyl's theorem.*

Proof. It follows by Corollary 2.10 and Lemma 2.11. □

For every $T \in B(H)$, the function $\sigma : T \longmapsto \sigma(T)$ is upper semi-continuous, but fails to be continuous in general. Conway and Morrel [6] made a detailed study of spectral continuity in $B(H)$. Duggal [8] proved that the function σ is continuous on the class of n-hyponormal operators. We now study the spectral continuity of quasi-n-hyponormal operators.

LEMMA 2.13. *Suppose that T is a quasi-n-hyponormal operator, $0 \neq \lambda \in \sigma_p(T)$ and*

$$T = \begin{pmatrix} \lambda I & A \\ 0 & B \end{pmatrix} \quad on \ H = N(T - \lambda I) \oplus N(T - \lambda I)^{\perp}.$$

Then $N(B - \lambda I) = \{0\}$.

Proof. Suppose $(B - \lambda I)x = 0$, where $x \in N(T - \lambda I)^{\perp}$. If $\lambda \neq 0$, then $B^n x = \lambda^n x$. Let

$$T = \begin{pmatrix} \lambda I & A \\ 0 & B \end{pmatrix}.$$

By simple calculations, we have

$$T^n = \begin{pmatrix} \lambda^n I & \sum_{j=0}^{n-1} \lambda^j A B^{n-1-j} \\ 0 & B^n \end{pmatrix}.$$

Since T is a quasi-n-hyponormal operator, T satisfies

$$T^*(T^{*n}T^n - T^n T^{*n})T \geqslant 0.$$

Thus

$$
\begin{aligned}
& T^*(T^{*n}T^n - T^nT^{*n})T \\
=& \begin{pmatrix} \lambda I & A \\ 0 & B \end{pmatrix}^* \\
& \times \left(\begin{pmatrix} \lambda I & A \\ 0 & B \end{pmatrix}^{*n} \begin{pmatrix} \lambda I & A \\ 0 & B \end{pmatrix}^{n} - \begin{pmatrix} \lambda I & A \\ 0 & B \end{pmatrix}^{n} \begin{pmatrix} \lambda I & A \\ 0 & B \end{pmatrix}^{*n} \right) \\
& \times \begin{pmatrix} \lambda I & A \\ 0 & B \end{pmatrix} \\
=& \begin{pmatrix} \overline{\lambda} I & 0 \\ A^* & B^* \end{pmatrix} \\
& \times \begin{pmatrix} -FF^* & \overline{\lambda}^n F - FB^{*n} \\ \lambda^n F^* - B^n F^* & F^*F + B^{*n}B^n - B^nB^{*n} \end{pmatrix} \\
& \times \begin{pmatrix} \lambda I & A \\ 0 & B \end{pmatrix} \\
=& \begin{pmatrix} -\overline{\lambda}\lambda FF^* & G \\ G^* & M \end{pmatrix} \\
\geqslant & 0,
\end{aligned}
$$

where

$$
\begin{aligned}
F &= \sum_{j=0}^{n-1} \lambda^j AB^{n-1-j}, \\
G &= -\overline{\lambda} FF^*A + \overline{\lambda}^{n+1} FB - \overline{\lambda} FB^{*n}B, \\
M &= -A^*FF^*A + B^*\lambda^n F^*A - B^*B^nF^*A + A^*\overline{\lambda}^n FB - A^*FB^{*n}B + B^*F^*FB \\
&\quad + B^*B^{*n}B^nB - B^*B^nB^{*n}B.
\end{aligned}
$$

Recall that $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geqslant 0$ if and only if $X \geqslant 0, Z \geqslant 0$ and $Y = X^{\frac{1}{2}} W Z^{\frac{1}{2}}$ for some contraction W. Since $-FF^* = -\sum_{j=0}^{n-1} \lambda^j AB^{n-1-j} (\sum_{j=0}^{n-1} \lambda^j AB^{n-1-j})^* \geqslant 0$, we have

$$
\sum_{j=0}^{n-1} \lambda^j AB^{n-1-j} = 0,
$$

hence $\sum_{j=0}^{n-1} \lambda^j AB^{n-1-j} x = 0$, which implies $Ax = 0$. Hence $(T - \lambda I)x = 0$, therefore, $x \in N(T - \lambda I)$ and $x = 0$. $\square$

LEMMA 2.14. [2] *Let H be a complex Hilbert space. Then there exists a Hilbert space K such that $H \subset K$ and a map $\varphi : B(H) \to B(K)$ such that*

(i) φ *is a faithful* $*$*-representation of the algebra* $B(H)$ *on* K*, i.e.,* $\varphi(T+S)=\varphi(T)+\varphi(S)$, $\varphi(\lambda T)=\lambda\varphi(T)$, $\varphi(TS)=\varphi(T)\varphi(S)$, $\varphi(T^*)=(\varphi(T))^*$, $\varphi(I)=I$ *and* $||\varphi(T)|| = ||T||$ *for any* $T,S\in B(H)$*;*
(ii) $\varphi(A)\geqslant 0$ *for any* $A\geqslant 0$ *in* $B(H)$*;*
(iii) $\sigma_a(T)=\sigma_a(\varphi(T))=\sigma_p(\varphi(T))$ *for any* $T\in B(H)$*.*

DEFINITION 2.15. [9] The set $C(i)$ consists of (all) the operators $T\in B(H)$ for which $\sigma(T)=\{0\}$ implies T is nilpotent (possibly, the 0 operator) and T° (the Berberian extension of T) satisfies the property:

$$T^\circ=\begin{pmatrix}\lambda I & A\\ 0 & B\end{pmatrix}\quad on\ \ H=N(T^\circ-\lambda I)\oplus N(T^\circ-\lambda I)^\perp$$

at every nonzero $\lambda\in\sigma_p(T^\circ)$ for some operators A and B such that $\lambda\notin\sigma_p(B)$ and $\sigma(T^\circ)=\sigma(B)\cup\{\lambda\}$.

THEOREM 2.16. *The function* σ *is continuous on the set of quasi-*n*-hyponormal operators.*

Proof. Suppose T is a quasi-n-hyponormal operator. Let $\varphi\colon\ B(H)\to B(K)$ be Berberian's faithful $*$-representation of Lemma 2.14. In the following, we show that $\varphi(T)$ is also a quasi-n-hyponormal operator. In fact, T is a quasi-n-hyponormal operator, we have $T^*(T^{*n}T^n-T^nT^{*n})T\geqslant 0$. Hence

$$\varphi(T)^*(\varphi(T)^{*n}\varphi(T)^n-\varphi(T)^n\varphi(T)^{*n})\varphi(T)=\varphi(T^*(T^{*n}T^n-T^nT^{*n})T)\geqslant 0,$$

so that $\varphi(T)$ is also a quasi-n-hyponormal operator. By Corollary 2.7 and Lemma 2.13, T belongs to the set $C(i)$. Therefore, the function σ is continuous on the set of quasi-n-hyponormal operators by [9, Theorem 1.1]. □

Finally we give an example to show that the class of n-hyponormal operators is properly contained in the class of quasi-n-hyponormal operators. The following lemma is needed.

LEMMA 2.17. *Let* $K=\bigoplus_{m=1}^{+\infty}H_m$*, and* $H_m\cong H$*. For given positive operators* A *and* B *on* H*, and any fixed positive integer* n*, define the operator* $T=T_{A,B,n}$ *on* K *as*

$$T(x_1,x_2,x_3,\ldots)=(0,Ax_1,Ax_2,\ldots,Ax_n,Bx_{n+1},Bx_{n+2},\ldots).$$

Then the following assertions hold
(i) T *belongs to* n*-hyponormal if and only if*

$$\begin{cases}B^{2n}-A^{2n}\geqslant 0,\\ B^{2n}-BA^{2n-2}B\geqslant 0,\\ B^{2n}-B^2A^{2n-4}B^2\geqslant 0,\\ \cdots\cdots\\ B^{2n}-B^{n-1}A^2B^{n-1}\geqslant 0.\end{cases}\tag{2.1}$$

(ii) *T belongs to quasi-n-hyponormal if and only if*

$$\begin{cases} A(B^{2n}-A^{2n})A \geqslant 0, \\ B(B^{2n}-BA^{2n-2}B)B \geqslant 0, \\ B(B^{2n}-B^2A^{2n-4}B^2)B \geqslant 0, \\ \cdots\cdots \\ B(B^{2n}-B^{n-1}A^2B^{n-1})B \geqslant 0. \end{cases} \tag{2.2}$$

Proof. Since

$$T(x_1,x_2,x_3,\ldots)=(0,Ax_1,Ax_2,\ldots,Ax_n,Bx_{n+1},Bx_{n+2},\ldots),$$

we obtain

$$T^*(x_1,x_2,x_3,\ldots)=(Ax_2,Ax_3,\ldots,Ax_{n+1},Bx_{n+2},Bx_{n+3},\ldots).$$

By simple calculations, the following equalities hold.

$$\begin{aligned} T^n(x_1,x_2,x_3,\ldots) = (&\overbrace{0,\ldots,0}^{n\ items},A^nx_1,BA^{n-1}x_2,B^2A^{n-2}x_3,\\ &\ldots,B^{n-1}Ax_n,B^nx_{n+1},B^nx_{n+2},\ldots); \end{aligned}$$

$$\begin{aligned} T^{*n}(x_1,x_2,x_3,\ldots) = (&A^nx_{n+1},A^{n-1}Bx_{n+2},A^{n-2}B^2x_{n+3},\\ &\ldots,AB^{n-1}x_{2n},B^nx_{2n+1},B^nx_{2n+2},\ldots). \end{aligned}$$

Hence

$$\begin{aligned} T^{*n}T^n(x_1,x_2,x_3,\ldots) = (&A^{2n}x_1,A^{n-1}B^2A^{n-1}x_2,A^{n-2}B^4A^{n-2}x_3,\\ &\ldots,AB^{2n-2}Ax_n,B^{2n}x_{n+1},B^{2n}x_{n+2},\ldots); \end{aligned}$$

$$\begin{aligned} T^nT^{*n}(x_1,x_2,x_3,\ldots) = (&\overbrace{0,\ldots,0}^{n\ items},A^{2n}x_{n+1},BA^{2n-2}Bx_{n+2},B^2A^{2n-4}B^2x_{n+3},\\ &\ldots,B^{n-1}A^2B^{n-1}x_{2n},B^{2n}x_{2n+1},B^{2n}x_{2n+2},\ldots); \end{aligned}$$

$$\begin{aligned} T^*T^{*n}T^nT(x_1,x_2,x_3,\ldots) = (&A^{2n+2}x_1,A^nB^2A^nx_2,A^{n-1}B^4A^{n-1}x_3,\\ &\ldots,AB^{2n}Ax_n,B^{2n+2}x_{n+1},B^{2n+2}x_{n+2},\ldots); \end{aligned}$$

$$\begin{aligned} T^*T^nT^{*n}T(x_1,x_2,x_3,\ldots) = (&\overbrace{0,\ldots,0}^{n-1\ items},A^{2n+2}x_n,BBA^{2n-2}BBx_{n+1},\\ &\ldots,BB^{n-1}A^2B^{n-1}Bx_{2n-1},B^{2n+2}x_{2n},B^{2n+2}x_{2n+1},\ldots). \end{aligned}$$

Therefore, T is n-hyponormal ($T^{*n}T^n \geqslant T^nT^{*n}$) if and only if

$$\begin{cases} B^{2n} - A^{2n} \geqslant 0, \\ B^{2n} - BA^{2n-2}B \geqslant 0, \\ B^{2n} - B^2A^{2n-4}B^2 \geqslant 0, \\ \cdots\cdots \\ B^{2n} - B^{n-1}A^2B^{n-1} \geqslant 0. \end{cases}$$

Similarly, T is quasi-n-hyponormal ($T^*(T^{*n}T^n - T^nT^{*n})T \geqslant 0$) if and only if

$$\begin{cases} A(B^{2n} - A^{2n})A \geqslant 0, \\ B(B^{2n} - BA^{2n-2}B)B \geqslant 0, \\ B(B^{2n} - B^2A^{2n-4}B^2)B \geqslant 0, \\ \cdots\cdots \\ B(B^{2n} - B^{n-1}A^2B^{n-1})B \geqslant 0. \end{cases} \qquad \square$$

EXAMPLE 2.18. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ be operators on $\mathbb{R}^2$, and let $H_m = \mathbb{R}^2$ for all positive integers m. Consider the operator T on $\oplus_{m=1}^{+\infty} H_m$ defined by

$$T(x_1, x_2, x_3, \ldots) = (0, Ax_1, Ax_2, \ldots, Ax_n, Bx_{n+1}, Bx_{n+2}, \ldots).$$

Then T is non-n-hyponormal and quasi-n-hyponormal.

Proof. Since

$$A^{2n} = A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B^{2n} = 2^{2n-1}B = \begin{pmatrix} 2^{2n-1} & 2^{2n-1} \\ 2^{2n-1} & 2^{2n-1} \end{pmatrix},$$

we have

$$B^{2n} - A^{2n} = \begin{pmatrix} 2^{2n-1} - 1 & 2^{2n-1} \\ 2^{2n-1} & 2^{2n-1} \end{pmatrix} \not\geqslant 0.$$

Hence T is non-n-hyponormal.

On the other hand,

$$\begin{aligned} A(B^{2n} - A^{2n})A &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2^{2n-1} - 1 & 2^{2n-1} \\ 2^{2n-1} & 2^{2n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2^{2n-1} - 1 & 0 \\ 0 & 0 \end{pmatrix} \geqslant 0, \end{aligned}$$

$$\begin{aligned} B(B^{2n} - BA^{2n-2}B)B &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^{2n-1} - 1 & 2^{2n-1} - 1 \\ 2^{2n-1} - 1 & 2^{2n-1} - 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{2n+1} - 4 & 2^{2n+1} - 4 \\ 2^{2n+1} - 4 & 2^{2n+1} - 4 \end{pmatrix} \geqslant 0, \end{aligned}$$

$$B(B^{2n}-B^2A^{2n-4}B^2)B = \begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}2^{2n-1}-4 & 2^{2n-1}-4\\ 2^{2n-1}-4 & 2^{2n-1}-4\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix} = \begin{pmatrix}2^{2n+1}-16 & 2^{2n+1}-16\\ 2^{2n+1}-16 & 2^{2n+1}-16\end{pmatrix} \geqslant 0,$$

$$\cdots\cdots$$

$$B(B^{2n}-B^{n-1}A^2B^{n-1})B = \begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}2^{2n-1}-2^{2n-4} & 2^{2n-1}-2^{2n-4}\\ 2^{2n-1}-2^{2n-4} & 2^{2n-1}-2^{2n-4}\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix} = \begin{pmatrix}2^{2n+1}-2^{2n-2} & 2^{2n+1}-2^{2n-2}\\ 2^{2n+1}-2^{2n-2} & 2^{2n+1}-2^{2n-2}\end{pmatrix} \geqslant 0.$$

Thus T is quasi-n-hyponormal. □

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