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REVERSE FORM OF THE MINKOWSKI INEQUALITIES WITH APPLICATIONS

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Abstract. In this paper, we obtain the reverse of Minkowski inequality for matrices, and from this we give reverse forms of the Oppenheim inequality for n-simplices and the Neuberg-Pedoe inequality for triangles.

1. Introduction

Let A and B be positive definite matrices of order n. Then we have

$$|A+B|^{\frac{1}{n}} \geqslant |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}, \tag{1.1}$$

where $|A|$ denotes the determinant of the matrix A. Moreover, equality holds if $A=\lambda B$ (for some $\lambda > 0$). Inequality (1.1) is called the Minkowski inequality for matrices; see [1, 2, 3]. Bergstrom [4] established an important inequality which is analogous to (1.1), as follows.

Let $A_{(j)}, B_{(j)}$ denote the sub-matrices of A and B obtained by deleting the j-th row and column, then

$$\frac{|A+B|}{|A_{(j)}+B_{(j)}|} \geqslant \frac{|A|}{|A_{(j)}|} + \frac{|B|}{|B_{(j)}|}.$$

Fan [5] gave a generalization of (1.1) and established an inequality for matrices. Yuan and Leng [6] gave a generalization of the inequality (1.1), as follows.

Let A_i denote the principal sub-matrix of A formed by taking the first i rows and columns of A, and I_{n-i} denote the unit matrix of order $n-i$, $(0 \leqslant i < n)$. If α and β are two nonnegative real numbers such that $A > aI_n$ and $B > bI_n$, and $D = A+B$, then

$$\left(\frac{|D|}{|D_i|} - |(\alpha+\beta)I_{n-i}|\right)^{1/(n-i)} \geqslant \left(\frac{|A|}{|A_i|} - |\alpha I_{n-i}|\right)^{1/(n-i)} + \left(\frac{|B|}{|B_i|} - |\beta I_{n-i}|\right)^{1/(n-i)},$$

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with equality if and only if $\alpha^{-1}A=\beta^{-1}B$.

There is a vast amount of work on the generalization of inequality (1.1); see [7–13]. In this paper, we study the problem on reverse form of inequality (1.1), and establish a reverse form of the Minkowski inequality. Moreover, we use this inequality and establish reverse forms of the Oppenheim inequality for n-simplices and the Neuberg-Pedoe inequality for triangles.

2. Main results

Let H_n be the set of all $n\times n$ real positive definite matrices and I_n be the $n\times n$ unit matrix. We use the notation A^T to denote the transpose of matrix A. We give a reverse of the Minkowski inequality (1.1) as follows.

THEOREM 2.1. *Let $A,B\in H_n$. If $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n>0$ are the eigenvalues of the matrix $B^{-1}A$, then*

$$|A+B|^{\frac{1}{n}}\leqslant\frac{1+\lambda_1}{1+\lambda_n}\left(|A|^{\frac{1}{n}}+|B|^{\frac{1}{n}}\right). \tag{2.1}$$

Moreover, equality holds if $A=\mu B$ for some $\mu>0$.

From Theorem 2.1 we state two corollaries as follows.

COROLLARY 2.1. *Under the conditions of the Theorem 2.1, if $x,y>0$ are two real numbers, then*

$$|xA+yB|^{\frac{1}{n}}\leqslant\frac{y+\lambda_1 x}{y+\lambda_n x}\left(x|A|^{\frac{1}{n}}+y|B|^{\frac{1}{n}}\right). \tag{2.2}$$

Moreover, if $A=\mu B$ for some $\mu>0$, then the above inequality turns into an equality.

COROLLARY 2.2. *Under the conditions of the Theorem 2.1, we have*

$$|A+B|^{\frac{1}{n}}\leqslant\frac{\lambda_1}{\lambda_n}\left(|A|^{\frac{1}{n}}+|B|^{\frac{1}{n}}\right). \tag{2.3}$$

Moreover, equality holds if $A=\mu B$ for some $\mu>0$.

We give another reverse for inequality (1.1) as follows.

THEOREM 2.2. *Let $A,B\in H_n$. If $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n>0$ are the eigenvalues of matrix $B^{-1}A$, then*

$$|A+B|^{\frac{1}{n}}\leqslant\sqrt{2}\left(\lambda_1+\lambda_n^{-1}\right)^{\frac{1}{2}}\left(|A|\cdot|B|\right)^{\frac{1}{2n}}\leqslant\frac{1}{\sqrt{2}}\left(\lambda_1+\lambda_n^{-1}\right)^{\frac{1}{2}}\left(|A|^{\frac{1}{n}}+|B|^{\frac{1}{n}}\right). \tag{2.4}$$

Moreover, equality holds if $A=\mu B$ for some $\mu>0$.

The other aim of this paper is to study the inverse of the Oppenheim inequality for n-simplices and to investigate of the Neuberg-Pedoe inequality for triangles. Suppose that ABC is a triangle with area S and sides a,b,c, and $A'B'C'$ is another triangle with area $S^{'}$ and sides $a^{'}, b^{'}, c^{'}$. Defined numbers $a^{''}, b^{''}, c^{''}$ by $a^{''} = (a^2 + a^{'2})^{\frac{1}{2}}$, etc., then $a^{''}, b^{''}, c^{''}$ are the sides of a triangle $A^{''}B^{''}C^{''}$ with area $S^{''}$. Then the following inequality is valid (see [14, 15]).

$$S^{''} \geqslant S + S^{'}, \tag{2.5}$$

with equality if the triangles ABC and $A^{'}B^{'}C^{'}$ are similar.

Inequality (2.5) is called the Oppenheim inequality. The concept of metric addition began with Oppenheim [14], and was first explicitly defined by Alexander in [16]. Let $\Omega_n = P_0P_1 \cdots P_n$ and $\Omega_n^{'} = P_0^{'}P_1^{'} \cdots P_n^{'}$ be two n-simplices in the n-dimensional Euclidean space E^n with vertices $P_0, P_1, \cdots, P_n$ and $P_0^{'}, P_1^{'}, \cdots, P_n^{'}$, respectively. Then there is an n-simplex $\Omega_n^{''} = P_0^{''}P_1^{''} \cdots P_n^{''}$ with vertices $P_i^{''}$ $(i = 0, 1, \cdots, n)$, such that $|P_i^{''}P_j^{''}|^2 = |P_iP_j|^2 + |P_i^{'}P_j^{'}|^2$ for $i, j = 0, 1, \cdots, n$. The simplex $\Omega_n^{''}$ is called matrix addition of the simplices Ω_n and $\Omega_n^{'}$ (see [16, 17]), and denoted by $\Omega_n^{''} = \Omega_n + \Omega_n^{'}$. Alexander conjectured the inequality

$$V^{''} \geqslant V^2 + V^{'2}, \tag{2.6}$$

holds, whence V, $V^{'}$ and $V^{''}$ are the volumes of Ω_n, $\Omega_n^{'}$ and $\Omega_n^{''}$, respectively.

However, Yang and Zhang [17] proved that (2.6) is not true, and proved the inequality

$$V^{''\frac{2}{n}} \geqslant V^{\frac{2}{n}} + V^{'\frac{2}{n}}, \tag{2.7}$$

where equality holds if the simplices Ω_n and $\Omega_n^{'}$ are similar.

Inequality (2.7) is called the n-dimensional Oppenheim inequality for simplices. Let $a_{ij} = |P_iP_j|$, $a_{ij}^{'} = |P_i^{'}P_j^{'}|$ $(i, j = 0, 1, \cdots, n)$ denote the edge-lengths of the simplices Ω_n and $\Omega_n^{'}$, respectively. And $a_{ij}^{''} = |P_i^{''}P_j^{''}| = (a_{ij}^2 + a_{ij}^{'2})^{\frac{1}{2}}$ $(i, j = 0, 1, \cdots, n)$ denote the edge-lengths of the simplex $\Omega_n^{''}$. We put $g_{ij} = a_{i0}^2 + a_{0j}^2 - a_{ij}^2$, $g_{ij}^{'} = a_{i0}^{'2} + a_{0j}^{'2} - a_{ij}^{'2}$ $(i, j = 1, 2, \cdots, n)$, $G = (g_{ij})$ and $G^{'} = (g_{ij}^{'})$ are $n \times n$ symmetric matrices. Let $g_{ij}^{''} = a_{i0}^{''2} + a_{0j}^{''2} - a_{ij}^{''2}$ $(1 \leqslant i, j \leqslant n)$ and $G^{''} = (g_{ij}^{''})$ be $n \times n$ symmetric matrix. It is easy to see that

$$G^{''} = G + G^{'}. \tag{2.8}$$

According to [17, 6], it is known that the matrices G and $G^{'}$ are both positive definite, and

$$|G| = 2^n(n!)^2V^2, \qquad |G^{'}| = 2^n(n!)^2V^{'2}. \tag{2.9}$$

Using Theorem 2.1, inequality (2.7) and inequality (2.8), we get the reverse of the Oppenheim inequality (2.6) as follows.

THEOREM 2.3. *Let Ω_n and $\Omega_n^{'}$ be n-dimensional simplices in E^n, and $\Omega_n^{''} = \Omega_n + \Omega_n^{'}$, $\mu_1 \geqslant \mu_2 \geqslant \cdots > \mu_n > 0$ be the eigenvalues of the matrix $G^{'-1}G$. Then*

$$V^{''\frac{2}{n}} \leqslant \frac{1 + \mu_1}{1 + \mu_n}\left(V^{\frac{2}{n}} + V^{'\frac{2}{n}}\right), \tag{2.10}$$

with equality if the simplices Ω_n and $\Omega_n^{'}$ are similar.

From Theorem 2.2 we get the following corollary.

COROLLARY 2.3. *Under the conditions of Theorem 2.2, we have*

$$V''^{\frac{2}{n}} \leqslant \frac{\mu_1}{\mu_n}\left(V^{\frac{2}{n}} + V'^{\frac{2}{n}}\right). \tag{2.11}$$

Moreover, equality holds if Ω_n and Ω_n' are similar.

By taking $n=2$ in Theorem 2.2 we get a reverse form of the Oppenheim inequality (2.4) as follows.

COROLLARY 2.4. *Let ABC and $A'B'C'$ be two triangles, and $A''B''C'' = ABC + A'B'C'$. Then*

$$S'' \leqslant \frac{1+\mu_1}{1+\mu_2}\left(S+S'\right). \tag{2.12}$$

The equalities hold if the triangles ABC and $A'B'C'$ are similar.

Using inequalities (2.4) and (2.9) we get another reverse form of inequality (2.7) as follows.

THEOREM 2.4. *Under the conditions of Theorem 2.3, the following inequalities hold:*

$$V''^{\frac{2}{n}} \leqslant \sqrt{2}(\mu_1+\mu_n^{-1})^{\frac{1}{2}}(VV')^{\frac{1}{n}} \leqslant \frac{1}{\sqrt{2}}(\mu_1+\mu_n^{-1})^{\frac{1}{2}}\left(V^{\frac{2}{n}} + V'^{\frac{2}{n}}\right), \tag{2.13}$$

with equality if the simplices Ω_n and Ω_n' are similar.

From Theorem 2.4 we get the following corollary.

COROLLARY 2.5. *Let the conditions of Corollary 2.4 be satisfied. Then*

$$S'' \leqslant \sqrt{2}\left(\mu_1+\mu_2^{-1}\right)^{\frac{1}{2}}(SS') \leqslant \frac{1}{\sqrt{2}}\left(\mu_1+\mu_n^{-1}\right)^{\frac{1}{2}}\left(S+S'\right). \tag{2.14}$$

The equalities hold if the triangles ABC and $A'B'C'$ are similar.

For two triangles ABC and $A'B'C'$, we have the well-known Neuberg-Pedoe inequality as (see [18,15])

$$H \equiv a'^2(b^2+c^2-a^2) + b'^2(c^2+a^2-b^2) + c'^2(a^2+b^2-c^2) \geqslant 16SS'. \tag{2.15}$$

Equality holds if the triangles ABC and $A'B'C'$ are similar.

For two triangles ABC and $A'B'C'$, we put

$$\begin{aligned} D &= \begin{pmatrix} b^2 & \frac{1}{2}(b^2+c^2-a^2) \\ \frac{1}{2}(b^2+c^2-a^2) & c^2 \end{pmatrix}, \\ D' &= \begin{pmatrix} b'^2 & \frac{1}{2}(b'^2+c'^2-a'^2) \\ \frac{1}{2}(b'^2+c'^2-a'^2) & c'^2 \end{pmatrix}. \end{aligned} \tag{2.16}$$

Now we prove that D and $D^{'}$ are both positive definite matrices. Using the formula for area of a triangle, we have

$$|D| = \frac{1}{4}(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4) = 4S^2 > 0. \tag{2.17}$$

From the above fact and since $b^2 > 0$, $c^2 > 0$, we reach that D is a symmetric positive definite matrix. Similarly, $D^{'}$ is also symmetric positive definite matrix and

$$|D^{'}| = 4S^{'2}. \tag{2.18}$$

Let $\alpha_1 \geqslant \alpha_2 > 0$ be the eigenvalues of the matrix $D^{'-1}D$. Using Theorem 2.1 we get

$$|D+D^{'}| \leqslant \left(\frac{1+\alpha_1}{1+\alpha_2}\right)^2 \left(|D|^{\frac{1}{2}} + |D^{'}|^{\frac{1}{2}}\right)^2. \tag{2.19}$$

An easy calculation shows that

$$\begin{aligned} |D+D^{'}| &= \frac{1}{4}\left(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\right) \\ &\quad + \frac{1}{4}\left(2a^{'2}b^{'2} + 2b^{'2}c^{'2} + 2c^{'2}a^{'2} - a^{'4} - b^{'4} - c^{'4}\right) + \frac{1}{2}H \\ &= \frac{1}{2}H + 4\left(S^2 + S^{'2}\right). \end{aligned} \tag{2.20}$$

Using (2.17), (2.18), (2.19) and (2.20), we get a reverse of the Neuberg-Pedoe inequality (2.12) as follows.

THEOREM 2.5. *Let ABC and $A^{'}B^{'}C^{'}$ be two triangles and $\alpha_1 \geqslant \alpha_2 > 0$ be the eigenvalues of the matrix $D^{'-1}D$. Then*

$$H \leqslant 8\left(\frac{1+\alpha_1}{1+\alpha_2}\right)^2 \left(S + S^{'2}\right) - 8\left(S^2 + S^{'2}\right). \tag{2.21}$$

Equality holds if the triangles ABC and $A^{'}B^{'}C^{'}$ are similar.

By Theorem 2.3 we get a corollary as follows.

COROLLARY 2.6. *Let the conditions in Theorem 2.3 be satisfied. Then*

$$H \leqslant 8\left[\left(\frac{\alpha_1}{\alpha_2}\right)^2 \left(S + S^{'}\right)^2 - \left(S^2 + S^{'2}\right)\right], \tag{2.22}$$

with equality if the triangles ABC and $A^{'}B^{'}C^{'}$ are similar.

THEOREM 2.6. *Let the conditions of Theorem 2.5 be satisfied. Then*

$$H \leqslant 16\left(\alpha_1 + \alpha_2^{-1} - 1\right)SS^{'}, \tag{2.23}$$

with equality if the triangles ABC and $A^{'}B^{'}C^{'}$ are similar.

NOTE.

$$\alpha_1+\alpha_2^{-1}-1 \geqslant 2\left(\alpha_1\alpha_2^{-1}\right)^{\frac{1}{2}}-1 \geqslant 1.$$

Proof of Theorem 2.6. Form Theorem 2.2 we have

$$|D+D^{'}| \leqslant 2\left(\alpha_1+\alpha_2^{-1}\right)\left(|D|\cdot|D^{'}|\right)^{\frac{1}{2}}. \tag{2.24}$$

Substituting (2.17), (2.18) and (2.20) into (2.24) we get the inequality (2.23). It is easy to see that equality holds in (2.23) if the triangles ABC and $A^{'}B^{'}C^{'}$ are similar. □

3. Proofs of Theorems

In this section we present the proof of Theorem 2.1.

Proof of Theorem 2.1. Since A and B are real positive definite, then there exists an invertible matrix P of order n such that $P^TBP=I_n$. It is easy to know that P^TAP is also positive definite matrix. Due to [1, Theorem 7.2], there exists an $n\times n$ unitary matrix U such that

$$U^*P^TAPU=\begin{bmatrix}\gamma_1 & & & 0\\ & \gamma_2 & & \\ & & \ddots & \\ 0 & & & \gamma_n\end{bmatrix}, \tag{3.1}$$

where $\gamma_i>0$ $(i=1,2,\cdots,n)$ are the eigenvalues of the matrix P^TAP.

From $P^TBP=I_n$ we get $B=(PP^T)^{-1}$, thus $B^{-1}=PP^T$ and $P(P^TAP)P^{-1}=(PP^T)A(PP^{-1})=B^{-1}A$. From this we know that γ_i $(i=1,2,\cdots,n)$ are eigenvalues of the matrix $B^{-1}A$. So

$$\begin{aligned}|P^TP|\cdot|A+B| &= |P^T(A+B)P| = |U^*P^T(A+B)PU|\\ &= |U^*P^TAPU+U^*P^TBPU| = |diag(\gamma_1,\gamma_2,\cdots,\gamma_n)+I_n|\\ &= \prod_{i=1}^{n}(\gamma_i+1).\end{aligned} \tag{3.2}$$

Since $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n>0$ are the eigenvalues of the matrix $B^{-1}A$, thus we have

$$|P^TP|\cdot|A+B| = \prod_{i=1}^{n}(\gamma_i+1) = \prod_{i=1}^{n}(\lambda_i+1). \tag{3.3}$$

By (3.3) we get

$$\prod_{i=1}^{n}(\lambda_i+1)^{\frac{1}{n}} = |P^TP|^{\frac{1}{n}}\cdot|A+B|^{\frac{1}{n}} = |B^{-1}|^{\frac{1}{n}}\cdot|A+B|^{\frac{1}{n}},$$

i.e.

$$\prod_{i=1}^{n}(\lambda_i+1)^{\frac{1}{n}}=\frac{|A+B|^{\frac{1}{n}}}{|B|^{\frac{1}{n}}}. \tag{3.4}$$

Besides, we have

$$\left(\prod_{i=1}^{n}\lambda_i\right)^{\frac{1}{n}}=|B^{-1}A|^{\frac{1}{n}}=\frac{|A|^{\frac{1}{n}}}{|B|^{\frac{1}{n}}}. \tag{3.5}$$

Using (3.4) and (3.5) we have

$$\frac{1+\left(\prod\limits_{i=1}^{n}\lambda_i\right)^{\frac{1}{n}}}{\prod\limits_{i=1}^{n}\left(\lambda_i+1\right)^{\frac{1}{n}}}=\frac{|A|^{\frac{1}{n}}+|B|^{\frac{1}{n}}}{|A+B|^{\frac{1}{n}}},$$

$$|A+B|^{\frac{1}{n}}=\frac{\prod\limits_{i=1}^{n}\left(1+\lambda_i\right)^{\frac{1}{n}}}{1+\left(\prod\limits_{i=1}^{n}\lambda_i\right)^{\frac{1}{n}}}\left(|A|^{\frac{1}{n}}+|B|^{\frac{1}{n}}\right). \tag{3.6}$$

It is easy to know that

$$\prod_{i=1}^{n}\left(1+\lambda_i\right)^{\frac{1}{n}}\leqslant 1+\lambda_1,\quad 1+\left(\prod_{i=1}^{n}\lambda_i\right)^{\frac{1}{n}}\geqslant 1+\lambda_n. \tag{3.7}$$

Equality holds if $\lambda_1=\lambda_2=\cdots=\lambda_n$.

By (3.6) and (3.7) we get

$$|A+B|^{\frac{1}{n}}\leqslant\frac{1+\lambda_1}{1+\lambda_n}\left(|A|^{\frac{1}{n}}+|B|^{\frac{1}{n}}\right). \tag{3.8}$$

If $A=\mu B$, then $B^{-1}A=\mu I_n$ and $\lambda_1=\lambda_2=\cdots=\lambda_n$, and equality holds in (3.8). Theorem 2.1 is complete. □

Proof of Theorem 2.2. We put $Q=B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$, then Q is a positive definite matrix. (In fact, by [19], since A is a positive definite matrix, then there exist an invertible matrix C satisfies $A=CC^T$. And since B is a positive definite matrix, we know that $B^{-\frac{1}{2}}$ is also a positive definite matrix. Then $Q=B^{-\frac{1}{2}}(CC^T)B^{-\frac{1}{2}}=(B^{-\frac{1}{2}}C)(B^{-\frac{1}{2}}C)^T$. So Q is a positive definite matrix.) It is known that matrix Q and $B^{-1}A$ are similar. Thus $\lambda_i(i=1,2,\cdots,n)$ are the eigenvalues of the matrix Q, and from this we get

$$|I_n+Q|=|Q|^{\frac{1}{2}}\cdot|Q^{-\frac{1}{2}}+Q^{\frac{1}{2}}|=|Q|^{\frac{1}{2}}\prod_{i=1}^{n}\left(\frac{1}{\sqrt{\lambda_i}}+\sqrt{\lambda_i}\right). \tag{3.9}$$

By the power mean inequality we have

$$\frac{1}{\sqrt{\lambda_i}}+\sqrt{\lambda_i}\leqslant\sqrt{2}\left(\frac{1}{\lambda_i}+\lambda_i\right)^{\frac{1}{2}},\qquad (i=1,2,\cdots,n). \tag{3.10}$$

Using (3.9) and (3.10) we get

$$|I_n+Q|\leqslant|Q|^{\frac{1}{2}}\prod_{i=1}^{n}\sqrt{2}\left(\lambda_i+\lambda_i^{-1}\right)^{\frac{1}{2}}\leqslant 2^{\frac{n}{2}}\left(\lambda_1+\lambda_n^{-1}\right)^{\frac{n}{2}}\cdot|Q|^{\frac{1}{2}}. \tag{3.11}$$

Besides, we have

$$|A+B|=|B|\cdot|I_n+B^{-\frac{1}{2}}AB^{-\frac{1}{2}}|=|B|\cdot|I_n+Q|. \tag{3.12}$$

Substituting (3.11) into (3.12) we get

$$|A+B|\leqslant 2^{\frac{n}{2}}\left(\lambda_1+\lambda_n^{-1}\right)^{\frac{n}{2}}\cdot|B|\cdot|Q|^{\frac{1}{2}}=2^{\frac{n}{2}}\left(\lambda_1+\lambda_n^{-1}\right)^{\frac{n}{2}}|A|^{\frac{1}{2}}\cdot|B|^{\frac{1}{2}},$$

i.e.

$$|A+B|^{\frac{1}{n}}\leqslant\sqrt{2}\left(\lambda_1+\lambda_n^{-1}\right)^{\frac{1}{2}}\left(|A|\cdot|B|\right)^{\frac{1}{2n}}. \tag{2.13}$$

Using the arithmetic-geometric mean inequality and (2.13) we get

$$|A+B|^{\frac{1}{2}}\leqslant\sqrt{2}\left(\lambda_1+\lambda_n^{-1}\right)^{\frac{1}{2}}\left(|A|\cdot|B|\right)^{\frac{1}{2n}}\leqslant\frac{1}{\sqrt{2}}\left(\lambda_1+\lambda_n^{-1}\right)^{\frac{1}{2}}\left(|A|^{\frac{1}{n}}+|B|^{\frac{1}{n}}\right). \tag{2.14}$$

If $A=\mu B$, then $B^{-1}A=\mu I_n$ and $\mu_1=\mu_2=\cdots=\mu_n$, and equality holds in (2.4). Theorem 2.2 is proved. □

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REFERENCES

[1] E. V. HAYNESWORTH, *Bounds for determinants with positive diagonals*, Trans. Amer. Math. Soc., 96 (1960), 395–413.
[2] E. V. HAYNESWORTH, *Note on bounds for certain determinats*, Duke. Math. J., 24 (1957), 313–320.
[3] Y. D. ZHANG, *Theory of Linear Algebra*, Shanghai: Shanghai Educational Press, 1980 (in Chinese).
[4] H. BERGSTROM, *A triangle inequality for matrices*, Den Elfte Skandiaviski Matematiker-kongress, Trondheim, 1949.
[5] K. FAN, *Some inequalities concerning positive-definite Hermitian matrices*, Proc. Cambrige Phil. Soc., 51 (1958), 414–421.
[6] J. YUAN, G. S. LENG, *A generalization of the matrix form of the Brunn-Minkowski inequality*, J. Aust. Math., 83 (2007), 125–134.
[7] C. BORELL, *Capacitary inequality of the Brunn-Minkowski inequality type*, Math. Ann., 263 (1993), 179–184.
[8] R. OSSERMAN, *The Brunn-Minkowski inequality for multiplictities*, Invent. Math., 125 (1996), 405–411.

[9] R. J. GARDNER, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc., 39 (2002), 355–405.

[10] M. FIEDLER, T. MARKHAM, *Some results on the Bergstrom and Minkowski inequality*, Linear Algebra Appl., 232 (1996), 199–211.

[11] R. J. GARDNER, P. GRONCHI, *A Brunn-Minkowski inequality for the integer lattice*, Trans. Amer. Math. Soc., 353 (2001), 3995–4024.

[12] J. C. BOURIN, M. UCHIYAMA, *Amatrix subadditivity inequality for* $f(A+B)$ *and* $f(A)+f(B)$, Linear Algebra Appl., 423 (2007), 512–518.

[13] R. SCHNEIDER, *Convex Bodies: The Brunn-Minkowski inequality theory*, Cambridge: Cambridge University Press, 1993.

[14] A. OPPEMHEIM, *Advanced problem 5092*, Amer. Math. Monthly, 70 (1963), 444.

[15] D. S. MITRINOVIC, J. E. PECARIC, *About the Neuberg-Pedoe and the Oppenheim inequalities*, J. Math. Anal. Appl., 129 (1988), 196–210.

[16] R. ALEXANDER, *The geometry of metric and linear space*, Berlin: Springer-Verlay, 1975.

[17] L. YANG, J. Z. ZHANG, *On Alexander's conjecture*, Chinese Sci. Bull., 27 (1982), 1–3.

[18] D. PEDOE, *An inequality for two triangles*, Proc. Cambridge philos, Soc., 38 (1943), 397–398.

[19] F. ZHANG, *Matrix theory: basic results and techniques*, Springer Science Business Media, 2011.

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