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ITERATIVE ALGORITHMS FOR COMMON SOLUTIONS OF SPLIT MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF λ–HYBRID MULTIVALUED MAPPINGS

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Abstract. In this paper, we present an iterative algorithm for solving split mixed equilibrium problems, fixed point problems of an infinite family of nonexpansive mappings and fixed point problems of λ-hybrid multivalued mappings in real Hilbert spaces. We prove that the proposed iterative algorithm converges weakly to a common solution of the considered problems under some mild assumptions.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and $\Theta : C\times C \to \mathbb{R}$ be a nonlinear bifunction. Let $\psi : H \to \mathbb{R}$ be a function and $F : C \to H$ be a nonlinear mapping. Let $T : C \to C$ be an operator and $\mathrm{Fix}(T) = \{u \in C | u = Tu\}$.

Recall that an equilibrium problem is to find an element $x^\dagger \in C$ such that

$$\Theta(x^\dagger, x) \geqslant 0, \quad \forall x \in C, \tag{1.1}$$

which was initially introduced by Blum and Oettli [3]. Equilibrium problems are interesting and useful, since they provide a novel and unified method to deal with various problems arising in pure and applied sciences such as image reconstruction, network, economics, finance, ecology, optimization, elasticity and transportation. A large number of important problems can be regarded as special cases of equilibrium problems, for instance, fixed point problem, variational inequalities problem, game theory and Nash equilibrium problem. The iterative methods has been studied for the equilibrium problem (1.1) by many authors (see [9], [11], [19], [22], [23]).

More generally, we consider the following mixed equilibrium problem: find $z^\sharp \in C$ such that

$$\Theta(z^\sharp, y) + \psi(y) - \psi(z^\sharp) + \langle Fz^\sharp, y - z^\sharp\rangle \geqslant 0, \quad \forall y \in C. \tag{1.2}$$

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The set of solutions of the mixed equilibrium problem (1.2) is denoted by $\mathrm{MEP}(\Theta, F, \psi)$. If $F \equiv 0$, then the mixed equilibrium problem (1.2) becomes the following mixed equilibrium problem: find $z^\sharp \in C$ such that

$$\Theta(z^\sharp, y) + \psi(y) - \psi(z^\sharp) \geqslant 0, \ \ \forall y \in C. \tag{1.3}$$

The set of solutions of the mixed equilibrium problem (1.3) is denoted by $\mathrm{MEP}(\Theta, \psi)$. The mixed equilibrium problem was studied by Ceng *et al.* [6].

The another motivation of this article is to be the split common fixed point problem which aims to find a point u such that

$$u \in \mathrm{Fix}(T) \quad \text{and} \quad Au \in \mathrm{Fix}(S). \tag{1.4}$$

The split common fixed point problem can be regarded as a generalization of the split feasibility problem. Recall that the split feasibility problem is to find a point satisfying

$$u \in C \quad \text{and} \quad Au \in Q,$$

where C and Q are two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively and $A : H_1 \to H_2$ is a bounded linear operator. Problem (1.4) was firstly introduced by Censor and Segal [7]. Note that solving (1.4) can be translated to solve the fixed point equation

$$u = S(u - \tau A^*(I - T)Au), \ \tau > 0.$$

Whereafter, Censor and Segal [7] proposed an algorithm for directed operators. Since then, there has been growing interest in the split common fixed point problem [1, 5].

In 2013, Kazmi and Rizvi [10] introduced and studied a split equilibrium problem. In 2014, Bnouhachem [4] suggested an iterative scheme for finding the approximate element of the common set of solutions of a split equilibrium problem and a hierarchical fixed point problem in a real Hilbert space. In 2016, Suantai *et al.* [16] introduced and studied iterative schemes for solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert spaces and proved that the modified Mann iteration converges weakly to a common solution of the considered problems.

In this article, we study the following split equilibrium problem and fixed point problem to find an element $z^\sharp$ such that

$$z^\natural \in \mathrm{Fix}(S) \bigcap \mathrm{MEP}(\Theta_1, F_1, \psi_1) \ \text{ and } \ Az^\natural \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \bigcap \mathrm{MEP}(\Theta_2, F_2, \psi_2). \tag{1.5}$$

Subsequently, we construct a new algorithm for solving the split common fixed point problem (1.5). Weak convergence theorems are established under some mild assumptions.

2. Preliminaries

In this section, we collect some tools including some definitions, useful inequalities and lemmas which will be used to derive our main results in the next section.

Now we give some definitions related to the involved operators.

DEFINITION 2.1. [2] An operator $T : C \longrightarrow C$ is called *nonexpansive* if $\|Tu - Tv\| \leqslant \|u - v\|$ for all $u, v \in C$.

DEFINITION 2.2. [2] An operator $T : C \longrightarrow C$ is called *firmly nonexpansive* if $\|Tu - Tv\|^2 \leqslant \|u - v\|^2 - \|(I - T)u - (I - T)v\|^2$ for all $u, v \in C$, or equivalently,

$$\langle Tu - Tv, u - v\rangle \geqslant \|Tu - Tv\|^2$$

for all $u, v \in C$.

DEFINITION 2.3. [2] An operator $T : C \longrightarrow C$ is called α-averaged if there exists a nonexpansive operator U such that $T = (1 - \alpha)I + \alpha U$, where I is an identity mapping.

DEFINITION 2.4. [2] An operator $T : C \longrightarrow C$ is said to be quasinonexpansive if $\|Tx - x^\dagger\| \leqslant \|x - x^\dagger\|$ for all $x \in C$ and $x^\dagger \in \text{Fix}\,(T)$, or equivalently,

$$\langle x - Tx, x - x^\dagger\rangle \geqslant \frac{1}{2}\|x - Tx\|^2$$

for all $x \in C$ and $x^\dagger \in \text{Fix}\,(T)$.

REMARK 2.5. Obviously, if $\text{Fix}(T) \neq \emptyset$, then the nonexpansive operator T is quasinonexpansive.

DEFINITION 2.6. [2] An operator $T : C \longrightarrow C$ is said to be strictly quasinonexpansive if $\|Tx - x^\dagger\| < \|x - x^\dagger\|$ for all $x \in C$ and $x^\dagger \in \text{Fix}\,(T)$.

REMARK 2.7. It is well known that an averaged operator T with $\text{Fix}(T) \neq \emptyset$ is strictly quasinonexpansive. For more details, see [2].

DEFINITION 2.8. [2] An operator $F : C \to H$ is said to be α-inverse strongly monotone if $\langle Fx - Fx^\dagger, x - x^\dagger\rangle \geqslant \alpha\|Fx - Fx^\dagger\|^2$ for some constant $\alpha > 0$ and all $x, x^\dagger \in C$.

Usually, some additional smoothness properties of mappings are required in the study of fixed point algorithms such as demiclosedness.

DEFINITION 2.9. [2] An operator T is said to be demiclosed at w if, for any sequence $\{u_n\}$ which converges weakly to u^* and $Tu_n \to w$, $Tu^* = w$.

Recall that the projection from H onto C, denoted by P_C, assigns to each $u \in H$, the unique point $P_C u \in C$ satisfying

$$\|u - P_C u\| = \inf\{\|u - v\| : v \in C\}.$$

Then P_C can be characterized by

$$\langle u - P_C u, v - P_C u\rangle \leqslant 0$$

for all $u \in H, v \in C$, and $P_C : H \to C$ is firmly nonexpansive, that is,

$$\begin{aligned}&\langle u - v, P_C u - P_C v\rangle \geqslant \|P_C u - P_C v\|^2\\ &\quad \Longleftrightarrow \|P_C u - P_C v\|^2 \leqslant \|u - v\|^2 - \|(I - P_C)u - (I - P_C)v\|^2\end{aligned}$$

for all $u, v \in H$ (see [15]).

For all $u, v \in H$, the following conclusions hold:

$$\|tu + (1-t)v\|^2 = t\|u\|^2 + (1-t)\|v\|^2 - t(1-t)\|u - v\|^2, \ \ t \in [0,1],$$

$$\|u + v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2$$

and

$$\|u + v\|^2 \leqslant \|u\|^2 + 2\langle v, u + v\rangle.$$

In the following text, we employ the following notations:

- $u_n \rightharpoonup u$ stands for that $\{u_n\}$ converges weakly to u;
- $u_n \to u$ stands for that $\{u_n\}$ converges strongly to u;
- $\mathrm{Fix}(T)$ means the set of fixed points of T;
- $\omega_w(u_n)$ means the set of cluster points in the weak topology, that is,

$$\omega_w(u_n) = \{u : \exists u_{n_j} \rightharpoonup u\}.$$

DEFINITION 2.10. [2] A sequence $\{x_n\}$ is called Fejér-monotone with respect to a given nonempty set Ω if for every $x^\dagger \in \Omega$,

$$\|x_{n+1} - x^\dagger\| \leqslant \|x_n - x^\dagger\|$$

for all $n \geqslant 0$.

LEMMA 2.11. [20] *Let H be a Hilbert space and $C(\neq \emptyset) \subset H$ be a closed convex set. If $F : C \to H$ is an α-inverse strongly monotone operator, then*

$$\|x - \gamma Fx - (y - \gamma Fy)\|^2 \leqslant \|x - y\|^2 + \gamma(\gamma - 2\alpha)\|Fx - Fy\|^2, \ \ \forall x, y \in C.$$

Especially, $I - \gamma F$ is nonexpansive provided $0 \leqslant \gamma \leqslant 2\alpha$.

LEMMA 2.12. [8] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demi-closed at zero. That is, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings and $\lambda_1, \lambda_2, \cdots$ be real numbers such that $0 \leqslant \lambda_i \leqslant 1$ for all $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, define a mapping W_n of C into C as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
&U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
&\vdots\\
&U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
&U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
&U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
&W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned}$$

Such a mapping W_n is called the W-mapping generated by $T_1, T_2, \cdots, T_n$ and $\lambda_1, \lambda_2, \cdots, \lambda_n$. We have the following crucial lemma concerning W_n.

LEMMA 2.13. [14] *Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings such that $\bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n) \neq \emptyset$. Let $\lambda_1, \lambda_2, \cdots$ be real numbers such that $0 \leqslant \lambda_i \leqslant b < 1$ for all $i \geqslant 1$. Then we have the following:*

(1) For any $x \in C$ and $k \geqslant 1$, $\lim_{n\to\infty} U_{n,k}x$ exists;

(2) $\operatorname{Fix}(W) = \bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n)$, where $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, $\forall x \in C$;

(3) For any bounded sequence $\{x_n\} \subset C$, $\lim_{n\to\infty} Wx_n = \lim_{n\to\infty} W_n x_n$.

LEMMA 2.14. [21] *If the sequence $\{x_n\}$ is Fejér monotone with respect to Ω, then we have the following conclusions:*

(i) $x_n \rightharpoonup x^\dagger \in \Omega$ if and only if $\omega_w(x_n) \subset \Omega$;

(ii) the sequence $\{P_\Omega x_n\}$ converges strongly;

(iii) if $x_n \rightharpoonup x^\dagger \in \Omega$, then $x^\dagger = \lim_{n\to\infty} P_\Omega x_n$.

We denote by $CB(C)$ and $K(C)$ the collection of all nonempty closed bounded subsets and nonempty compact subsets of C, respectively. The Hausdorff metric $\mathscr{H}$ on $CB(C)$ is defined by

$$\mathscr{H}(A,B) := \max\{\sup_{x\in A} \operatorname{dist}(x,B), \sup_{y\in B} \operatorname{dist}(y,A)\}, \ \forall A,B \in CB(C),$$

where $\operatorname{dist}(x,B) = \inf\{d(x,y) : y \in B\}$. Let $S : C \to CB(C)$ be a multivalued mapping. An element $x \in C$ is said to be a fixed point of S if $x \in Sx$. The set of all fixed points of S is denoted by $\mathrm{Fix}(S)$, that is, $\mathrm{Fix}(S)=\{x\in C: x\in Sx\}$. Recall that a multivalued mapping $S: C\to CB(C)$ is said to be

(i) nonexpansive if

$$\mathscr{H}(Sx,Sy)\leqslant \|x-y\|,\ \ x,y\in C;$$

(ii) quasinonexpansive if $\mathrm{Fix}(S)\neq\emptyset$ and

$$\mathscr{H}(Sx,Sx^{\natural})\leqslant \|x-x^{\natural}\|,\ \ x\in C, x^{\natural}\in \mathrm{Fix}(S);$$

(iii) nonspreading if

$$2\mathscr{H}(Sx,Sy)\leqslant \mathrm{dist}(y,Sx)^2+\mathrm{dist}(x,Sy)^2,\ \ \forall x,y\in C;$$

(iv) λ-hybrid if there exists $\lambda\in R$ such that

$$(1+\lambda)\mathscr{H}(Sx,Sy)^2\leqslant (1-\lambda)\|x-y\|^2+\lambda\,\mathrm{dist}(y,Sx)^2+\lambda\,\mathrm{dist}(x,Sy)^2,\ \forall x,y\in C.$$

REMARK 2.15. It can be readily seen that 0-hybrid is nonexpansive, 1-hybrid is nonspreading, and if S is λ-hybrid with $\mathrm{Fix}(S)\neq\emptyset$, then S is quasinonexpansive. It is well known that if S is λ-hybrid, then $\mathrm{Fix}(S)$ is closed. In addition, if S satisfies the condition: $Sx^{\natural}=x^{\natural}$ for all $x^{\natural}\in\mathrm{Fix}(S)$, then $\mathrm{Fix}(S)$ is also convex. For more details, please see [17].

The following result is a demiclosedness principle for λ-hybrid multivalued mapping in a real Hilbert space.

LEMMA 2.16. [17] *Let C be a nonempty closed convex subset of a real Hilbert space H and $S: C\to K(C)$ be a λ-hybrid multivalued mapping. If $\{x_n\}$ is a sequence in C such that $x_n\rightharpoonup x$ and $y_n\in Sx_n$ with $\|x_n-y_n\|\to 0$, then $x\in Sx$.*

From now on, we assume that $\Theta: C\times C\to\mathbb{R}$ satisfies the following conditions:
(H1) $\Theta(x,x)\geqslant 0$ for all $x\in C$;
(H2) Θ is monotone, that is, $\Theta(x,y)+\Theta(y,x)\leqslant 0$ for all $x,y\in C$;
(H3) For each $y\in C$, the function $x\to\Theta(x,y)$ is upper-hemicontinuous, that is,

$$\limsup_{t\to 0}\Theta(tz+(1-t)x,y)\leqslant\Theta(x,y),\ \forall x,y,z\in C,\ t\in[0,1];$$

(H4) For each $x\in C$, the function $y\to\Theta(x,y)$ is convex and lower semi-continuous.

LEMMA 2.17. [13] *Let $\Theta: C\times C\to\mathbb{R}$ be a nonlinear bifunction. Let $\psi: C\to\mathbb{R}$ be a convex lower semi-continuous function. For any $r>0$ and $x\in H$, define a mapping $T_r: H\to C$ as follows:*

$$T_r^{\Theta}(x)=\Big\{z\in C:\Theta(z,y)+\psi(y)-\psi(z)+\frac{1}{r}\langle y-z,z-x\rangle\geqslant 0,\ \ \forall y\in C\Big\}.$$

Suppose that the following conditions are satisfied:

(i) Θ *satisfies the condition* $(H1)$–$(H4)$*;*

(ii) For each $x \in H$*, there exists a compact subset* $D_x \subset H$ *and* $y_x \in C \bigcap D_x$ *such that, for all* $z \in C \backslash D_x$*,*

$$\Theta(z, y_x) + \psi(y_x) - \psi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0.$$

Then we have the following results:

(1) For each $x \in H$*,* $T_r^{\Theta}(x) \neq \emptyset$ *and* $T_r^{\Theta}(x)$ *is single-valued;*

(2) $T_r^{\Theta} : H \to C$ *is firmly nonexpansive, that is, for all* $x, y \in H$*,*

$$\|T_r x - T_r y\|^2 \leqslant \langle T_r x - T_r y, x - y\rangle;$$

(3) $\mathrm{Fix}(\mathrm{T_r^{\Theta}}) = \mathrm{MEP}(\Theta, \psi)$.

REMARK 2.18. It is easy to see that $\mathrm{Fix}(T_r^{\Theta}(I - rF)) = \mathrm{MEP}(\Theta, F, \psi)$.

LEMMA 2.19. [12, 18] *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leqslant (1 - \gamma_n)\alpha_n + \delta_n, \ \ n \in N,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$*;*

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Main results

In this section, assume that H_1 and H_2 are two real Hilbert spaces. Let $C(\neq \emptyset) \subset H_1$ and $Q(\neq \emptyset) \subset H_2$ be closed convex sets. We use $\langle \cdot, \cdot \rangle$ to denote the inner product and $\|\cdot\|$ stands for the corresponding norm.

Next, we show a lemma which will be very useful for our main theorem.

LEMMA 3.1. *Let* $T_1 : C \to C$ *be a strictly quasinonexpansive operator and* $T_2 : C \to C$ *be a quasinonexpansive operator. Suppose that* $\mathrm{Fix}\,(T_1) \bigcap \mathrm{Fix}\,(T_2) \neq \emptyset$*. Then*

$$\mathrm{Fix}(\mathrm{T_1}) \bigcap \mathrm{Fix}(\mathrm{T_2}) = \mathrm{Fix}(\mathrm{T_1 T_2}).$$

Proof. $\mathrm{Fix}\,(T_1) \bigcap \mathrm{Fix}\,(T_2) \subset \mathrm{Fix}\,(T_1 T_2)$ is obvious. We only need to prove that $\mathrm{Fix}\,(T_1 T_2) \subset \mathrm{Fix}\,(T_1) \bigcap \mathrm{Fix}\,(T_2)$. Let $x^\sharp \in \mathrm{Fix}(T_1 T_2)$ and $z \in \mathrm{Fix}(T_1) \bigcap \mathrm{Fix}(T_2)$. We consider the following two cases:

(i) The case $T_2 x^\sharp \in \mathrm{Fix}(T_1)$. Then $T_2 x^\sharp = T_1 T_2 x^\sharp = x^\sharp$. Thus $x^\sharp \in \mathrm{Fix}(T_2) \bigcap \mathrm{Fix}(T_1)$.

(ii) The case $T_2 x^\sharp \notin \mathrm{Fix}(T_1)$ and hence $x^\sharp \notin \mathrm{Fix}(T_2)$. Since T_1 is strictly quasinonexpansive, we have that $\|x^\sharp - z\| = \|T_1 T_2 x^\sharp - z\| < \|T_2 x^\sharp - z\| \leqslant \|x^\sharp - z\|$, which is a contradiction. □

In the sequel, we state several assumptions and symbols:

(A1): $A : H_1 \longrightarrow H_2$ is a bounded linear operator with its adjoint A^*.

(A2): $\{T_n\}_{n=1}^{\infty} : H_2 \to H_2$ is an infinite family of nonexpansive mappings given in the statement of Lemma 2.13.

(A3): $S : H_1 \longrightarrow K(H_1)$ is a λ-hybrid multivalued mapping with $Sp = \{p\}$ for all $p \in \mathrm{Fix}(S)$.

(A4): $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$.

(A5): $F_1 : H_1 \to H_1$ is a θ_1-inverse strongly monotone operator.

(A6): $F_2 : H_2 \to H_2$ is a θ_2-inverse strongly monotone operator.

(A7): $\Omega = \{z^\natural | z^\natural \in \mathrm{Fix}(S) \bigcap \mathrm{MEP}(\Theta_1, F_1, \psi_1)$ and $Az^\natural \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \bigcap \mathrm{MEP}(\Theta_2, F_2, \psi_2)\}$.

Throughout this paper, we assume that $\Omega \neq \emptyset$.

In the sequel, we present the following iterative algorithm to solve (1.5).

ALGORITHM 3.2. Let $x_1 \in H_1$ be an initial value. Let $\{\alpha_n\}$ be a real number sequence in $(0,1)$. Let $\varsigma \in (0,2)$, $r_1 \in (0, 2\theta_1)$ and $r_2 \in (0, 2\theta_2)$ be three real constants. Assume that a sequence $\{x_n\}$ is given. For each x_n, compute

$$y_n = x_n - T_{r_1}^{\Theta_1}(I - r_1 F_1)x_n,$$
$$z_n = Ax_n - W_n T_{r_2}^{\Theta_2}(I - r_2 F_2)Ax_n.$$

Case 1. If $\|y_n + A^* z_n\| \neq 0$, then we continue to construct u_n and x_{n+1} via the following manner:

$$u_n = x_n - \varsigma \tau_n (y_n + A^* z_n),$$
$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) w_n, \ w_n \in Su_n,$$

where

$$\tau_n = \frac{\|y_n\|^2 + \|z_n\|^2}{\|y_n + A^* z_n\|^2}.$$

Case 2. If $\|y_n + A^* z_n\| = 0$, then we continue and construct x_{n+1} via the following manner:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) w_n, \ w_n \in Sx_n.$$

REMARK 3.3. Note that W_n and W are averaged operators and therefore strictly quasinonexpansive operators (see [2]). By Lemma 3.1, we obtain that

$$\mathrm{Fix}(W_n T_{r_2}^{\Theta_2}(I - r_2 F_2)) = \mathrm{Fix}(W_n) \bigcap \mathrm{MEP}(\Theta_2, F_2, \psi_2)$$

and

$$\mathrm{Fix}(W T_{r_2}^{\Theta_2}(I - r_2 F_2)) = \mathrm{Fix}(W) \bigcap \mathrm{MEP}(\Theta_2, F_2, \psi_2).$$

From now on, we will divide our main result into several propositions.

PROPOSITION 3.4. $\|y_n + A^*z_n\| = 0$ *if and only if* $x_n \in \Omega_n$, *where*

$$\Omega_n = \{z^\natural | z^\natural \in \mathrm{MEP}(\Theta_1, \mathrm{F}_1, \psi_1) \text{ and } \mathrm{A}z^\natural \in \mathrm{Fix}(\mathrm{W_n})\bigcap \mathrm{MEP}(\Theta_2, \mathrm{F}_2, \psi_2)\}.$$

Evidently, $\Omega \subset \Omega_n$.

Proof. Note that $\mathrm{Fix}(W_n) = \bigcap_{j=1}^n \mathrm{Fix}(T_j)$. It is obvious that if $x_n \in \Omega_n$ then $\|y_n + A^*z_n\| = 0$. To see the converse, assume that $\|y_n + A^*z_n\| = 0$. By Lemma 2.17, we have that $T_{r_1}^{\Theta_1}(I - r_1F_1)$ and $W_nT_{r_2}^{\Theta_2}(I - r_2F_2)$ are nonexpansive. Then, for each $x^\dagger \in \Omega_n$, in the light of Definition 2.4, we have

$$\begin{aligned} 0 &= \langle y_n + A^*z_n, x_n - x^\dagger\rangle \\ &= \langle x_n - T_{r_1}^{\Theta_1}(I - r_1F_1)x_n, x_n - x^\dagger\rangle + \langle A^*(I - W_nT_{r_2}^{\Theta_2}(I - r_2F_2))Ax_n, x_n - x^\dagger\rangle \\ &= \langle x_n - T_{r_1}^{\Theta_1}(I - r_1F_1)x_n, x_n - x^\dagger\rangle + \langle (I - W_nT_{r_2}^{\Theta_2}(I - r_2F_2))Ax_n, Ax_n - Ax^\dagger\rangle \\ &\geqslant \frac{1}{2}(\|x_n - T_{r_1}^{\Theta_1}(I - r_1F_1)x_n\|^2 + \|(I - W_nT_{r_2}^{\Theta_2}(I - r_2F_2))Ax_n\|^2) \\ &= \frac{1}{2}(\|y_n\|^2 + \|z_n\|^2). \end{aligned} \tag{3.1}$$

It follows that

$$\|x_n - T_{r_1}^{\Theta_1}(I - r_1F_1)x_n\|^2 + \|(I - W_nT_{r_2}^{\Theta_2}(I - r_2F_2))Ax_n\|^2 = 0,$$

which implies that $x_n \in \mathrm{Fix}(T_{r_1}^{\Theta_1}(I - r_1F_1))$ and $Ax_n \in \mathrm{Fix}(W_nT_{r_2}^{\Theta_2}(I - r_2F_2))$. According to Remark 3.3, we have that $x_n \in \mathrm{MEP}(\Theta_1, F_1, \psi_1)$ and $Ax_n \in \mathrm{Fix}(W_n)\bigcap \mathrm{MEP}(\Theta_2, F_2, \psi_2)$. That is, $x_n \in \Omega_n$. This completes the proof. □

PROPOSITION 3.5. *The sequence* $\{x_n\}$ *generated by Algorithm 3.2 is bounded.*

Proof. Let $x^\flat \in \Omega$. Then $T_{r_1}^{\Theta_1}(I - r_1F_1)x^\flat = x^\flat$, $Sx^\flat = \{x^\flat\}$ and $T_{r_2}^{\Theta_2}(I - r_2F_2)Ax^\flat = T_nAx^\flat = Ax^\flat$ for all $n \geqslant 1$. First, in Case 1, by virtue of (3.1), we have

$$\begin{aligned} \|u_n - x^\flat\| &= \|x_n - x^\flat - \varsigma\tau_n(y_n + A^*z_n)\|^2 \\ &\leqslant \|x_n - x^\flat\|^2 + \varsigma^2\tau_n^2\|y_n + A^*z_n\|^2 - 2\langle\varsigma\tau_n(y_n + A^*z_n), x_n - x^\flat\rangle \\ &\leqslant \|x_n - x^\flat\|^2 + \varsigma^2\tau_n^2\|y_n + A^*z_n\|^2 - \varsigma\tau_n(\|y_n\|^2 + \|z_n\|^2) \\ &\leqslant \|x_n - x^\flat\|^2 - \varsigma(1 - \varsigma)\frac{(\|y_n\|^2 + \|z_n\|^2)^2}{\|y_n + A^*z_n\|^2}. \end{aligned} \tag{3.2}$$

Since $\|w_n - x^\flat\| \leqslant \mathscr{H}(Su_n, Sx^\flat) \leqslant \|u_n - x^\flat\|$, by Remark 2.15 and (3.2), we have that

$$\begin{aligned} \|x_{n+1} - x^\flat\| &= \|\alpha_nx_n + (1 - \alpha_n)w_n - x^\flat\| \\ &= \|\alpha_n(x_n - x^\flat) + (1 - \alpha_n)(w_n - x^\flat)\| \\ &\leqslant \alpha_n\|x_n - x^\flat\| + (1 - \alpha_n)\|w_n - x^\flat\| \\ &\leqslant \alpha_n\|x_n - x^\flat\| + (1 - \alpha_n)\|u_n - x^\flat\| \\ &\leqslant \alpha_n\|x_n - x^\flat\| + (1 - \alpha_n)\|x_n - x^\flat\| \\ &\leqslant \|x_n - x^\flat\|. \end{aligned}$$

In Case 2, we can obtain that $\|w_n - x^\flat\| \leqslant \mathscr{H}(Sx_n, Sx^\flat) \leqslant \|x_n - x^\flat\|$ and consequently,

$$\begin{aligned}\|x_{n+1} - x^\flat\| &= \|\alpha_n x_n + (1-\alpha_n)w_n - x^\flat\| \\ &\leqslant \alpha_n\|x_n - x^\flat\| + (1-\alpha_n)\|x_n - x^\flat\| \\ &\leqslant \|x_n - x^\flat\|.\end{aligned}$$

Hence, $\|x_{n+1} - x^\flat\| \leqslant \|x_n - x^\flat\|$, which implies that the sequence $\{x_n\}$ is Fejér monotone with respect to Ω and $\lim_{n\to\infty}\|x_n - x^\flat\|$ exists. So the sequence $\{x_n\}$ is bounded. □

THEOREM 3.6. *Under the assuptions* (A1)–(A7), *the sequence* $\{x_n\}$ *generated by Algorithm* 3.2 *converges weakly to a solution* $\hat{x} \in \Omega$, *where* $\hat{x} = \lim_{n\to\infty} P_\Omega x_n$.

Proof. Let $x^\flat \in \Omega$. We will show that every weak cluster point of the sequence $\{x_n\}$ belongs to the solution set, that is, $\omega_w(x_n) \subset \Omega$. By the boundedness of the sequence $\{x_n\}$, we can assume that there exists a subsequence $\{x_{n_j}\}$ such that $x_{n_j} \rightharpoonup \hat{x}$. Furthermore, we can also assumee that the sequence $\{x_{n_j}\}$ satisfies Case 1. Hence

$$\begin{aligned}&\|x_{n_j+1} - x^\flat\|^2 \\ &\quad = \|\alpha_{n_j}(x_{n_j} - x^\flat) + (1-\alpha_{n_j})(w_n - x^\flat)\|^2 \\ &\quad \leqslant \alpha_{n_j}\|(x_{n_j} - x^\flat)\|^2 + (1-\alpha_{n_j})\|w_{n_j} - x^\flat\|^2 - \alpha_{n_j}(1-\alpha_{n_j})\|x_{n_j} - w_{n_j}\|^2 \\ &\quad \leqslant \alpha_{n_j}\|(x_{n_j} - x^\flat)\|^2 + (1-\alpha_{n_j})\|u_{n_j} - x^\flat\|^2 - \alpha_{n_j}(1-\alpha_{n_j})\|x_{n_j} - w_{n_j}\|^2 \\ &\quad \leqslant \|x_{n_j} - x^\flat\|^2 - (1-\alpha_{n_j})\varsigma(1-\varsigma)\frac{(\|y_{n_j}\|^2 + \|z_{n_j}\|^2)^2}{\|y_{n_j} + A^* z_{n_j}\|^2} \\ &\qquad - \alpha_{n_j}(1-\alpha_{n_j})\|x_{n_j} - w_{n_j}\|^2.\end{aligned}$$

Thus we have

$$\begin{aligned}(1-\alpha_{n_j})\varsigma(1-\varsigma)&\frac{(\|y_{n_j}\|^2 + \|z_{n_j}\|^2)^2}{\|y_{n_j} + A^* z_{n_j}\|^2} + \alpha_{n_j}(1-\alpha_{n_j})\|x_{n_j} - w_{n_j}\|^2 \\ &\leqslant \|x_{n_j} - x^\flat\|^2 - \|x_{n_j+1} - x^\flat\|^2,\end{aligned}$$

which implies that

$$\lim_{j\to\infty} \frac{(\|y_{n_j}\|^2 + \|z_{n_j}\|^2)^2}{\|y_{n_j} + A^* z_{n_j}\|^2} = 0 \tag{3.3}$$

and

$$\lim_{j\to\infty} \|x_{n_j} - w_{n_j}\| = 0 \tag{3.4}$$

due to the assumption (A4). It follows from (3.3) that

$$\lim_{j\to\infty}\|x_{n_j}-u_{n_j}\| = \lim_{j\to\infty}\|\varsigma\tau_{n_j}(y_{n_j}+A^*z_{n_j})\| = \varsigma\lim_{j\to\infty}\frac{\|y_{n_j}\|^2+\|z_{n_j}\|^2}{\|y_{n_j}+A^*z_{n_j}\|}=0.$$

This together with (3.4) implies that

$$\lim_{j\to\infty}\|u_{n_j}-w_{n_j}\|=0. \tag{3.5}$$

By the boundedness of the sequence $\{y_n+A^*z_n\}$, it follows from (3.3) that

$$\lim_{j\to\infty}\|y_{n_j}\|^2+\|z_{n_j}\|^2=0,$$

which implies that

$$\lim_{j\to\infty}\|y_{n_j}\| = \lim_{j\to\infty}\|x_{n_j}-T_{r_1}^{\Theta_1}(I-r_1F_1)x_{n_j}\|=0$$

and

$$\lim_{j\to\infty}\|z_{n_j}\| = \lim_{j\to\infty}\|Ax_{n_j}-W_{n_j}T_{r_2}^{\Theta_2}(I-r_2F_2)Ax_{n_j}\|=0. \tag{3.6}$$

Note that $T_{r_1}^{\Theta_1}(I-r_1F_1)$ and $W_nT_{r_2}^{\Theta_2}(I-r_2F_2)$ is nonexpansive. By Lemma 2.12, we have that $\hat{x}\in \mathrm{Fix}(T_{r_1}^{\Theta_1}(I-r_1F_1)$ and $A\hat{x}\in \mathrm{Fix}(W_nT_{r_2}^{\Theta_2}(I-r_2F_2))$. By the boundedness of the sequence $\{T_{r_2}^{\Theta_2}(I-r_2F_2)Ax_{n_j}\}$ and Lemma 2.13, we obtain that

$$\lim_{j\to\infty}\|WT_{r_2}^{\Theta_2}(I-r_2F_2)Ax_{n_j}-W_nT_{r_2}^{\Theta_2}(I-r_2F_2)Ax_{n_j}\|=0.$$

This together with (3.6) implies that

$$\lim_{j\to\infty}\|Ax_{n_{k_j}}-WT_{r_2}^{\Theta_2}(I-r_2F_2)Ax_{n_j}\|=0.$$

Since $WT_{r_2}^{\Theta_2}(I-r_2F_2)$ is nonexpansive, by Lemma 2.12, we have that

$$A\hat{x}=WT_{r_2}^{\Theta_2}(I-r_2F_2)A\hat{x}.$$

By Remark 3.3, we can also get that $A\hat{x}\in \mathrm{Fix}(W)\bigcap \mathrm{MEP}(\Theta_2,F_2,\psi_2)$. By (3.5) and $x_{n_j}\rightharpoonup \hat{x}$, we get that $u_{n_j}\rightharpoonup\hat{x}$. According to Lemma 2.16, we have that $S\hat{x}=\{\hat{x}\}$. Hence we obtain that $\hat{x}\in\Omega$.

Suppose that there exists a subsequence $\{x_{n_j}\}$ satisfying Case 2. By Proposition 3.4, we get that $x_{n_j}\in\Omega_{n_j}$, which implies that $x_{n_j}\in \mathrm{Fix}(T_{r_1}^{\Theta_1}(I-r_1F_1)$ and $Ax_{n_j}\in \mathrm{Fix}(W_nT_{r_2}^{\Theta_2}(I-r_2F_2))$. By employing an argument similar to the above, we can obtain that $A\hat{x}\in \mathrm{Fix}(W)\bigcap \mathrm{MEP}(\Theta_2,F_2,\psi_2)$. We can also get that

$$\begin{aligned}
&\|x_{n_j+1}-x^\flat\|^2\\
&\quad=\|\alpha_{n_j}(x_{n_j}-x^\flat)+(1-\alpha_{n_j})(w_{n_j}-x^\flat)\|^2\\
&\quad\leqslant \alpha_{n_j}\|(x_{n_j}-x^\flat)\|^2+(1-\alpha_{n_j})\|w_{n_j}-x^\flat\|^2-\alpha_{n_j}(1-\alpha_{n_j})\|x_{n_j}-w_{n_j}\|^2\\
&\quad\leqslant \alpha_{n_j}\|(x_{n_j}-x^\flat)\|^2+(1-\alpha_{n_j})\|x_{n_j}-x^\flat\|^2-\alpha_{n_j}(1-\alpha_{n_j})\|x_{n_j}-w_{n_j}\|^2\\
&\quad\leqslant \|x_{n_j}-x^\flat\|^2-\alpha_{n_j}(1-\alpha_{n_j})\|x_{n_j}-w_{n_j}\|^2
\end{aligned}$$

and consequently,

$$\alpha_{n_j}(1-\alpha_{n_j})\|x_{n_j}-w_{n_j}\|^2 \leqslant \|x_{n_j}-x^\flat\|^2 - \|x_{n_j+1}-x^\flat\|^2,$$

which implies that

$$\lim_{j\to\infty}\|x_{n_j}-w_{n_j}\| = 0.$$

By Lemma 2.16, we have that $S\hat{x} = \{\hat{x}\}$. Hence we obtain that $\hat{x}\in\Omega$. By Lemma 2.14, we obtain that the sequence $\{x_n\}$ converges weakly to the solution $\hat{x}\in\Omega$, where $\hat{x} = \lim_{n\to\infty} P_\Omega x_n$. □

ALGORITHM 3.7. Let $x_1 \in H_1$ be an initial value. Let T be a nonexpansive operator. Let $\{\alpha_n\}$ be a real number sequence in $(0,1)$. Let $\delta\in(0,1)$, $\varsigma\in(0,2)$, $r_1\in(0,2\theta_1)$ and $r_2\in(0,2\theta_2)$ be three real constants. Assume that a sequence $\{x_n\}$ is given as follows. For each x_n, compute

$$y_n = x_n - T_{r_1}^{\Theta_1}(I-r_1F_1)x_n,$$
$$z_n = Ax_n - (\delta T + (1-\delta)I)T_{r_2}^{\Theta_2}(I-r_2F_2)Ax_n.$$

Case 1. If $\|y_n + A^*z_n\| \neq 0$, then we continue to construct u_n and x_{n+1} via the following manner:

$$u_n = x_n - \varsigma\tau_n(y_n + A^*z_n),$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)w_n,\ w_n\in Su_n,$$

where

$$\tau_n = \frac{\|y_n\|^2 + \|z_n\|^2}{\|y_n + A^*z_n\|^2}.$$

Case 2. If $\|y_n + A^*z_n\| = 0$, then we continue and construct x_{n+1} via the following manner:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)w_n,\ w_n\in Sx_n.$$

COROLLARY 3.8. *Let*

$$\hat{\Omega} = \{z^\natural | z^\natural \in Fix(S)\bigcap MEP(\Theta_1, F_1, \psi_1)$$

and

$$Az^\natural \in Fix(T)\bigcap MEP(\Theta_2, F_2, \psi_2)\}.$$

Suppose that $\hat{\Omega}\neq\emptyset$. *Under the assumptions* (A1), (A3) − (A6), *the sequence* $\{x_n\}$ *generated by Algorithm 3.7 converges weakly to a solution* $\hat{x}\in\Omega$, *where* $\hat{x} = \lim_{n\to\infty} P_{\hat{\Omega}} x_n$.

Declarations

Availablity of data and materials. Not applicable.

Competing interests. The authors declare that they have no competing interests.

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