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A MORE ACCURATE EXTENDED HARDY–HILBERT'S INEQUALITY WITH PARAMETERS

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Abstract. In this paper, in virtue of the symmetry principle, applying the techniques of real analysis and Euler-Maclaurin summation formula, we construct proper weight coefficients and use them to establish a more accurate extended Hardy-Hilbert's inequality with parameters. Then, we obtain the equivalent forms and some equivalent statements of the best possible constant factor related to several parameters. Finally, we illustrate the operator expressions and how the obtained results can generate some particular Hardy-Hilbert's inequalities.

1. Introduction

Assuming that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, a_m, $b_n \geqslant 0$, $0<\sum_{m=1}^{\infty} a_m^p<\infty$ and $0<\sum_{n=1}^{\infty} b_n^q<\infty$, we have the following well known Hardy-Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

In 2006, by introducing parameters $\lambda_i\in(0,2]$ $(i=1,2)$, $\lambda_1+\lambda_2=\lambda\in(0,4]$ an extension of (1) was provided by Krnić et al. [2] as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{p(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{2}$$

where, the constant factor $B(\lambda_1,\lambda_2)$ is the best possible, and

$$B(u,v):=\int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}dt \quad (u,v>0)$$

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is the beta function. For $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, inequality (2) reduces to (1); for $p = q = 2$, (2) reduces to Yang's inequality in [3]. Recently, applying (2), Adiyasuren at el. [4] also gave a new Hardy-Hilbert's inequality involving partial sums.

If $f(x),\ g(y) \geqslant 0$, $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(y)dy < \infty$, we still have the following analogue of (1) named in Hardy-Hilbert's integral inequality (cf. [1], Theorem 316):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \tag{3}$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequalities (1)–(3) with their extensions played an important role in analysis and its applications (cf. [5–15]).

In 1934, a half-discrete Hilbert-type inequality was given as follows (cf. [1], Theorem 351): If $K(t)$ $(t > 0)$ is a decreasing function, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi(s) = \int_0^\infty K(t)t^{s-1}dt < \infty$, $a_n \geqslant 0$, $0 < \sum_{n=1}^\infty a_n^p < \infty$, then

$$\int_0^\infty x^{p-2} \left(\sum_{n=1}^\infty K(nx)a_n\right)^p dx < \phi^p\left(\frac{1}{p}\right) \sum_{n=1}^\infty a_n^p. \tag{4}$$

Some new extensions of (4) were provided by [16–20]. In 2016, Hong et al. [21] obtained some equivalent statements of the extension of (1) with the best possible constant factor related to a few parameters. The other similar works were given by [22–28].

In this paper, following the way of [2, 21], by means of the weight coefficients, the idea of introduced parameters and Euler-Maclaurin summation formula, a more accurate extension of (1) with parameters are given. The equivalent forms and the equivalent statements of the best possible constant factor related to several parameters are obtained, and the operator expressions as well as some particular inequalities are considered.

2. Some lemmas

In what follows, we suppose that $p > 1$ $(q > 1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\xi,\ \eta \in [0, \frac{1}{4}]$, $\lambda \in (0, \frac{3}{2}]$, $\lambda_i \in (0,1] \cap (0,\lambda)$,

$$K_\lambda(\lambda_i) := \frac{\pi}{\lambda \sin(\pi\lambda_i/\lambda)} \quad (i = 1,2),$$

$\widehat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\widehat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, $a_m,\ b_n \geqslant 0$, $m,\ n \in N = \{1,2,\cdots\}$, such that

$$0 < \sum_{m=1}^\infty (m-\xi)^{p[1-\widehat{\lambda}_1]-1} a_m^p < \infty \ \text{ and } \ 0 < \sum_{n=1}^\infty (n-\eta)^{q[1-\widehat{\lambda}_2]-1} b_n^q < \infty. \tag{5}$$

LEMMA 1. (cf. [5], (2.2.13)) *If $g(t)$ is a positive decreasing function in $[m,\infty)$ $(m \in \mathbf{N})$ with $g(\infty) = 0$, $P_i(t)$ $(t \in \mathbf{N})$, B_i are Bernoulli functions and Bernoulli numbers of i-order, then we have*

$$\int_m^\infty P_{2q-1}(t)g(t)dt = \varepsilon \frac{B_{2q}}{q}\left(\frac{1}{2^{2q}} - 1\right)g(m) \ \ (0 < \varepsilon < 1;\ q = 1,2,\cdots) \tag{6}$$

In particular, for $q=1$, in view of $B_2=\frac{1}{2}$, we have

$$-\frac{1}{8}g(m)<\int_m^\infty P_1(t)g(t)dt<0. \tag{7}$$

LEMMA 2. *For $\lambda_2\in(0,1]\cap(0,\lambda)$, define the following weight coefficient:*

$$\varpi(\lambda_2,m)=:(m-\xi)^{\lambda-\lambda_2}\sum_{n=1}^\infty\frac{(n-\eta)^{\lambda_2-1}}{(m-\xi)^\lambda+(n-\eta)^\lambda}\ (m\in\mathbf{N}). \tag{8}$$

We have the following inequalities:

$$\begin{aligned}&K_\lambda(\lambda_2)\left(1-O\left(\frac{1}{(m-\xi)^{\lambda_2}}\right)\right)\\&<\varpi(\lambda_2,m)<K_\lambda(\lambda_2)=\frac{\pi}{\lambda\sin(\pi\lambda_2/\lambda)}\ \ (m\in\mathbf{N}),\end{aligned} \tag{9}$$

where, for $a=1-\eta\,(\in[\frac{3}{4},1])$, $O\left(\frac{1}{(m-\xi)^{\lambda_2}}\right)$ is indicated as

$$O\left(\frac{1}{(m-\xi)^{\lambda_2}}\right):=\frac{\lambda\sin(\pi\lambda_2/\lambda)}{\pi}\int_0^{\frac{a^\lambda}{(m-\xi)^\lambda}}\frac{u^{\frac{\lambda_2}{\lambda}-1}du}{(1+u)^\lambda}\in(0,1)\ \ (m\in\mathbf{N}). \tag{10}$$

Proof. For fixed $m\in\mathbf{N}$, we set the following real function:

$$g(m,t):=\frac{(t-\eta)^{\lambda_2-1}}{(m-\xi)^\lambda+(t-\eta)^\lambda}\ \ (t>\eta).$$

By means of Euler-Maclaurin summation formula as follows (cf. [2, 3]), we have

$$\begin{aligned}\sum_{n=1}^\infty g(m,n)&=\int_1^\infty g(m,t)dt+\frac{1}{2}g(m,1)+\int_1^\infty P_1(t)g'(m,t)dt\\&=\int_\eta^\infty g(m,t)dt-h_m(a,t),\\h_m(a,t)&:=\int_\eta^1 g(m,t)dt-\frac{1}{2}g(m,1)-\int_1^\infty P_1(t)g'(m,t)dt.\end{aligned}$$

We find $\frac{1}{2}g(m,1)=\frac{a^{\lambda_2-1}}{2[(m-\xi)^\lambda+a^\lambda]}$, and

$$\begin{aligned}\int_\eta^1 g(m,t)dt&=\int_\eta^1\frac{(t-\eta)^{\lambda_2-1}dt}{(m-\xi)^\lambda+(t-\eta)^\lambda}=\int_0^a\frac{u^{\lambda_2-1}du}{(m-\xi)^\lambda+u^\lambda}\\&=\frac{1}{\lambda_2}\int_0^a\frac{du^{\lambda_2}}{(m-\xi)^\lambda+u^\lambda}\\&=\frac{1}{\lambda_2}\frac{u^{\lambda_2}}{(m-\xi)^\lambda+u^\lambda}\Big|_0^a+\frac{\lambda}{\lambda_2}\int_0^a\frac{u^{\lambda+\lambda_2-1}du}{[(m-\xi)^\lambda+u^\lambda]^2}\\&>\frac{a^{\lambda_2}}{\lambda_2[(m-\xi)^\lambda+a^\lambda]}.\end{aligned}$$

We also obtain

$$\begin{aligned} g'(m,t) &:= \frac{(\lambda_2-1)(t-\eta)^{\lambda_2-2}}{(m-\xi)^\lambda+(t-\eta)^\lambda} - \frac{\lambda(t-\eta)^{\lambda+\lambda_2-2}}{[(m-\xi)^\lambda+(t-\eta)^\lambda]^2} \\ &= -\frac{(1-\lambda_2)(t-\eta)^{\lambda_2-2}}{(m-\xi)^\lambda+(t-\eta)^\lambda} \\ &\quad -\frac{\lambda[(m-\xi)^\lambda+(t-\eta)^\lambda-(m-\xi)^\lambda](t-\eta)^{\lambda_2-2}}{[(m-\xi)^\lambda+(t-\eta)^\lambda]^2} \\ &= -\frac{(\lambda+1-\lambda_2)(t-\eta)^{\lambda_2-2}}{(m-\xi)^\lambda+(t-\eta)^\lambda} + \frac{\lambda(m-\xi)^\lambda(t-\eta)^{\lambda_2-2}}{[(m-\xi)^\lambda+(t-\eta)^\lambda]^2}. \end{aligned}$$

For $0<\lambda_2\leqslant 1$, $\lambda_2<\lambda\leqslant\frac{3}{2}$, it follows that $\frac{d}{dt}[\frac{(t-\eta)^{\lambda_2-2}}{[(m-\xi)^\lambda+(t-\eta)^\lambda]^i}]<0\ (i=1,2)$. By (7), we obtain

$$(\lambda+1-\lambda_2)\int_1^\infty P_1(t)\frac{(t-\eta)^{\lambda_2-2}dt}{(m-\xi)^\lambda+(t-\eta)^\lambda} > -\frac{(\lambda+1-\lambda_2)a^{\lambda_2-2}}{8[(m-\xi)^\lambda+a^\lambda]},$$

$$-(m-\xi)^\lambda\lambda\int_1^\infty P_1(t)\frac{(t-\eta)^{\lambda_2-2}dt}{[(m-\xi)^\lambda+(t-\eta)^\lambda]^2} > 0.$$

Hence we find

$$-\int_1^\infty P_1(t)g'(m,t)dt > \frac{(\lambda+1-\lambda_2)a^{\lambda_2-2}}{8[(m-\xi)^\lambda+a^\lambda]},$$

and then for $\lambda\in(0,\frac{3}{2}]$, it follows that

$$\begin{aligned} h_m(a,t) &> \frac{a^{\lambda_2}}{\lambda_2[(m-\xi)^\lambda+a^\lambda]} - \frac{a^{\lambda_2-1}}{2[(m-\xi)^\lambda+a^\lambda]} - \frac{(\lambda+1-\lambda_2)a^{\lambda_2-2}}{8[(m-\xi)^\lambda+a^\lambda]} \\ &= \frac{8a^2-(4a+\lambda+1)\lambda_2+\lambda_2^2}{8\lambda_2[(m-\xi)^\lambda+a^\lambda]}a^{\lambda_2-2} \geqslant \frac{8a^2-(4a+\frac{5}{2})\lambda_2+\lambda_2^2}{8\lambda_2[(m-\xi)^\lambda+a^\lambda]}a^{\lambda_2-2}. \end{aligned}$$

Since we have $\frac{\partial}{\partial a}[8a^2-(4a+\lambda+1)\lambda_2+\lambda_2^2]=4(4a-\lambda_2)>0$, and

$$\frac{\partial}{\partial\lambda_2}[8a^2-(4a+\lambda+1)\lambda_2+\lambda_2^2] = -4a-\left(\frac{3}{2}-\lambda_2\right)<0,$$

it follows that $h_m(a,t)>\frac{8(\frac{3}{4})^2-(4\times\frac{3}{4}+\frac{5}{2})+1}{8\lambda_2[(m-\xi)^\lambda+a^\lambda]}a^{\lambda_2-2}=0$.

Setting $t=(m-\xi)u^{1/\lambda}+\eta$, we find

$$\begin{aligned} \varpi(\lambda_2,m) &= (m-\xi)^{\lambda-\lambda_2}\sum_{n=1}^\infty g(m,n) < (m-\xi)^{\lambda-\lambda_2}\int_\eta^\infty g(m,t)dt \\ &= (m-\xi)^{\lambda-\lambda_2}\int_\eta^\infty \frac{(t-\eta)^{\lambda_2-1}dt}{(m-\xi)^\lambda+(t-\eta)^\lambda} = \frac{\pi}{\lambda\sin(\pi\lambda_2/\lambda)}. \end{aligned}$$

On the other hand, we also have

$$\begin{aligned}\sum_{n=1}^{\infty} g(m,n) &= \int_{1}^{\infty} g(m,t)dt + \frac{1}{2}g(m,1) + \int_{1}^{\infty} P_1(t)g'(m,t)dt \\ &= \int_{1}^{\infty} g(m,t)dt + H(m), \\ H(m) &:= \frac{1}{2}g(m,1) + \int_{1}^{\infty} P_1(t)g'(m,t)dt.\end{aligned}$$

Since we find $\frac{1}{2}g(m,1) = \frac{a^{\lambda_2-1}}{2[(m-\xi)^{\lambda}+a^{\lambda}]}$ and

$$g'(m,t) = -\frac{(\lambda+1-\lambda_2)(t-\eta)^{\lambda_2-2}}{(m-\xi)^{\lambda}+(t-\eta)^{\lambda}} + \frac{\lambda(m-\xi)^{\lambda}(t-\eta)^{\lambda_2-2}}{[(m-\xi)^{\lambda}+(t-\eta)^{\lambda}]^2},$$

in view of (7), we obtain

$$\begin{aligned}&-(\lambda+1-\lambda_2)\int_{1}^{\infty} P_1(t)\frac{(t-\eta)^{\lambda_2-2}dt}{(m-\xi)^{\lambda}+(t-\eta)^{\lambda}} > 0, \text{ and} \\ &(m-\xi)^{\lambda}\lambda\int_{1}^{\infty} P_1(t)\frac{(t-\eta)^{\lambda_2-2}dt}{[(m-\xi)^{\lambda}+(t-\eta)^{\lambda}]^2} > -\frac{(m-\xi)^{\lambda}\lambda a^{\lambda_2-2}}{8[(m-\xi)^{\lambda}+a^{\lambda}]^2}.\end{aligned}$$

Hence, we have

$$\begin{aligned}H(m) &> \frac{a^{\lambda_2-1}}{2[(m-\xi)^{\lambda}+a^{\lambda}]} - \frac{[(m-\xi)^{\lambda}+a^2]2a^{\lambda_2-2}}{8[(m-\xi)^{\lambda}+a^{\lambda}]^2} \\ &= \frac{a^{\lambda_2-1}(2a-1)}{4[(m-\xi)^{\lambda}+a^{\lambda}]} > 0 \ \ (\lambda < 2).\end{aligned}$$

Setting $t = (m-\xi)u^{1/\lambda}+\eta$, we obtain

$$\begin{aligned}\varpi(\lambda_2,m) &= (m-\xi)^{\lambda-\lambda_2}\sum_{n=1}^{\infty} g(m,n) > (m-\xi)^{\lambda-\lambda_2}\int_{1}^{\infty} g(m,t)dt \\ &= (m-\xi)^{\lambda-\lambda_2}\int_{\eta}^{\infty} g(m,t)dt - (m-\xi)^{\lambda-\lambda_2}\int_{\eta}^{1} g(m,t)dt \\ &= \frac{\pi}{\lambda\sin(\pi\lambda_2/\lambda)} \\ &\quad \times\left[1 - \frac{\lambda\sin(\pi\lambda_2/\lambda)}{\pi}(m-\xi)^{\lambda-\lambda_2}\int_{\eta}^{1}\frac{(t-\eta)^{\lambda_2-1}dt}{(m-\xi)^{\lambda}+(t-\eta)^{\lambda}}\right] \\ &= \frac{\pi}{\lambda\sin(\pi\lambda_2/\lambda)}\left(1 - O(\frac{1}{(m-\xi)^{\lambda_2}})\right) > 0,\end{aligned}$$

where, $O\left(\frac{1}{(m-\xi)^{\lambda_2}}\right)$ (>0) is indicated by (10), satisfying

$$0 < \int_{0}^{\frac{a^{\lambda}}{(m-\xi)^{\lambda}}} \frac{u^{\frac{\lambda_2}{\lambda}-1}du}{(1+u)^{\lambda}} < \int_{0}^{\frac{a^{\lambda}}{(m-\xi)^{\lambda}}} u^{\frac{\lambda_2}{\lambda}-1}du = \frac{\lambda a^{\lambda_2}}{\lambda_2(m-\xi)^{\lambda_2}} \ \ (m \in \mathbf{N}).$$

Therefore, (9) and (10) follow.

The lemma is proved. □

LEMMA 3. *We have the following more accurate extended Hardy-Hilbert's inequality:*

$$I := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m-\xi)^{\lambda}+(n-\eta)^{\lambda}} < K_{\lambda}^{\frac{1}{p}}(\lambda_2)K_{\lambda}^{\frac{1}{q}}(\lambda_1)\left[\sum_{m=1}^{\infty}(m-\xi)^{p[1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}} \times\left[\sum_{n=1}^{\infty}(n-\eta)^{q[1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{11}$$

Proof. In the same way, for $\lambda_1 \in (0,1)\cap(0,\lambda)$, $n\in\mathbf{N}$, we have the following inequalities for the other weight coefficient:

$$\omega(\lambda_1,n) := (n-\eta)^{\lambda-\lambda_1}\sum_{m=1}^{\infty}\frac{(m-\xi)^{\lambda_1-1}}{(m-\xi)^{\lambda}+(n-\eta)^{\lambda}}\ \ (n\in\mathbf{N}), \tag{12}$$

$$K_{\lambda}(\lambda_1)\left(1-O\left(\frac{1}{(n-\eta)^{\lambda_1}}\right)\right) < \omega(\lambda_1,n) < K_{\lambda}(\lambda_1) = \frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}\ \ (n\in\mathbf{N}), \tag{13}$$

$$O\left(\frac{1}{(n-\eta)^{\lambda_1}}\right) := \frac{\lambda\sin(\pi\lambda_1/\lambda)}{\pi}\int_0^{\frac{b^{\lambda}}{(n-\eta)^{\lambda}}}\frac{u^{\frac{\lambda_1}{\lambda}-1}}{(1+u)^{\lambda}}du \in (0,1)\ \ (b := 1-\xi, m\in\mathbf{N}). \tag{14}$$

By Hölder's inequality, we find

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m-\xi)^{\lambda}+(n-\eta)^{\lambda}}\left[\frac{(n-\eta)^{(\lambda_2-1)/p}}{(m-\xi)^{(\lambda_1-1)/q}}a_m\right]\left[\frac{(m-\xi)^{(\lambda_1-1)/q}}{(n-\eta)^{(\lambda_2-1)/p}}b_n\right] \\ &\leqslant \left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m-\xi)^{\lambda}+(n-\eta)^{\lambda}}\frac{(n-\eta)^{(\lambda_2-1)}}{(m-\xi)^{(\lambda_1-1)(p-1)}}a_m^p\right]^{\frac{1}{p}} \\ &\quad\times\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m-\xi)^{\lambda}+(n-\eta)^{\lambda}}\frac{(m-\xi)^{(\lambda_1-1)}}{(n-\eta)^{(\lambda_2-1)(q-1)}}b_n^q\right]^{\frac{1}{q}} \\ &= \left[\sum_{m=1}^{\infty}\varpi(\lambda_2,m)(m-\xi)^{p[1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\omega(\lambda_1,n)(n-\eta)^{q[1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \end{aligned}$$

Then by (9) and (13), we have (11).

The lemma is proved. □

REMARK 1. By (11), for $\lambda_1+\lambda_2=\lambda,$ we find

$$0<\sum_{m=1}^{\infty}(m-\xi)^{p[1-\lambda_1)-1}a_m^p<\infty, 0<\sum_{n=1}^{\infty}(n-\eta)^{q[1-\lambda_2)-1}b_n^q<\infty.$$

and the following more accurate extended Hardy-Hilbert's inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m-\xi)^{\lambda}+(n-\eta)^{\lambda}} < \frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}\left[\sum_{m=1}^{\infty}(m-\xi)^{p[1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\eta)^{q[1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \quad (15)$$

LEMMA 4. *The constant factor* $\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}$ *in* (15) *is the best possible.*

Proof. For any $0<\varepsilon<p\lambda_1$, we set $\widetilde{a}_m:=m^{\lambda_1-\frac{\varepsilon}{p}-1}$, $\widetilde{b}_n:=n^{\lambda_2-\frac{\varepsilon}{q}-1}$ $(m,n\in\mathbf{N})$. If there exists a constant $M\leqslant\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}$, such that (12) is valid when we replace $\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}$ by M, then in particular, for $\xi=\eta=0$ in (15), we still have

$$\widetilde{I}_0:=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{m^{\lambda}+n^{\lambda}} < M\left[\sum_{m=1}^{\infty}m^{p[1-\lambda_1)-1}\widetilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q[1-\lambda_2)-1}\widetilde{b}_n^q\right]^{\frac{1}{q}}. \quad (16)$$

By (16) and the decreasingness property of series, we obtain

$$\widetilde{I}_0<M\left[\sum_{m=1}^{\infty}m^{p[1-\lambda_1)-1}m^{p(\lambda_1-\frac{\varepsilon}{p}-1)}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q[1-\lambda_2)-1}n^{q(\lambda_2-\frac{\varepsilon}{q}-1)}\right]^{\frac{1}{q}}$$
$$=M\left(1+\sum_{m=2}^{\infty}m^{-\varepsilon-1}\right)<M(1+\int_1^{\infty}x^{-\varepsilon-1}dx)=\frac{M}{\varepsilon}(\varepsilon+1).$$

By (12) and (13) (for $\xi=\eta=0,\ \lambda-\lambda_1=\lambda_2$), setting $\widetilde{\lambda}_1:=\lambda_1-\frac{\varepsilon}{p}\in(0,1)\cap(0,\lambda)$ $(0<\widetilde{\lambda}_2=\lambda_2+\frac{\varepsilon}{p}=\lambda-\widetilde{\lambda}_1<\lambda)$, we find

$$\widetilde{I}_0:=\sum_{n=1}^{\infty}[n^{\lambda_2+\frac{\varepsilon}{p}}\sum_{m=1}^{\infty}\frac{m^{(\lambda_1-\frac{\varepsilon}{p})-1}}{m^{\lambda}+n^{\lambda}}]n^{-\varepsilon-1}=\sum_{n=1}^{\infty}\omega_0(\widetilde{\lambda}_1,n)n^{-\varepsilon-1}$$
$$>K_{\lambda}(\widetilde{\lambda}_1)\sum_{n=1}^{\infty}\left(1-O\left(\frac{1}{n^{\widetilde{\lambda}_1}}\right)\right)n^{-\varepsilon-1}=K_{\lambda}(\widetilde{\lambda}_1)\sum_{n=1}^{\infty}n^{-\varepsilon-1}-O(1)$$
$$>K_{\lambda}(\widetilde{\lambda}_1)\int_1^{\infty}y^{-\varepsilon-1}dy-O(1)=\frac{K_{\lambda}(\widetilde{\lambda}_1)}{\varepsilon}(1-\varepsilon O(1)).$$

In view of the above results, we have the following inequality:

$$K_\lambda\left(\lambda_1-\frac{\varepsilon}{p}\right)(1-\varepsilon O(1))\leqslant \varepsilon\widetilde{I}_0<M(\varepsilon+1).$$

For $\varepsilon\to 0^+$, we find that $K_\lambda(\lambda_1)\leqslant M$. Hence, $M=K_\lambda(\lambda_1)$ is the best possible constant factor of (15).

The lemma is proved. □

REMARK 2. For $\widehat{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\widehat{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$ in (11), we find

$$\widehat{\lambda}_1+\widehat{\lambda}_2=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\lambda,$$
$$0<\widehat{\lambda}_i<\lambda \ \ (i=1,2).$$

If $\lambda-\lambda_i\leqslant 1\ \ (i=1,2)$, then it follows that $\widehat{\lambda}_i\leqslant 1\ \ (i=1,2)$, and we can rewrite (15) as follows:

$$\begin{aligned} I&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m-\xi)^\lambda+(n-\eta)^\lambda}\\ &<K_\lambda(\widehat{\lambda}_1)\left[\sum_{m=1}^{\infty}(m-\xi)^{p[1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\eta)^{q[1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \end{aligned}\tag{17}$$

LEMMA 5. *If the constant factor* $K_\lambda^{\frac{1}{p}}(\lambda_2)K_\lambda^{\frac{1}{q}}(\lambda_1)$ *in* (11) *is the best possible, then for* $\lambda-\lambda_i\leqslant 1\ \ (i=1,2)$, *we have* $\lambda_1+\lambda_2=\lambda$.

Proof. If the constant factor $K_\lambda^{\frac{1}{p}}(\lambda_2)K_\lambda^{\frac{1}{q}}(\lambda_1)$ in (11) is the best possible, then in view of the assumption and (17), we have the following inequality:

$$K_\lambda^{\frac{1}{p}}(\lambda_2)K_\lambda^{\frac{1}{q}}(\lambda_1)\leqslant K_\lambda(\widehat{\lambda}_1)(<\infty)\tag{18}$$

By Hölder's inequality (cf. [29]), we find

$$\begin{aligned} 0<K_\lambda(\widehat{\lambda}_1)&=K_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)\\ &=\frac{1}{\lambda}\int_0^\infty\frac{1}{1+u}u^{\frac{\lambda-\lambda_2}{p\lambda}+\frac{\lambda_1}{q\lambda}-1}du\\ &=\frac{1}{\lambda}\int_0^\infty\frac{1}{1+u}(u^{\frac{-\lambda_2}{p\lambda}})(u^{\frac{\lambda_1-\lambda}{q\lambda}})du\\ &\leqslant\left(\frac{1}{\lambda}\int_0^\infty\frac{v^{\frac{\lambda_2}{\lambda}-1}}{v+1}dv\right)^{\frac{1}{p}}\left(\frac{1}{\lambda}\int_0^\infty\frac{u^{\frac{\lambda_1}{\lambda}-1}}{1+u}du\right)^{\frac{1}{q}}\\ &=K_\lambda^{\frac{1}{p}}(\lambda_2)K_\lambda^{\frac{1}{q}}(\lambda_1)<\infty. \end{aligned}\tag{19}$$

In view of (18), we have $K_{\lambda}^{\frac{1}{p}}(\lambda_2)K_{\lambda}^{\frac{1}{q}}(\lambda_1) = K_{\lambda}(\widehat{\lambda}_1)$, namely, (19) keeps the form of equality.

We observe that (19) keeps the form of equality if and only if there exist constants A and B, such that they are not both zero and (cf. [29])

$$Au^{\frac{-\lambda_2}{\lambda}} = Bu^{\frac{\lambda_1}{\lambda}-1} \text{ a.e. in } \mathbf{R}_+.$$

Assuming that $A \neq 0$, it follows that $u^{-\frac{\lambda_2+\lambda_1}{\lambda}+1} = \frac{B}{A}$ a.e. in $\mathbf{R}_+$, and $-\frac{\lambda_2+\lambda_1}{\lambda}+1 = 0$, namely, $\lambda_1+\lambda_2 = \lambda$.

The lemma is proved. □

3. Main results

THEOREM 1. *Inequality* (11) *is equivalent to the following inequality:*

$$J := \left\{\sum_{n=1}^{\infty}(n-\eta)^{p\widehat{\lambda}_2-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m-\xi)^{\lambda}+(n-\eta)^{\lambda}}\right]^p\right\}^{\frac{1}{p}}$$
$$< K_{\lambda}^{\frac{1}{p}}(\lambda_2)K_{\lambda}^{\frac{1}{q}}(\lambda_1)\left[\sum_{m=1}^{\infty}(m-\xi)^{p(1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}. \tag{20}$$

If the constant factor in (11) *is the best possible, then so is the constant factor in* (20).

Proof. Suppose that (11) is valid. By Hölder's inequality (cf. [29]), we find

$$I = \sum_{n=1}^{\infty}\left[(n-\eta)^{\widehat{\lambda}_2-\frac{1}{p}}\sum_{m=1}^{\infty}\frac{a_m}{(m-\xi)^{\lambda}+(n-\eta)^{\lambda}}\right]\left[(n-\eta)^{\frac{1}{p}-\widehat{\lambda}_2}b_n\right]$$
$$\leqslant J\left[\sum_{n=1}^{\infty}(n-\eta)^{q[1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{21}$$

Then by (17), we obtain (11).

On the other hand, assuming that (11) is valid, we set

$$b_n := (n-\eta)^{p\widehat{\lambda}_2-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m-\xi)^{\lambda}+(n-\eta)^{\lambda}}\right]^{p-1}, \quad n \in \mathbf{N}.$$

If $J=0$, then (11) is naturally valid; if $J=\infty$, then it is impossible that makes (11) valid, namely, $J<\infty$. Suppose that $0<J<\infty$. By (11), it follows that

$$\sum_{n=1}^{\infty}(n-\eta)^{q[1-\widehat{\lambda}_2)-1}b_n^q = J^p = I$$

$$< K_\lambda^{\frac{1}{p}}(\lambda_2)K_\lambda^{\frac{1}{q}}(\lambda_1)\left[\sum_{m=1}^{\infty}(m-\xi)^{p(1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}$$

$$\times\left[\sum_{n=1}^{\infty}(n-\eta)^{q(1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}},$$

$$J=\left[\sum_{n=1}^{\infty}(n-\eta)^{q[1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{p}}$$

$$< K_\lambda^{\frac{1}{p}}(\lambda_2)K_\lambda^{\frac{1}{q}}(\lambda_1)\left[\sum_{m=1}^{\infty}(m-\xi)^{p(1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}},$$

namely, (17) follows, which is equivalent to (11).

If the constant factor in (11) is the best possible, then so is constant factor in (17). Otherwise, by (21), we would reach a contradiction that the constant factor in (11) is not the best possible.

The theorem is proved. □

THEOREM 2. *The following statements (i), (ii), (iii) and (iv) are equivalent:*

(i) Both $K_\lambda^{\frac{1}{p}}(\lambda_2)K_\lambda^{\frac{1}{q}}(\lambda_1)$ *and* $K_\lambda(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ *are independent of* p,q*;*

(ii)

$$K_\lambda^{\frac{1}{p}}(\lambda_2)K_\lambda^{\frac{1}{q}}(\lambda_1)=K_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right); \tag{22}$$

(iii) $K_\lambda^{\frac{1}{p}}(\lambda_2)K_\lambda^{\frac{1}{q}}(\lambda_1)$ *in* (11) *is the best possible constant factor;*

(iv) if $\lambda-\lambda_i\leqslant 1$ $(i=1,2)$*, then* $\lambda_1+\lambda_2=\lambda$*.*

If the statement (iv) follows, namely, $\lambda_1+\lambda_2=\lambda$*, then we have* (12) *and the following equivalent inequality with the best possible constant factor* $\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}$*:*

$$\left\{\sum_{n=1}^{\infty}(n-\eta)^{p\lambda_2-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m-\xi)^\lambda+(n-\eta)^\lambda}\right]^p\right\}^{\frac{1}{p}}$$

$$<\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}\left[\sum_{m=1}^{\infty}(m-\xi)^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}. \tag{23}$$

Proof. $(i) \Rightarrow (ii)$. By (i), we have

$$K_\lambda^{\frac{1}{p}}(\lambda_2)K_\lambda^{\frac{1}{q}}(\lambda_1) = \lim_{p\to\infty}\lim_{q\to 1^+} K_\lambda^{\frac{1}{p}}(\lambda_2)K_\lambda^{\frac{1}{q}}(\lambda_1) = K_\lambda(\lambda_1),$$

$$K_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right) = \lim_{p\to\infty}\lim_{q\to 1^+} K_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right) = K_\lambda(\lambda_1),$$

namely, (22) follows.

$(ii) \Rightarrow (iv)$. If (22) follows, then (19) keeps the form of equality. In view of the proof of Lemma 5, it follows that $\lambda_1+\lambda_2=\lambda$.

$(iv) \Rightarrow (i)$. If $\lambda_1+\lambda_2=\lambda$, then we have

$$K_\lambda^{\frac{1}{p}}(\lambda_2)K_\lambda^{\frac{1}{q}}(\lambda_1) = K_\lambda(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}) = K_\lambda(\lambda_1),$$

which is independent of p,q.

Hence, it follows that $(i) \Leftrightarrow (ii) \Leftrightarrow (iv)$.

$(iii) \Rightarrow (iv)$. By the assumption and Lemma 5, we have $\lambda_1+\lambda_2=\lambda$.

$(iv) \Rightarrow (iii)$. By Lemma 4, for $\lambda_1+\lambda_2=\lambda$,

$$K_\lambda^{\frac{1}{p}}(\lambda_2)K_\lambda^{\frac{1}{q}}(\lambda_1)(=K_\lambda(\lambda_1))$$

is the best possible constant factor of (11).

Therefore, we have $(iii) \Leftrightarrow (iv)$.

Hence, the statements (i), (ii), (iii) and (iv) are equivalent.

The theorem is proved. □

4. Operator expressions and some particular inequalities

We set functions

$$\varphi(m) := (m-\xi)^{p(1-\widehat{\lambda}_1)-1}, \quad \psi(n) := (n-\eta)^{q(1-\widehat{\lambda}_2)-1},$$

wherefrom,

$$\psi^{1-p}(n) = (n-\eta)^{p\widehat{\lambda}_2-1} \ \ (m,n\in\mathbf{N}).$$

Define the following real normed spaces:

$$l_{p,\varphi} := \left\{ a=\{a_m\}_{m=1}^{\infty}; ||a||_{p,\varphi} := \left(\sum_{m=1}^{\infty}\varphi(m)|a_m|^p\right)^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\psi} := \left\{ b=\{b_n\}_{n=1}^{\infty}; ||b||_{q,\psi} := \left(\sum_{n=1}^{\infty}\psi(n)|b_n|^q\right)^{\frac{1}{q}} < \infty \right\},$$

$$l_{p,\psi^{1-p}} := \left\{ c=\{c_n\}_{n=1}^{\infty}; ||c||_{p,\psi^{1-p}} := \left(\sum_{n=1}^{\infty}\psi^{1-p}(n)|c_n|^p\right)^{\frac{1}{p}} < \infty \right\}.$$

Assuming that $a \in l_{p,\varphi}$, setting

$$c = \{c_n\}_{n=1}^{\infty}, c_n := \sum_{m=1}^{\infty} \frac{a_m}{(m-\xi)^{\lambda} + (n-\eta)^{\lambda}}, n \in \mathbf{N},$$

we can rewrite (20) as follows:

$$||c||_{p,\psi^{1-p}} < K_{\lambda}^{\frac{1}{p}}(\lambda_2) K_{\lambda}^{\frac{1}{q}}(\lambda_1) ||a||_{p,\varphi} < \infty,$$

namely, $c \in l_{p,\psi^{1-p}}$.

DEFINITION 1. Define a more accurate extended Hardy-Hilbert's operator $T : l_{p,\varphi} \to l_{p,\psi^{1-p}}$ as follows: For any $a \in l_{p,\varphi}$, there exists a unique representation $c \in l_{p,\psi^{1-p}}$. Define the formal inner product of Ta and $b \in l_{q,\psi}$, and the norm of T as follows:

$$(Ta, b) := \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m-\xi)^{\lambda} + (n-\eta)^{\lambda}} \right] b_n,$$

$$||T|| := \sup_{a(\neq 0) \in l_{p,\varphi}} \frac{||Ta||_{p,\psi^{1-p}}}{||a||_{p,\varphi}}.$$

By Theorem 1 and Theorem 2, we have

THEOREM 3. *If* $a \in l_{p,\varphi}$, $b \in l_{q,\psi}$, $||a||_{p,\varphi}$, $||b||_{q,\psi} > 0$, *then we have the following equivalent inequalities:*

$$(Ta, b) < K_{\lambda}^{\frac{1}{p}}(\lambda_2) K_{\lambda}^{\frac{1}{q}}(\lambda_1) ||a||_{p,\varphi} ||b||_{q,\psi}, \tag{24}$$

$$||Ta||_{p,\psi^{1-p}} < k_{\lambda}^{\frac{1}{p}}(\lambda_2) k_{\lambda}^{\frac{1}{q}}(\lambda_1) ||a||_{p,\varphi}. \tag{25}$$

Moreover, if $\lambda_1 + \lambda_2 = \lambda$, *then the constant factor* $K_{\lambda}^{\frac{1}{p}}(\lambda_2) K_{\lambda}^{\frac{1}{q}}(\lambda_1)$ *in* (24) *and* (25) *is the best possible, namely,*

$$||T|| = K_{\lambda}(\lambda_1) = \frac{\pi}{\lambda \sin(\pi \lambda_1/\lambda)}. \tag{26}$$

On the other hand, if the constant factor $K_{\lambda}^{\frac{1}{p}}(\lambda_2) K_{\lambda}^{\frac{1}{q}}(\lambda_1)$ in (24) and (25) is the best possible, then for $\lambda - \lambda_i \leqslant 1$ $(i = 1, 2)$, we have $\lambda_1 + \lambda_2 = \lambda$.

REMARK 3. (i) For $\lambda = 1$, $\lambda_1 = \frac{1}{r}$, $\lambda_2 = \frac{1}{s}$ $(r > 1, \frac{1}{r} + \frac{1}{s} = 1)$, $\tau = \xi + \eta \in [0, \frac{1}{2}]$ in (12) and (23), we have the following equivalent inequalities with the best possible constant factor $\frac{\pi}{\sin(\pi/r)}$:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n-\tau} < \frac{\pi}{\sin(\pi/r)} \left[\sum_{m=1}^{\infty} (m-\xi)^{\frac{p}{s}-1} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} (n-\eta)^{\frac{q}{r}-1} b_n^q \right]^{\frac{1}{q}}. \tag{27}$$

$$\left[\sum_{n=1}^{\infty}(n-\eta)^{\frac{p}{s}-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n-\tau}\right)^p\right]^{\frac{1}{p}} < \frac{\pi}{\sin(\pi/r)}\left[\sum_{m=1}^{\infty}(m-\xi)^{\frac{p}{s}-1}a_m^p\right]^{\frac{1}{p}}. \tag{28}$$

In particular, for $\xi=\eta=0,\ s=p,\ r=q$ in (27) and (28), we have (1) and the following equivalent inequality:

$$\left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p\right]^{\frac{1}{p}} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}. \tag{29}$$

Hence, (27) (resp. (15)) is a more accurate extension of (1).

(ii) For $\lambda=\frac{3}{2},\ \lambda_1=\lambda_2=\frac{3}{4}$ in (12) and (23), we have the following equivalent inequalities with the best possible constant factor $\frac{2\pi}{3}$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m-\xi)^{3/2}+(n-\eta)^{3/2}} < \frac{2\pi}{3}\left[\sum_{m=1}^{\infty}(m-\xi)^{\frac{p}{4}-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\eta)^{\frac{q}{4}-1}b_n^q\right]^{\frac{1}{q}}. \tag{30}$$

$$\left[\sum_{n=1}^{\infty}(n-\eta)^{\frac{3p}{4}-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{(m-\xi)^{3/2}+(n-\eta)^{3/2}}\right)^p\right]^{\frac{1}{p}} < \frac{2\pi}{3}\left[\sum_{m=1}^{\infty}(m-\xi)^{\frac{p}{4}-1}a_m^p\right]^{\frac{1}{p}}. \tag{31}$$

5. Conclusions

In this paper, by virtue of the symmetry principle, applying the techniques of real analysis and Euler-Maclaurin summation formula, we construct proper weight coefficients and use them to establish a more accurate extended Hardy-Hilbert's inequality with parameters in Lemma 3. Then, we obtain the equivalent forms and some equivalent statements of the best possible constant factor related to several parameters in Theorem 1 and Theorem 2. Finally, we illustrate the operator expressions and how the obtained results can generate some new Hardy-Hilbert's inequalities in Theorem 3 and Remark 3. The lemmas and theorems provide an extensive account of this type of reverse inequalities.

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