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SOME PARAMETERIZED INEQUALITIES ARISING FROM THE TEMPERED FRACTIONAL INTEGRALS INVOLVING THE (μ,η)–INCOMPLETE GAMMA FUNCTIONS

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Abstract. We utilize the definition of a class of the tempered fractional integral operators, proposed by Sabzikar et al. in [J. Comput. Phys., 293:14–28, 2015], to establish a fractional-type integral identity with parameters. We present some parameterized integral inequalities for differentiable mappings in terms of the constructed identity, and give two examples to identify the correctness of the obtained results as well.

1. Introduction

The convexity of functions is a powerful tool to deal with a variety of issues of pure and applied science. Recently, many authors have devoted themselves to studying the properties and inequalities related to convexity in various directions, see the published articles [9, 10, 13, 21, 23, 24, 33] and the references therein. One of the most important mathematical inequalities considering convex mappings is Hermite–Hadamard inequality, which is also utilized widely in many other disciplines of applied mathematics. Let us state it as below.

Suppose that $h:\mathscr{K}\subseteq\mathbb{R}\to\mathbb{R}$ is a convex mappings defined on the interval $\mathscr{K}$ of real numbers, and $\tau_1,\tau_2\in\mathscr{K}$ with $\tau_1<\tau_2$. The subsequent inequality is named Hermite–Hadamard inequality:

$$h\left(\frac{\tau_1+\tau_2}{2}\right)\leqslant\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}h(\rho)\mathrm{d}\rho\leqslant\frac{h(\tau_1)+h(\tau_2)}{2}. \tag{1.1}$$

This inequality has received considerable attention from the authors and other mathematicians. There have been many studies on the Hermite–Hadamard-type inequality for other kinds of convex functions. For example, one can refer to [27] for convex functions, to [12] for exponential-type convex functions, to [31] for s-convex functions, to [18] for h-convex mappings, to [35] for h-preinvex functions, to [17] for harmonically convex functions, to [2] for N-quasiconvex functions, and so on. For more development about this topic, the reader may consult [14, 15, 29, 32, 34, 28, 40] and the references therein.

For the sake of further discussion, let us evoke the following definitions.

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DEFINITION 1.1. [7] Considering the mapping $\eta : \mathscr{K} \times \mathscr{K} \to \mathbb{R}^n$, if for each $\tau_1, \tau_2 \in \mathscr{K}$ and $\xi \in [0,1]$, $\tau_1 + \xi\eta(\tau_2, \tau_1) \in \mathscr{K}$, the set $\mathscr{K} \subseteq \mathbb{R}^n$ is called to be an invex set of η.

DEFINITION 1.2. [41] Suppose that $\mathscr{K} \subseteq \mathbb{R}^n$ is an invex set regarding $\eta : \mathscr{K} \times \mathscr{K} \to \mathbb{R}^n$. The mapping h defined on the invex set $\mathscr{K} \subseteq \mathbb{R}^n$ is named to be preinvex regarding η if for every $\tau_1, \tau_2 \in \mathscr{K}$ and $\xi \in [0,1]$ we have that

$$h(\tau_1 + \xi\eta(\tau_2, \tau_1)) \leqslant (1-\xi)h(\tau_1) + \xi h(\tau_2).$$

If the mapping $\eta(\tau_2, \tau_1) = \tau_2 - \tau_1$, then the preinvex mapping transfers to the classically convex mapping.

DEFINITION 1.3. [22] A set $\mathscr{K} \subseteq \mathbb{R}^n$ is named m-invex with regard to the mapping $\eta : \mathscr{K} \times \mathscr{K} \times (0,1] \to \mathbb{R}^n$ for certain fixed $m \in (0,1]$ if $m\tau_1 + \xi\eta(\tau_2, \tau_1, m) \in \mathscr{K}$ holds for all $\tau_1, \tau_2 \in \mathscr{K}$ and $\xi \in [0,1]$.

DEFINITION 1.4. [42] Assume that $\mathscr{K} \subseteq \mathbb{R}^n$ is an open m-invex set with regard to $\eta : \mathscr{K} \times \mathscr{K} \times (0,1] \to \mathbb{R}^n$. For any $\tau_1, \tau_2 \in \mathscr{K}$ and $m \in (0,1]$, the η_m-path $P_{\tau_1\tau_3}$ linking the points $m\tau_1$ and $\tau_3 = m\tau_1 + \eta(\tau_2, \tau_1, m)$ is defined as

$$P_{\tau_1\tau_3} = \Big\{\theta | \theta = m\tau_1 + \xi\eta(\tau_2, \tau_1, m), \xi \in [0,1]\Big\}.$$

DEFINITION 1.5. [22] Assume that $\mathscr{K} \subseteq \mathbb{R}^n$ is an open m-invex set with regard to $\eta : \mathscr{K} \times \mathscr{K} \times (0,1] \to \mathbb{R}^n$. For certain fixed $s, m \in (0,1]$, the mapping $h : \mathscr{K} \to \mathbb{R}$ is called the generalized (s,m)-preinvex mapping if

$$h\big(m\tau_1 + \xi\eta(\tau_2, \tau_1, m)\big) \leqslant m(1-\xi)^s h(\tau_1) + \xi^s h(\tau_2)$$

is available for all $\tau_1, \tau_2 \in \mathscr{K}$ and $\xi \in [0,1]$.

Notice that if the mapping $\eta(\tau_2, \tau_1, m) = \tau_2 - \tau_1 m$ with $m = 1$, then the conception of generalized (s,m)-preinvexity transfers to the conception of s-convexity.

DEFINITION 1.6. [36] For any real numbers $\alpha > 0$ and $\lambda, \mu \geqslant 0$, the μ-incomplete gamma function is defined as

$$\gamma_\mu(\alpha, \lambda) = \int_0^\lambda \rho^{\alpha-1} e^{-\mu\rho} d\rho.$$

Obviously, if we take $\mu = 1$, then it transfers to the incomplete gamma function [16]:

$$\gamma(\alpha, \lambda) = \int_0^\lambda \rho^{\alpha-1} e^{-\rho} d\rho, \alpha > 0.$$

DEFINITION 1.7. [30] Assume that $[\tau_1,\tau_2]$ is a real interval and $\tau_1 \geqslant 0$, $\alpha > 0$. For a mapping $h \in L^1([\tau_1,\tau_2])$, the left and right Reimann–Liouvile fractional integrals, respectively, are defined by

$$\mathscr{I}^{\alpha}_{\tau_1^+} h(x) = \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{x} (x-\rho)^{\alpha-1} h(\rho) \mathrm{d}\rho, \; x > \tau_1,$$

and

$$\mathscr{I}^{\alpha}_{\tau_2^-} h(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\tau_2} (\rho-x)^{\alpha-1} h(\rho) \mathrm{d}\rho, \; x < \tau_2.$$

Here, $\Gamma(\alpha)$ is the Gamma function, $\Gamma(\alpha) = \int_0^\infty e^{-\rho}\rho^{\alpha-1}\mathrm{d}\rho$, and $\mathscr{I}^{0}_{\tau_1^+} h(x) = \mathscr{I}^{0}_{\tau_2^-} h(x) = h(x)$.

In terms of the Riemann–Liouville fractional integral operators above, Sarikaya et al. extended the classical Hermite–Hadamard inequality to the version of fractional integrals as below.

THEOREM 1.1. [39] *Assume that* $h : [\tau_1,\tau_2] \to \mathbb{R}$ *is a positive mapping along with* $0 \leqslant \tau_1 < \tau_2$ *and* $h \in L^1([\tau_1,\tau_2])$. *If* h *is a convex mapping defined on* $[\tau_1,\tau_2]$, *then the following fractional integral inequalities*

$$h\left(\frac{\tau_1+\tau_2}{2}\right) \leqslant \frac{\Gamma(\alpha+1)}{2(\tau_2-\tau_1)^\alpha}\left[\mathscr{I}^{\alpha}_{\tau_1^+} h(\tau_2) + \mathscr{I}^{\alpha}_{\tau_2^-} h(\tau_1)\right] \leqslant \frac{h(\tau_1)+h(\tau_2)}{2} \qquad (1.2)$$

are valid with $\alpha > 0$.

Recently, Sabzikar et al. in [38] introduced the notion of the tempered fractional integrals.

DEFINITION 1.8. Assume that $[\tau_1,\tau_2]$ is a real interval and $\mu \geqslant 0$, $\alpha > 0$. Then for a mapping $h \in L^1([\tau_1,\tau_2])$, the left and right tempered fractional integrals, respectively, are defined by

$$\mathscr{I}^{(\alpha,\mu)}_{\tau_1^+} h(x) = \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{x} (x-\rho)^{\alpha-1} e^{-\mu(x-\rho)} h(\rho) \mathrm{d}\rho, \; x > \tau_1,$$

and

$$\mathscr{I}^{(\alpha,\mu)}_{\tau_2^-} h(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\tau_2} (\rho-x)^{\alpha-1} e^{-\mu(\rho-x)} h(\rho) \mathrm{d}\rho, \; x < \tau_2.$$

For recent related development pertaining to the tempered fractional integrals, see the published articles [26, 36, 37, 38] and the references therein.

It is undeniable that the fractional integral operators has a great influence on realizing the differentiation and integration of real order and complex order. Moreover, it emerged rapidly due to its applications in modelling plenty of issues, especially in tackling the dynamics of the complex systems, decision making in structural engineering and stochastic problems and so on, see the published articles [4, 30]. This subject has attracted much attention from scholars during the last few decades. There have been many studies on the Hermite–Hadamard-type inequalities considering different types of fractional integral operators. For instance, refer to Hadamard fractional integrals [3] and to conformable fractional integrals [1] and so on. For more development about this subject, the interested readers may refer to [19, 25, 11, 5] and references therein.

Inspired by the reports mentioned above, in particular, the result displayed in [36], the present study aims to investigate some parameterized inequalities of Hermite–Hadamard type, which involves the tempered fractional integrals and the introduced (μ,η)-incomplete gamma functions. To achieve this purpose, on the basis of the discovered integral identity in the paper, we consider the following three cases: (i) the derivative of the considered mapping satisfies the Lipschitz condition; (ii) the derivative of the considered mapping is bounded; (iii) the derivative of the considered mapping is generalized (s,m)-preinvex. The obtained results here can be reduced to the Riemann–Liouville fractional integral inequalities for $\mu=0$ and the Riemann integral inequalities for $\alpha=1$ with $\mu=0$.

2. Main results

Throughout this work, we assume that $\mathscr{K}\subseteq\mathbb{R}$ is an open m-invex set regarding $\eta:\mathscr{K}\times\mathscr{K}\times(0,1]\to\mathbb{R}$ for certain fixed $m\in(0,1]$, $a,b\in\mathscr{K}$ with $ma<ma+\eta(b,a,m)$, and $f:\mathscr{K}\to\mathbb{R}$ is a differentiable mapping satisfying that f' is integrable on η_m-path $P_{\tau_1\tau_3}:\theta=m\tau_1+\lambda\eta(\tau_2,\tau_1,m)$ for all $\tau_1,\tau_2\in[a,b]$.

2.1. A new definition and a lemma

We now define the following (μ,η)-incomplete gamma function, which is the extension of μ-incomplete gamma function and incomplete gamma function.

DEFINITION 2.1. Let the mapping $\eta:\mathscr{K}\times\mathscr{K}\times(0,1]\to\mathbb{R}$, where $\mathscr{K}\subseteq\mathbb{R}$ be an open m-invex subset with some fixed $m\in(0,1]$. For any real numbers $\alpha>0$ along with $\lambda,\mu\geqslant 0$, the (μ,η)-incomplete gamma function is defined by

$$\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda)=\int_0^{\lambda}x^{\alpha-1}e^{-\mu\eta(b,a,m)x}\mathrm{d}x.$$

Clearly, if the mapping $\eta(b,a,m)=1$, then the definition of (μ,η)-incomplete gamma function reduces to the definition of μ-incomplete gamma function. In particular, if we take $\mu=1$, then it transfers to the incomplete gamma function.

REMARK 2.1. For the real numbers $\alpha>0$ and $\lambda,\mu\geqslant 0$, we have the following results.

(1) $\gamma_{\mu\eta(b,a,m)}(\alpha,1)=\displaystyle\int_0^1 u^{\alpha-1}e^{-\mu\eta(b,a,m)u}\mathrm{d}u=\frac{1}{\eta^{\alpha}(b,a,m)}\gamma_{\mu}\Big(\alpha,\eta(b,a,m)\Big).$

(2) $\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda)\mathrm{d}u = \frac{\gamma_\mu\left(\alpha,\eta(b,a,m)\right)}{\eta^\alpha(b,a,m)} - \frac{\gamma_\mu\left(\alpha+1,\eta(b,a,m)\right)}{\eta^{\alpha+1}(b,a,m)}.$

Proof. (1) If we change the variable $t=\eta(b,a,m)u$, then the first item's proof yields from Definition 2.1.

(2) In view of Definition 2.1, we have

$$\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda)\mathrm{d}u = \int_0^1\int_0^\lambda t^{\alpha-1}e^{-\mu\eta(b,a,m)t}\mathrm{d}t\mathrm{d}u.$$

By changing the order of the integration, we deduce

$$\begin{aligned}\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda)\mathrm{d}u &= \int_0^1\int_t^1 \left(t^{\alpha-1}e^{-\mu\eta(b,a,m)t}\mathrm{d}u\right)\mathrm{d}t\\ &= \int_0^1 (1-t)t^{\alpha-1}e^{-\mu\eta(b,a,m)t}\mathrm{d}t\\ &= \int_0^1 t^{\alpha-1}e^{-\mu\eta(b,a,m)t}\mathrm{d}t - \int_0^1 t^{\alpha}e^{-\mu\eta(b,a,m)t}\mathrm{d}t.\end{aligned}$$

In the light of Remark 2.1 (1), we deduce

$$\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda)\mathrm{d}u = \frac{\gamma_\mu\left(\alpha,\eta(b,a,m)\right)}{\eta^\alpha(b,a,m)} - \frac{\gamma_\mu\left(\alpha+1,\eta(b,a,m)\right)}{\eta^{\alpha+1}(b,a,m)}. \qquad \square$$

LEMMA 2.1. *The following tempered fractional integral equality with* $0<\lambda\leqslant 1$, $\alpha>0$, $\mu\geqslant 0$ *holds:*

$$\begin{aligned}&\mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b)\\ &\quad= \frac{\eta(b,a,m)}{2}\left[\int_0^{\frac{\lambda}{2}} \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2t)f'\left(ma+t\eta(b,a,m)\right)\mathrm{d}t\right.\\ &\qquad\left. -\int_{1-\frac{\lambda}{2}}^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2+2t)f'\left(ma+t\eta(b,a,m)\right)\mathrm{d}t\right],\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}&\mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b)\\ &:= \frac{2^{\alpha-1}\Gamma(\alpha)}{\eta^\alpha(b,a,m)}\left[\mathscr{J}_{ma^+}^{(\alpha,2\mu)}f\left(ma+\frac{\lambda}{2}\eta(b,a,m)\right)\right.\\ &\quad\left.+\mathscr{J}_{(ma+\eta(b,a,m))^-}^{(\alpha,2\mu)}f\left(ma+\left(1-\frac{\lambda}{2}\right)\eta(b,a,m)\right)\right]\\ &\quad-\frac{1}{2}\left[\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda)f(ma)+\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda)f\left(ma+\eta(b,a,m)\right)\right].\end{aligned}$$

Proof. Integrating by parts, it yields that

$$
\begin{aligned}
I_1 &= \int_0^{\frac{\lambda}{2}} \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2t) f'\big(ma+t\eta(b,a,m)\big)\mathrm{d}t \\
&= \frac{1}{\eta(b,a,m)}\Bigg[\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2t) f\big(ma+t\eta(b,a,m)\big)\Big|_0^{\frac{\lambda}{2}} \\
&\quad + 2\int_0^{\frac{\lambda}{2}} (\lambda-2t)^{\alpha-1} e^{-\mu\eta(b,a,m)(\lambda-2t)} f\big(ma+t\eta(b,a,m)\big)\mathrm{d}t\Bigg] \\
&= \frac{1}{\eta(b,a,m)}\Bigg[-\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda) f(ma) + \frac{2^\alpha}{\eta^\alpha(b,a,m)} \\
&\quad \times \int_{ma}^{ma+\frac{\lambda}{2}\eta(b,a,m)} \left[ma+\frac{\lambda}{2}\eta(b,a,m)-u\right]^{\alpha-1} e^{-2\mu\left[ma+\frac{\lambda}{2}\eta(b,a,m)-u\right]} f(u)\mathrm{d}u\Bigg] \\
&= -\frac{1}{\eta(b,a,m)}\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda) f(ma) + \frac{2^\alpha\Gamma(\alpha)}{\eta^{\alpha+1}(b,a,m)} \mathscr{I}_{ma^+}^{(\alpha,2\mu)} f\left(ma+\frac{\lambda}{2}\eta(b,a,m)\right).
\end{aligned} \tag{2.2}
$$

Similarly,

$$
\begin{aligned}
I_2 &= \int_{1-\frac{\lambda}{2}}^{1} \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2+2t) f'\big(ma+t\eta(b,a,m)\big)\mathrm{d}t \\
&= \frac{1}{\eta(b,a,m)}\Bigg[\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2+2t) f\big(ma+t\eta(b,a,m)\big)\Big|_{1-\frac{\lambda}{2}}^{1} \\
&\quad - 2\int_{1-\frac{\lambda}{2}}^{1} (\lambda-2+2t)^{\alpha-1} e^{-\mu\eta(b,a,m)(\lambda-2+2t)} f\big(ma+t\eta(b,a,m)\big)\mathrm{d}t\Bigg] \\
&= \frac{1}{\eta(b,a,m)}\Bigg[\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda) f\big(ma+\eta(b,a,m)\big) \\
&\quad - \frac{2^\alpha}{\eta^\alpha(b,a,m)} \int_{ma+(1-\frac{\lambda}{2})\eta(b,a,m)}^{ma+\eta(b,a,m)} \left[u-\left(ma+\left(1-\frac{\lambda}{2}\right)\eta(b,a,m)\right)\right]^{\alpha-1} \\
&\quad \times e^{-2\mu\left[u-\left(ma+\left(1-\frac{\lambda}{2}\right)\eta(b,a,m)\right)\right]} f(u)\mathrm{d}u\Bigg] \\
&= \frac{1}{\eta(b,a,m)}\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda) f\big(ma+\eta(b,a,m)\big) \\
&\quad - \frac{2^\alpha\Gamma(\alpha)}{\eta^{\alpha+1}(b,a,m)} \mathscr{I}_{(ma+\eta(b,a,m))^-}^{(\alpha,2\mu)} f\left(ma+\left(1-\frac{\lambda}{2}\right)\eta(b,a,m)\right).
\end{aligned} \tag{2.3}
$$

If we subtract (2.3) from (2.2) and multiply by $\frac{\eta(b,a,m)}{2}$, then we obtain the required equality in (2.1). This ends the proof. □

COROLLARY 2.1. *If we consider*

$$f(ma)=f\left(ma+\frac{\lambda}{2}\eta(b,a,m)\right)=f\left(ma+\left(1-\frac{\lambda}{2}\right)\eta(b,a,m)\right)=f\left(ma+\eta(b,a,m)\right)$$

with $\lambda=1$ *in Lemma* 2.1*, then one has*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha)}{\eta^{\alpha}(b,a,m)}\left[\mathscr{J}_{ma^+}^{(\alpha,2\mu)}f\left(ma+\frac{1}{2}\eta(b,a,m)\right)+\mathscr{J}_{(ma+\eta(b,a,m))^-}^{(\alpha,2\mu)}f\left(ma+\frac{1}{2}\eta(b,a,m)\right)\right]\\&\quad-\frac{1}{2}\left[\gamma_{\mu\eta(b,a,m)}(\alpha,1)f\left(ma+\frac{1}{2}\eta(b,a,m)\right)+\gamma_{\mu\eta(b,a,m)}(\alpha,1)f\left(ma+\frac{1}{2}\eta(b,a,m)\right)\right]\\&=\frac{\eta(b,a,m)}{2}\left[\int_0^{\frac{1}{2}}\gamma_{\mu\eta(b,a,m)}(\alpha,1-2t)f'\left(ma+t\eta(b,a,m)\right)\mathrm{d}t\right.\\&\quad\left.-\int_{\frac{1}{2}}^{1}\gamma_{\mu\eta(b,a,m)}(\alpha,2t-1)f'\left(ma+t\eta(b,a,m)\right)\mathrm{d}t\right].\end{aligned}$$

COROLLARY 2.2. *If we take* $\mu=0$ *in Lemma* 2.1*, then the identity* (2.1) *reduces to*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^{\alpha}(b,a,m)}\left[\mathscr{J}_{ma^+}^{\alpha}f\left(ma+\frac{\lambda}{2}\eta(b,a,m)\right)\right.\\&\quad\left.+\mathscr{J}_{(ma+\eta(b,a,m))^-}^{\alpha}f\left(ma+\left(1-\frac{\lambda}{2}\right)\eta(b,a,m)\right)\right]\\&\quad-\frac{1}{2}\left[\lambda^{\alpha}f\left(ma+\eta(b,a,m)\right)+\lambda^{\alpha}f(ma)\right]\\&=\frac{\eta(b,a,m)}{2}\left[\int_0^{\frac{\lambda}{2}}(\lambda-2t)^{\alpha}f'\left(ma+t\eta(b,a,m)\right)\mathrm{d}t\right.\\&\quad\left.-\int_{1-\frac{\lambda}{2}}^{1}(\lambda-2+2t)^{\alpha}f'\left(ma+t\eta(b,a,m)\right)\mathrm{d}t\right].\end{aligned}$$

COROLLARY 2.3. *If we take* $\lambda=1$ *in Lemma* 2.1*, then the identity* (2.1) *can be written as*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha)}{\eta(b,a,m)^{\alpha}\gamma_{\mu\eta(b,a,m)}(\alpha,1)}\left[\mathscr{J}_{ma^+}^{(\alpha,2\mu)}f\left(ma+\frac{1}{2}\eta(b,a,m)\right)\right.\\&\quad\left.+\mathscr{J}_{(ma+\eta(b,a,m))^-}^{(\alpha,2\mu)}f\left(ma+\frac{1}{2}\eta(b,a,m)\right)\right]-\frac{f(ma)+f(ma+\eta(b,a,m))}{2}\\&=\frac{\eta(b,a,m)}{2\gamma_{\mu\eta(b,a,m)}(\alpha,1)}\left[\int_0^{\frac{1}{2}}\gamma_{\mu\eta(b,a,m)}(\alpha,1-2t)f'\left(ma+t\eta(b,a,m)\right)\mathrm{d}t\right.\\&\quad\left.-\int_{\frac{1}{2}}^{1}\gamma_{\mu\eta(b,a,m)}(\alpha,2t-1)f'\left(ma+t\eta(b,a,m)\right)\mathrm{d}t\right].\end{aligned}$$

COROLLARY 2.4. *If we take* $\lambda = 1$ *and* $\eta(b,a,m) = b - ma$ *with* $m = 1$ *in Lemma 2.1, then the identity (2.1) transfers to*

$$\frac{2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}\gamma_{\mu(b-a)}(\alpha,1)}\left[\mathscr{J}_{a^+}^{(\alpha,2\mu)}f\left(\frac{a+b}{2}\right)+\mathscr{J}_{b^-}^{(\alpha,2\mu)}f\left(\frac{a+b}{2}\right)\right]-\frac{f(a)+f(b)}{2}$$
$$=\frac{b-a}{2\gamma_{\mu(b-a)}(\alpha,1)}\left[\int_0^{\frac{1}{2}}\gamma_{\mu(b-a)}(\alpha,1-2t)f'\big(a+t(b-a)\big)\mathrm{d}t\right.$$
$$\left.-\int_{\frac{1}{2}}^{1}\gamma_{\mu(b-a)}(\alpha,2t-1)f'\big(a+t(b-a)\big)\mathrm{d}t\right].$$

In particular, if we take $\alpha = 1$ *and* $\mu = 0$*, then we get*

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{f(a)+f(b)}{2}=\frac{b-a}{2}\int_0^1(1-2t)f'\big(a+t(b-a)\big)\mathrm{d}t,$$

which is proved by Dragomir and Agarwal in [20] to establish several Hermite–Hadamard's integral inequalities for convex mappings.

2.2. f' is a Lipschitzian function

Applying Lemma 2.1, we obtain the tempered fractional integral inequality in the case that first derivative of defined mapping is Lipschitzian.

THEOREM 2.1. *Let* $f : [ma, ma+\eta(b,a,m)] \to \mathbb{R}$ *be a differentiable function. Assume that* f' *is integrable on* $[ma, ma+\eta(b,a,m)]$ *and satisfies a Lipschitz condition for some* $L > 0$*. Then we have*

$$\left|\mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b)\right| \leqslant \frac{L\eta^2(b,a,m)}{2}\cdot\Delta_1(\alpha,\lambda,\mu), \tag{2.4}$$

where,

$$\Delta_1(\alpha,\lambda,\mu)=\frac{\lambda}{2}\left[\frac{\gamma_{\mu}\big(\alpha,\lambda\eta(b,a,m)\big)}{\eta^{\alpha}(b,a,m)}-\frac{\gamma_{\mu}\big(\alpha+1,\lambda\eta(b,a,m)\big)}{\lambda\eta^{\alpha+1}(b,a,m)}\right].$$

Proof. It yields from Lemma 2.1 that

$$\mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b)$$
$$=\frac{\lambda\eta(b,a,m)}{4}\left\{\int_0^1\gamma_{\mu\eta(b,a,m)}\big(\alpha,\lambda(1-t)\big)\left[f'\left(ma+\frac{\lambda}{2}t\eta(b,a,m)\right)-f'(ma)+f'(ma)\right]\mathrm{d}t\right.$$
$$\left.-\int_0^1\gamma_{\mu\eta(b,a,m)}\big(\alpha,\lambda(1-t)\big)\left[f'\left(ma+\left(1-\frac{\lambda}{2}t\right)\eta(b,a,m)\right)-f'(ma)+f'(ma)\right]\mathrm{d}t\right\}$$
$$=\frac{\lambda\eta(b,a,m)}{4}\left\{\int_0^1\gamma_{\mu\eta(b,a,m)}\big(\alpha,\lambda(1-t)\big)\left[f'\left(ma+\frac{\lambda}{2}t\eta(b,a,m)\right)-f'(ma)\right]\mathrm{d}t\right.$$
$$\left.-\int_0^1\gamma_{\mu\eta(b,a,m)}\big(\alpha,\lambda(1-t)\big)\left[f'\left(ma+\left(1-\frac{\lambda}{2}t\right)\eta(b,a,m)\right)-f'(ma)\right]\mathrm{d}t\right\}.$$

Since f' satisfies Lipschitz conditions for some $L>0$, we have

$$\left|f'\left(ma+\frac{\lambda}{2}t\eta(b,a,m)\right)-f'(ma)\right|\leqslant L\left|ma+\frac{\lambda}{2}t\eta(b,a,m)-ma\right| \\ = L\frac{\lambda}{2}t\eta(b,a,m),$$

and

$$\left|f'\left(ma+\left(1-\frac{\lambda}{2}t\right)\eta(b,a,m)\right)-f'(ma)\right|\leqslant L\left|ma+\left(1-\frac{\lambda}{2}t\right)\eta(b,a,m)-ma\right| \\ = L\left(1-\frac{\lambda}{2}t\right)\eta(b,a,m).$$

Hence,

$$\begin{aligned}&\left|\mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b)\right| \\ &\leqslant \frac{\lambda\eta(b,a,m)}{4}\left[\int_0^1\left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\right|\left|f'\left(ma+\frac{\lambda}{2}t\eta(b,a,m)\right)-f'(ma)\right|\mathrm{d}t\right. \\ &\quad\left.+\int_0^1\left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\right|\left|f'\left(ma+\left(1-\frac{\lambda}{2}t\right)\eta(b,a,m)\right)-f'(ma)\right|\mathrm{d}t\right] \\ &\leqslant \frac{L\eta^2(b,a,m)}{2}\left(\frac{\lambda}{2}\int_0^1\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t\right).\end{aligned}$$

Changing the order of the integration, we have

$$\begin{aligned}\Delta_1(\alpha,\lambda,\mu) &:= \frac{\lambda}{2}\int_0^1\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t \\ &= \frac{\lambda}{2}\int_0^1\left(\int_0^{\lambda(1-t)}y^{\alpha-1}e^{-\mu\eta(b,a,m)y}\mathrm{d}y\right)\mathrm{d}t \\ &= \frac{\lambda}{2}\int_0^1\left(\int_t^1\lambda^{\alpha}(1-u)^{\alpha-1}e^{-\mu\eta(b,a,m)\lambda(1-u)}\mathrm{d}u\right)\mathrm{d}t \\ &= \frac{\lambda^{\alpha+1}}{2}\int_0^1\left(\int_0^u(1-u)^{\alpha-1}e^{-\mu\eta(b,a,m)\lambda(1-u)}\mathrm{d}t\right)\mathrm{d}u \\ &= \frac{\lambda^{\alpha+1}}{2}\int_0^1u(1-u)^{\alpha-1}e^{-\mu\eta(b,a,m)\lambda(1-u)}\mathrm{d}u \\ &= \frac{\lambda^{\alpha+1}}{2}\left[\int_0^1u^{\alpha-1}e^{-\lambda\mu\eta(b,a,m)u}\mathrm{d}u-\int_0^1u^{\alpha}e^{-\lambda\mu\eta(b,a,m)u}\mathrm{d}u\right] \\ &= \frac{\lambda}{2}\left[\frac{\gamma_\mu(\alpha,\lambda\eta(b,a,m))}{\eta^{\alpha}(b,a,m)}-\frac{\gamma_\mu(\alpha+1,\lambda\eta(b,a,m))}{\lambda\eta^{\alpha+1}(b,a,m)}\right].\end{aligned}$$

This finishes the proof. □

COROLLARY 2.5. *If we take* $\mu = 0$ *in Theorem* 2.1*, then we have*

$$\begin{aligned}&\Bigg|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^{\alpha}(b,a,m)}\left[\mathscr{I}^{\alpha}_{ma^{+}}f\left(ma+\frac{\lambda}{2}\eta(b,a,m)\right)\right.\\&\quad\left.+\mathscr{I}^{\alpha}_{(ma+\eta(b,a,m))^{-}}f\left(ma+\left(1-\frac{\lambda}{2}\right)\eta(b,a,m)\right)\right]\\&\quad-\frac{1}{2}\left[\lambda^{\alpha}f(ma)+\lambda^{\alpha}f\big(ma+\eta(b,a,m)\big)\right]\Bigg|\\&\leqslant\frac{\lambda^{\alpha+1}L\eta^{2}(b,a,m)}{4(\alpha+1)},\end{aligned}$$

specially for $\alpha = 1$ *and* $\lambda = 1$*, we get*

$$\begin{aligned}&\left|\frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)}f(t)\mathrm{d}t-\frac{f(ma)+f\big(ma+\eta(b,a,m)\big)}{2}\right|\\&\leqslant\frac{L\eta^{2}(b,a,m)}{8}.\end{aligned}\tag{2.5}$$

COROLLARY 2.6. *Suppose that all the hypothesis of Corollary* 2.5 *are satisfied, we have*

$$\left|\frac{1}{b-a}\int_{a}^{b}f(t)\mathrm{d}t-\frac{f(a)+f(b)}{2}\right|\leqslant\frac{L(b-a)^{2}}{8}.$$

Proof. If we take $\eta(b,a,m) = b - ma$ with $m = 1$ in inequality (2.5), then we have the desired inequality. □

2.3. f' is a bounded function

Our next goal is to obtain the error bounds of the tempered fractional integral inequality when the derivative of the considered functions f' are bounded.

THEOREM 2.2. *If there exist constants* $\gamma < \Upsilon$ *satisfying that* $-\infty < \gamma \leqslant f'(x) \leqslant \Upsilon < +\infty$ *for all* $x \in [ma, ma+\eta(b,a,m)]$*, then the following inequality for the tempered fractional integral operators with* $0 < \lambda \leqslant 1$*,* $\alpha > 0$ *and* $\mu \geqslant 0$ *holds:*

$$\left|\mathscr{T}_{f}(\alpha,\mu,\lambda;\eta,m,a,b)\right|\leqslant\frac{\eta(b,a,m)(\Upsilon-\gamma)}{2}\cdot\Delta_{1}(\alpha,\lambda,\mu),\tag{2.6}$$

where $\Delta_1(\alpha,\lambda,\mu)$ *is defined as in Theorem* 2.1.

Proof. Utilizing Lemma 2.1, we have

$$\begin{aligned}
&\left|\mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b)\right| \\
&= \frac{\eta(b,a,m)}{2}\left|\frac{\lambda}{2}\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))f'\left(ma+\frac{\lambda}{2}t\eta(b,a,m)\right)\mathrm{d}t\right. \\
&\quad \left.-\frac{\lambda}{2}\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))f'\left(ma+\left(1-\frac{\lambda}{2}t\right)\eta(b,a,m)\right)\mathrm{d}t\right| \\
&= \left|\frac{\lambda\eta(b,a,m)}{4}\left[\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\left[f'\left(ma+\frac{\lambda}{2}t\eta(b,a,m)\right)-\frac{\gamma+\Upsilon}{2}+\frac{\gamma+\Upsilon}{2}\right]\mathrm{d}t\right.\right. \\
&\quad \left.\left.-\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\left[f'\left(ma+\left(1-\frac{\lambda}{2}t\right)\eta(b,a,m)\right)-\frac{\gamma+\Upsilon}{2}+\frac{\gamma+\Upsilon}{2}\right]\mathrm{d}t\right]\right| \\
&= \left|\frac{\lambda\eta(b,a,m)}{4}\left[\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\left[f'\left(ma+\frac{\lambda}{2}t\eta(b,a,m)\right)-\frac{\gamma+\Upsilon}{2}\right]\mathrm{d}t\right.\right. \\
&\quad \left.\left.-\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\left[f'\left(ma+\left(1-\frac{\lambda}{2}t\right)\eta(b,a,m)\right)-\frac{\gamma+\Upsilon}{2}\right]\mathrm{d}t\right]\right| \\
&\leqslant \frac{\lambda\eta(b,a,m)}{4}\left[\int_0^1 \left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\right|\left|f'\left(ma+\frac{\lambda}{2}t\eta(b,a,m)\right)-\frac{\gamma+\Upsilon}{2}\right|\mathrm{d}t\right. \\
&\quad \left.+\int_0^1 \left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\right|\left|f'\left(ma+\left(1-\frac{\lambda}{2}t\right)\eta(b,a,m)\right)-\frac{\gamma+\Upsilon}{2}\right|\mathrm{d}t\right] \\
&\leqslant \frac{\eta(b,a,m)(\Upsilon-\gamma)}{2}\left(\frac{\lambda}{2}\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t\right) \\
&= \frac{\eta(b,a,m)(\Upsilon-\gamma)}{2}\cdot\Delta_1(\alpha,\lambda,\mu).
\end{aligned}$$

This finishes the proof. □

COROLLARY 2.7. *If we take $\mu=0$ in Theorem 2.2, then we have*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^{\alpha}(b,a,m)}\left[\mathscr{J}^{\alpha}_{ma^+}f\left(ma+\frac{\lambda}{2}\eta(b,a,m)\right)\right.\right. \\
&\qquad \left.+\mathscr{J}^{\alpha}_{(ma+\eta(b,a,m))^-}f\left(ma+\left(1-\frac{\lambda}{2}\right)\eta(b,a,m)\right)\right] \\
&\qquad \left.-\frac{1}{2}\left[\lambda^{\alpha}f(ma)+\lambda^{\alpha}f(ma+\eta(b,a,m))\right]\right| \\
&\leqslant \frac{\lambda^{\alpha+1}\eta(b,a,m)(\Upsilon-\gamma)}{4\alpha(\alpha+1)},
\end{aligned}$$

specially for $\alpha = 1$ *and* $\lambda = 1$*, we get*

$$\left| \frac{1}{\eta(b,a,m)} \int_{ma}^{ma+\eta(b,a,m)} f(t)\mathrm{d}t - \frac{f(ma) + f\big(ma + \eta(b,a,m)\big)}{2} \right| \leqslant \frac{\eta(b,a,m)(\Upsilon - \gamma)}{8}. \tag{2.7}$$

COROLLARY 2.8. *Suppose that all the hypothesis of Corollary 2.7 are satisfied, we have*

$$\left| \frac{1}{b-a} \int_{a}^{b} f(t)\mathrm{d}t - \frac{f(a) + f(b)}{2} \right| \leqslant \frac{(b-a)(\Upsilon - \gamma)}{8}.$$

Proof. If we take $\eta(b,a,m) = b - ma$ with $m = 1$ in inequality (2.7), then we have the required inequality. □

For deducing the following results, we consider another case of boundedness of f'.

THEOREM 2.3. *Under all assumptions of Lemma 2.1, suppose that* f' *is bounded on* $(ma, ma+\eta(b,a,m))$*, i.e.,* $||f'||_\infty = \sup|f'(x)| < \infty$*. Then the following inequality for the tempered fractional integral operators with* $0 < \lambda \leqslant 1$*,* $\alpha > 0$ *and* $\mu \geqslant 0$ *holds:*

$$\left| \mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b) \right| \leqslant \eta(b,a,m)||f'||_\infty \cdot \Delta_1(\alpha,\lambda,\mu), \tag{2.8}$$

where $\Delta_1(\alpha,\lambda,\mu)$ *is defined as in Theorem 2.1.*

Proof. Utilizing Lemma 2.1, we have

$$\begin{aligned}
&\left| \mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b) \right| \\
&= \frac{\eta(b,a,m)}{2} \left\{ \left| \frac{\lambda}{2} \int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha, \lambda(1-t)) f'\left(ma + \frac{\lambda}{2} t \eta(b,a,m) \right) \mathrm{d}t \right.\right. \\
&\qquad \left.\left. - \frac{\lambda}{2} \int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha, \lambda(1-t)) f'\left(ma + \left(1 - \frac{\lambda}{2} t\right) \eta(b,a,m) \right) \mathrm{d}t \right| \right\} \\
&\leqslant \frac{\eta(b,a,m)}{2} \left\{ \frac{\lambda}{2} \int_0^1 \left| \gamma_{\mu\eta(b,a,m)}(\alpha, \lambda(1-t)) \right| \left| f'\left(ma + \frac{\lambda}{2} t \eta(b,a,m) \right) \right| \mathrm{d}t \right. \\
&\qquad \left. + \frac{\lambda}{2} \int_0^1 \left| \gamma_{\mu\eta(b,a,m)}(\alpha, \lambda(1-t)) \right| \left| f'\left(ma + \left(1 - \frac{\lambda}{2} t\right) \eta(b,a,m) \right) \right| \mathrm{d}t \right\} \\
&\leqslant \eta(b,a,m)||f'||_\infty \left(\frac{\lambda}{2} \int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha, \lambda(1-t)) \mathrm{d}t \right) \\
&= \eta(b,a,m)||f'||_\infty \cdot \Delta_1(\alpha,\lambda,\mu).
\end{aligned}$$

This finishes the proof. □

COROLLARY 2.9. *If we take* $\mu = 0$ *in Theorem* 2.3*, then we have*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^{\alpha}(b,a,m)}\left[\mathscr{J}^{\alpha}_{ma^+}f\left(ma+\frac{\lambda}{2}\eta(b,a,m)\right)\right.\right.$$
$$\left.+\mathscr{J}^{\alpha}_{(ma+\eta(b,a,m))^-}f\left(ma+\left(1-\frac{\lambda}{2}\right)\eta(b,a,m)\right)\right]$$
$$\left.-\frac{1}{2}\left[\lambda^{\alpha}f(ma)+\lambda^{\alpha}f\big(ma+\eta(b,a,m)\big)\right]\right|$$
$$\leqslant \frac{\lambda^{\alpha+1}\eta(b,a,m)||f'||_{\infty}}{2(\alpha+1)},$$

specially for $\alpha = 1$ *and* $\lambda = 1$*, we get*

$$\left|\frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)} f(t)\mathrm{d}t - \frac{f(ma)+f\big(ma+\eta(b,a,m)\big)}{2}\right| \leqslant \frac{\eta(b,a,m)||f'||_{\infty}}{4}. \tag{2.9}$$

COROLLARY 2.10. *Suppose that all the hypothesis of Corollary* 2.9 *are satisfied, we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{f(a)+f(b)}{2}\right| \leqslant \frac{(b-a)||f'||_{\infty}}{4}.$$

Proof. If we take $\eta(b,a,m) = b - ma$ with $m = 1$ in inequality (2.9), then we have the desired inequality. □

2.4. $|f'|$ and $|f'|^q$ are generalized (s,m)-preinvex

Considering the mappings whose the absolute values of the first derivative are generalized (s,m)-preinvex, we are capable of establishing certain tempered fractional integral inequalities with regard to such kind of mapping.

THEOREM 2.4. *If* $|f'|$ *is generalized* (s,m)*-preinvex with certain fixed* $s,m \in (0,1]$*, then the following inequality for the tempered fractional integral operators with* $0 < \lambda \leqslant 1$*,* $\alpha > 0$ *and* $\mu \geqslant 0$ *holds:*

$$\left|\mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b)\right| \leqslant \frac{m|f'(a)|+|f'(b)|}{2^s}\eta(b,a,m)\cdot\Delta_1(\alpha,\lambda,\mu), \tag{2.10}$$

where $\Delta_1(\alpha,\lambda,\mu)$ *is defined as in Theorem* 2.1.

Proof. In terms of Lemma 2.1 and the generalized (s,m)-preinvexity of $|f'|$, we have

$$
\begin{aligned}
&\left|\mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b)\right|\\
&\leqslant \frac{\eta(b,a,m)}{2}\left[\int_0^{\frac{\lambda}{2}}\left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2t)\right|\left|f'\left(ma+t\eta(b,a,m)\right)\right|\mathrm{d}t\right.\\
&\quad\left.+\int_{1-\frac{\lambda}{2}}^{1}\left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2+2t)\right|\left|f'\left(ma+t\eta(b,a,m)\right)\right|\mathrm{d}t\right]\\
&\leqslant \frac{\eta(b,a,m)}{2}\left[\int_0^{\frac{\lambda}{2}}\left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2t)\right|\left[m(1-t)^s\left|f'(a)\right|+t^s\left|f'(b)\right|\right]\mathrm{d}t\right.\\
&\quad\left.+\int_{1-\frac{\lambda}{2}}^{1}\left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2+2t)\right|\left[m(1-t)^s\left|f'(a)\right|+t^s\left|f'(b)\right|\right]\mathrm{d}t\right]\\
&=\frac{\eta(b,a,m)}{2}\left[m\left|f'(a)\right|\left(\int_0^{\frac{\lambda}{2}}(1-t)^s\left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2t)\right|\mathrm{d}t\right.\right.\\
&\quad\left.+\int_{1-\frac{\lambda}{2}}^{1}(1-t)^s\left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2+2t)\right|\mathrm{d}t\right)\\
&\quad\left.+\left|f'(b)\right|\left(\int_0^{\frac{\lambda}{2}}t^s\left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2t)\right|\mathrm{d}t+\int_{1-\frac{\lambda}{2}}^{1}t^s\left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2+2t)\right|\mathrm{d}t\right)\right]\\
&=\frac{\eta(b,a,m)}{2}\left(m\left|f'(a)\right|+\left|f'(b)\right|\right)\int_0^{\frac{\lambda}{2}}\left(t^s+(1-t)^s\right)\left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2t)\right|\mathrm{d}t\\
&\leqslant \frac{\eta(b,a,m)}{2^s}\left(m\left|f'(a)\right|+\left|f'(b)\right|\right)\int_0^{\frac{\lambda}{2}}\left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2t)\right|\mathrm{d}t.
\end{aligned}
$$

To prove the last inequality above, we use the fact

$$t^s+(1-t)^s\leqslant 2^{1-s},$$

for all $t\in[0,1]$ and $s\in[0,1]$.

Also, by means of the change of the variable $u=\frac{2}{\lambda}t$, we have

$$
\begin{aligned}
&\int_0^{\frac{\lambda}{2}}\left|\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2t)\right|\mathrm{d}t\\
&=\frac{\lambda}{2}\int_0^1\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t\\
&:=\Delta_1(\alpha,\lambda,\mu).
\end{aligned}
$$

Thus, the proof is completed. □

COROLLARY 2.11. *If we take* $\lambda = 1$ *in Theorem 2.4, then we have*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{\eta(b,a,m)^{\alpha}\gamma_{\eta(b,a,m)}(\alpha,1)}\left[\mathscr{J}^{(\alpha,2\mu)}_{ma^{+}}f\left(ma+\frac{1}{2}\eta(b,a,m)\right)\right.\right.\\&\qquad\left.\left.+\mathscr{J}^{(\alpha,2\mu)}_{(ma+\eta(b,a,m))^{-}}f\left(ma+\frac{1}{2}\eta(b,a,m)\right)\right]-\frac{f(ma)+f(ma+\eta(b,a,m))}{2}\right|\\&\quad\leqslant\frac{\eta(b,a,m)}{2^{s+1}\gamma_{\eta(b,a,m)}(\alpha,1)}\left[\frac{\gamma_{\mu}\big(\alpha,\eta(b,a,m)\big)}{\eta^{\alpha}(b,a,m)}-\frac{\gamma_{\mu}\big(\alpha+1,\eta(b,a,m)\big)}{\eta^{\alpha+1}(b,a,m)}\right]\\&\quad\times\Big[m\big|f'(a)\big|+\big|f'(b)\big|\Big],\end{aligned}$$

specially for $\eta(b,a,m) = b - ma$ *with* $m = 1$*, we get*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}\gamma_{\mu(b-a)}(\alpha,1)}\left[\mathscr{J}^{(\alpha,2\mu)}_{a^{+}}f\left(\frac{a+b}{2}\right)+\mathscr{J}^{(\alpha,2\mu)}_{b^{-}}f\left(\frac{a+b}{2}\right)\right]-\frac{f(a)+f(b)}{2}\right|\\&\quad\leqslant\frac{b-a}{2^{s+1}\gamma_{(b-a)}(\alpha,1)}\left[\frac{\gamma_{\mu}\big(\alpha,b-a\big)}{(b-a)^{\alpha}}-\frac{\gamma_{\mu}\big(\alpha+1,b-a\big)}{(b-a)^{\alpha+1}}\right]\Big[\big|f'(a)\big|+\big|f'(b)\big|\Big].\end{aligned}$$

COROLLARY 2.12. *If we take* $\mu = 0$ *in Theorem 2.4, then following inequality for Riemann–Liouville fractional integrals holds:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^{\alpha}(b,a,m)}\left[\mathscr{J}^{\alpha}_{ma^{+}}f\left(ma+\frac{\lambda}{2}\eta(b,a,m)\right)\right.\right.\\&\qquad\left.+\mathscr{J}^{\alpha}_{(ma+\eta(b,a,m))^{-}}f\left(ma+\left(1-\frac{\lambda}{2}\right)\eta(b,a,m)\right)\right]\\&\qquad\left.-\frac{1}{2}\Big[\lambda^{\alpha}f(ma)+\lambda^{\alpha}f\big(ma+\eta(b,a,m)\big)\Big]\right|\\&\quad\leqslant\frac{\lambda^{\alpha+1}\eta(b,a,m)}{\alpha(\alpha+1)2^{s+1}}\Big[m\big|f'(a)\big|+\big|f'(b)\big|\Big],\end{aligned}$$

specially for $\alpha = 1 = \lambda$*, if the mapping* $\eta(b,a,m)$ *with* $m = 1$ *degenerates into* $\eta(b,a)$ *and we choose* $s = 1$*, then we have*

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(t)\mathrm{d}t-\frac{\big[f(a)+f\big(a+\eta(b,a)\big)\big]}{2}\right|\\&\quad\leqslant\frac{\eta(b,a)}{8}\Big[\big|f'(a)\big|+\big|f'(b)\big|\Big].\end{aligned}$$

This is one of the inequalities established in [8, Theorem 2.1].

THEOREM 2.5. *If $|f'|^2$ is generalized (s,m)-preinvex with certain fixed $s,m \in (0,1]$, then the following inequality for the tempered fractional integral operators with $0<\lambda \leqslant 1$, $\alpha>0$ and $\mu \geqslant 0$ holds:*

$$\begin{aligned} &\left|\mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b)\right| \\ &\leqslant \frac{\eta(b,a,m)}{2}\cdot \mathscr{L}^{\frac{1}{2}}(\alpha,\lambda,\mu)\left\{\left(\frac{m}{s+1}\left[1-\left(1-\frac{\lambda}{2}\right)^{s+1}\right]\left|f'(a)\right|^2\right.\right. \\ &\quad \left. + \frac{1}{s+1}\left(\frac{\lambda}{2}\right)^{s+1}\left|f'(b)\right|^2\right)^{\frac{1}{2}} \\ &\quad \left. + \left(\frac{m}{s+1}\left(\frac{\lambda}{2}\right)^{s+1}\left|f'(a)\right|^2 + \frac{1}{s+1}\left[1-\left(1-\frac{\lambda}{2}\right)^{s+1}\right]\left|f'(b)\right|^2\right)^{\frac{1}{2}}\right\}, \end{aligned} \tag{2.11}$$

where,

$$\mathscr{L}(\alpha,\lambda,\mu) = \frac{1}{4(2\alpha-1)\mu\eta(b,a,m)}\left(\frac{\lambda^{2\alpha}}{2\alpha} - \frac{1}{(2\eta(b,a,m))^{2\alpha}}\gamma_\mu\left(2\alpha,2\lambda\eta(b,a,m)\right)\right).$$

Proof. By utilizing the properties of modulus on Lemma 2.1 and using the Cauchy integral inequality, we obtain

$$\begin{aligned} &\left|\mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b)\right| \\ &\leqslant \frac{\eta(b,a,m)}{2}\left(\left[\int_0^{\frac{\lambda}{2}} \gamma^2_{\mu\eta(b,a,m)}(\alpha,\lambda-2t)\mathrm{d}t\right]^{\frac{1}{2}}\left[\int_0^{\frac{\lambda}{2}} \left|f'\left(ma+t\eta(b,a,m)\right)\right|^2\mathrm{d}t\right]^{\frac{1}{2}}\right. \\ &\quad \left. + \left[\int_{1-\frac{\lambda}{2}}^1 \gamma^2_{\mu\eta(b,a,m)}(\alpha,\lambda-2+2t)\mathrm{d}t\right]^{\frac{1}{2}}\left[\int_{1-\frac{\lambda}{2}}^1 \left|f'\left(ma+t\eta(b,a,m)\right)\right|^2\mathrm{d}t\right]^{\frac{1}{2}}\right). \end{aligned}$$

Due to the generalized (s,m)-preinvexity of $|f'|^2$ on $[a,b]$, we have

$$\begin{aligned} &\int_0^{\frac{\lambda}{2}} \left|f'\left(ma+t\eta(b,a,m)\right)\right|^2\mathrm{d}t \\ &\leqslant \int_0^{\frac{\lambda}{2}}\left[m(1-t)^s\left|f'(a)\right|^2 + t^s\left|f'(b)\right|^2\right]\mathrm{d}t \\ &= \frac{m}{s+1}\left[1-\left(1-\frac{\lambda}{2}\right)^{s+1}\right]\left|f'(a)\right|^2 + \frac{1}{s+1}\left(\frac{\lambda}{2}\right)^{s+1}\left|f'(b)\right|^2, \end{aligned}$$

and

$$\begin{aligned}
&\int_{1-\frac{\lambda}{2}}^{1} \left|f'(ma+t\eta(b,a,m))\right|^2 \mathrm{d}t \\
&\quad \leqslant \int_{1-\frac{\lambda}{2}}^{1} \left[m(1-t)^s \left|f'(a)\right|^2 + t^s \left|f'(b)\right|^2\right] \mathrm{d}t \\
&\quad = \frac{m}{s+1}\left(\frac{\lambda}{2}\right)^{s+1} \left|f'(a)\right|^2 + \frac{1}{s+1}\left[1-\left(1-\frac{\lambda}{2}\right)^{s+1}\right] \left|f'(b)\right|^2.
\end{aligned}$$

Using the change of the variable $u = 1-t$ and the Cauchy integral inequality, we have

$$\begin{aligned}
&\int_0^{\frac{\lambda}{2}} \gamma^2_{\mu\eta(b,a,m)}(\alpha,\lambda-2t)\mathrm{d}t \\
&= \int_{1-\frac{\lambda}{2}}^{1} \gamma^2_{\mu\eta(b,a,m)}(\alpha,\lambda-2+2t)\mathrm{d}t \\
&= \int_0^{\frac{\lambda}{2}} \left(\int_0^{\lambda-2t} x^{\alpha-1} e^{-\mu\eta(b,a,m)x} \mathrm{d}x\right)^2 \mathrm{d}t \\
&\leqslant \int_0^{\frac{\lambda}{2}} \left[\int_0^{\lambda-2t} \left(x^{\alpha-1}\right)^2 \mathrm{d}x \int_0^{\lambda-2t} \left(e^{-\mu\eta(b,a,m)x}\right)^2 \mathrm{d}x\right] \mathrm{d}t \\
&= \frac{1}{2(2\alpha-1)\mu\eta(b,a,m)} \left(\int_0^{\frac{\lambda}{2}} (\lambda-2t)^{2\alpha-1} \left(1-e^{-2\mu\eta(b,a,m)(\lambda-2t)}\right) \mathrm{d}t\right) \\
&= \frac{1}{4(2\alpha-1)\mu\eta(b,a,m)} \left(\frac{\lambda^{2\alpha}}{2\alpha} - \gamma_{2\mu\eta(b,a,m)}(2\alpha,\lambda)\right).
\end{aligned}$$

Also,

$$\begin{aligned}
\gamma_{2\mu\eta(b,a,m)}(2\alpha,\lambda) &= \int_0^{\lambda} u^{2\alpha-1} e^{-2\mu\eta(b,a,m)u} \mathrm{d}u \\
&= \frac{1}{(2\eta(b,a,m))^{2\alpha}} \int_0^{2\lambda\eta(b,a,m)} t^{2\alpha-1} e^{-t\mu} \mathrm{d}t \\
&= \frac{1}{(2\eta(b,a,m))^{2\alpha}} \gamma_\mu\left(2\alpha, 2\lambda\eta(b,a,m)\right),
\end{aligned}$$

where we use the change of the variable $t = 2\eta(b,a,m)u$. A simple combination leads to the desired inequality (2.11). This ends the proof. □

COROLLARY 2.13. *If we take $\lambda = 1$ in Theorem 2.5, then we have*

$$\begin{aligned}
&\Bigg|\frac{2^{\alpha-1}\Gamma(\alpha)}{\eta(b,a,m)^\alpha \gamma_{\eta(b,a,m)}(\alpha,1)} \left[\mathscr{I}^{(\alpha,2\mu)}_{ma^+} f\left(ma+\frac{1}{2}\eta(b,a,m)\right)\right. \\
&\qquad \left. + \mathscr{I}^{(\alpha,2\mu)}_{(ma+\eta(b,a,m))^-} f\left(ma+\frac{1}{2}\eta(b,a,m)\right)\right] - \frac{f(ma)+f(ma+\eta(b,a,m))}{2}\Bigg|
\end{aligned}$$

$$\leqslant \frac{\eta(b,a,m)}{2}\mathscr{L}^{\frac{1}{2}}(\alpha,1,\mu)\left[\left(\frac{m}{s+1}\left(\frac{2^{s+1}-1}{2^{s+1}}\right)\left|f'(a)\right|^2+\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}\left|f'(b)\right|^2\right)^{\frac{1}{2}}\right.$$
$$\left.+\left(\frac{m}{s+1}\left(\frac{1}{2}\right)^{s+1}\left|f'(a)\right|^2+\frac{1}{s+1}\left(\frac{2^{s+1}-1}{2^{s+1}}\right)\left|f'(b)\right|^2\right)^{\frac{1}{2}}\right],$$

specially for $\eta(b,a,m)=b-ma$ *with* $m=1$*, we get*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}\gamma_{(b-a)}(\alpha,1)}\left[\mathscr{J}_{a^+}^{(\alpha,2\mu)}f\left(\frac{a+b}{2}\right)+\mathscr{J}_{b^-}^{(\alpha,2\mu)}f\left(\frac{a+b}{2}\right)\right]-\frac{f(a)+f(b)}{2}\right|$$
$$\leqslant \frac{b-a}{2}\mathscr{L}^{\frac{1}{2}}(\alpha,1,\mu)\left[\left(\frac{1}{s+1}\left(\frac{2^{s+1}-1}{2^{s+1}}\right)\left|f'(a)\right|^2+\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}\left|f'(b)\right|^2\right)^{\frac{1}{2}}\right.$$
$$\left.+\left(\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}\left|f'(a)\right|^2+\frac{1}{s+1}\left(\frac{2^{s+1}-1}{2^{s+1}}\right)\left|f'(b)\right|^2\right)^{\frac{1}{2}}\right],$$

specially for $s=1$*, we get*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}\gamma_{(b-a)}(\alpha,1)}\left[\mathscr{J}_{a^+}^{(\alpha,2\mu)}f\left(\frac{a+b}{2}\right)+\mathscr{J}_{b^-}^{(\alpha,2\mu)}f\left(\frac{a+b}{2}\right)\right]-\frac{f(a)+f(b)}{2}\right|$$
$$\leqslant \frac{b-a}{2}\mathscr{L}^{\frac{1}{2}}(\alpha,1,\mu)\left[\left(\frac{3}{8}\left|f'(a)\right|^2+\frac{1}{8}\left|f'(b)\right|^2\right)^{\frac{1}{2}}+\left(\frac{1}{8}\left|f'(a)\right|^2+\frac{3}{8}\left|f'(b)\right|^2\right)^{\frac{1}{2}}\right].$$

COROLLARY 2.14. *If we take* $\mu=0$ *and* $s=1$ *in Theorem 2.5, then we have*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^{\alpha}(b,a,m)}\left[\mathscr{J}_{ma^+}^{\alpha}f\left(ma+\frac{\lambda}{2}\eta(b,a,m)\right)\right.\right.$$
$$\left.+\mathscr{J}_{(ma+\eta(b,a,m))^-}^{\alpha}f\left(ma+\left(1-\frac{\lambda}{2}\right)\eta(b,a,m)\right)\right]$$
$$\left.-\frac{1}{2}\left[\lambda^{\alpha}f(ma)+\lambda^{\alpha}f\big(ma+\eta(b,a,m)\big)\right]\right|$$
$$\leqslant \frac{\eta(b,a,m)}{2}\left(\frac{\lambda^{2\alpha+1}}{2(2\alpha+1)}\right)^{\frac{1}{2}}\left[\left(\frac{m(4\lambda-\lambda^2)}{8}\left|f'(a)\right|^2+\frac{\lambda^2}{8}\left|f'(b)\right|^2\right)^{\frac{1}{2}}\right.$$
$$\left.+\left(\frac{m\lambda^2}{8}\left|f'(a)\right|^2+\frac{4\lambda-\lambda^2}{8}\left|f'(b)\right|^2\right)^{\frac{1}{2}}\right],$$

specially for $\alpha=1$ *and* $\lambda=1$*, if the mapping* $\eta(b,a,m)$ *with* $m=1$ *degenerates into*

$\eta(b,a)$, then we have

$$\left|\frac{1}{\eta(b,a)}\int_0^1 f\big(a+t\eta(b,a)\big)\mathrm{d}t-\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}\right|$$
$$\leqslant \frac{\eta(b,a)}{2\sqrt{6}}\left[\left(\frac{3}{8}\left|f'(a)\right|^2+\frac{1}{8}\left|f'(b)\right|^2\right)^{\frac{1}{2}}+\left(\frac{1}{8}\left|f'(a)\right|^2+\frac{3}{8}\left|f'(b)\right|^2\right)^{\frac{1}{2}}\right].$$

In the next theorem, we use the following confluent hypergeometric functions [6].

$$ {}_1F_1(a;b;z)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 u^{a-1}(1-u)^{b-a-1}e^{zu}\mathrm{d}u, \quad b>a>0, \quad -\infty<z<+\infty.$$

THEOREM 2.6. *If $|f'|^q$ for $q>1$ is generalized (s,m)-preinvex with certain fixed $s,m\in(0,1]$, then the following inequality for the tempered fractional integrals with $0<\lambda\leqslant 1$, $\alpha>0$ and $\mu\geqslant 0$ holds:*

$$\begin{aligned}&\left|\mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b)\right|\\ &\leqslant \frac{\eta(b,a,m)\lambda}{4}\left(\frac{2}{\lambda}\right)^{1-\frac{1}{q}}\Delta_1^{1-\frac{1}{q}}(\alpha,\lambda,\mu)\\ &\quad\times\left[\left(m\Delta_3(\alpha,\lambda,\mu)\left|f'(a)\right|^q+\Delta_2(\alpha,\lambda,\mu)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.\\ &\quad\left.+\left(m\Delta_2(\alpha,\lambda,\mu)\left|f'(a)\right|^q+\Delta_3(\alpha,\lambda,\mu)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right],\end{aligned} \tag{2.12}$$

where,

$$\Delta_2(\alpha,\lambda,\mu)=\frac{\lambda^{s+\alpha}\Gamma(s+2)\Gamma(\alpha)}{2^s(s+1)\Gamma(s+\alpha+2)e^{\mu\eta(b,a,m)\lambda}}\cdot {}_1F_1(s+2;s+\alpha+2;\mu\eta(b,a,m)\lambda),$$

$$\Delta_3(\alpha,\lambda,\mu)=\frac{2^{2-s}}{\lambda}\Delta_1(\alpha,\lambda,\mu)-\Delta_2(\alpha,\lambda,\mu),$$

and $\Delta_1(\alpha,\lambda,\mu)$ is defined as in Theorem 2.1.

Proof. For $q>1$, if we use the Lemma 2.1, the power-mean integral inequality and the generalized (s,m)-preinvexity of $|f'|^q$, then we have

$$\begin{aligned}&\left|\mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b)\right|\\ &=\frac{\eta(b,a,m)}{2}\left|\frac{\lambda}{2}\int_0^1\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))f'\left(ma+\frac{\lambda}{2}t\eta(b,a,m)\right)\mathrm{d}t\right.\\ &\quad\left.-\frac{\lambda}{2}\int_0^1\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))f'\left(ma+\left(1-\frac{\lambda}{2}t\right)\eta(b,a,m)\right)\mathrm{d}t\right|\end{aligned}$$

$$\leqslant \frac{\eta(b,a,m)\lambda}{4}\left(\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\left|f'\left(ma+\frac{\lambda}{2}t\eta(b,a,m)\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\left|f'\left(ma+\left(1-\frac{\lambda}{2}t\right)\eta(b,a,m)\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right]$$
$$\leqslant \frac{\eta(b,a,m)\lambda}{4}\left(\frac{2}{\lambda}\int_0^{\frac{\lambda}{2}} \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda-2t)\mathrm{d}t\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\left(m\left(1-\frac{\lambda t}{2}\right)^s\left|f'(a)\right|^q+\left(\frac{\lambda t}{2}\right)^s\left|f'(b)\right|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\left(m\left(\frac{\lambda t}{2}\right)^s\left|f'(a)\right|^q+\left(1-\frac{\lambda t}{2}\right)^s\left|f'(b)\right|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}\right\}$$
$$=\frac{\eta(b,a,m)\lambda}{4}\left(\frac{2}{\lambda}\right)^{1-\frac{1}{q}}\Delta_1^{1-\frac{1}{q}}(\alpha,\lambda,\mu)$$
$$\times\left\{\left[m\left|f'(a)\right|^q\int_0^1\left(1-\frac{\lambda t}{2}\right)^s\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t\right.\right.$$
$$\left.+\left|f'(b)\right|^q\int_0^1\left(\frac{\lambda t}{2}\right)^s\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t\right]^{\frac{1}{q}}$$
$$+\left[m\left|f'(a)\right|^q\int_0^1\left(\frac{\lambda t}{2}\right)^s\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t\right.$$
$$\left.\left.+\left|f'(b)\right|^q\int_0^1\left(1-\frac{\lambda t}{2}\right)^s\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t\right]^{\frac{1}{q}}\right\}.$$

Using the change of the variable $u=1-\frac{1}{\lambda}y$ and changing the order of the integration, we have

$$\int_0^1\left(\frac{\lambda t}{2}\right)^s\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t$$
$$=\left(\frac{\lambda}{2}\right)^s\int_0^1 t^s\gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t$$
$$=\left(\frac{\lambda}{2}\right)^s\int_0^1 t^s\left(\int_0^{\lambda(1-t)} y^{\alpha-1}e^{-\mu\eta(b,a,m)y}\mathrm{d}y\right)\mathrm{d}t$$
$$=\left(\frac{\lambda}{2}\right)^s\int_0^1 t^s\left(\int_1^t \lambda^{\alpha-1}(1-u)^{\alpha-1}e^{-\mu\eta(b,a,m)\lambda(1-u)}(-\lambda)\mathrm{d}u\right)\mathrm{d}t$$
$$=\left(\frac{\lambda}{2}\right)^s\lambda^\alpha\int_0^1(1-u)^{\alpha-1}e^{-\mu\eta(b,a,m)\lambda(1-u)}\left(\int_0^u t^s\mathrm{d}t\right)\mathrm{d}u$$

$$
\begin{aligned}
&= \left(\frac{\lambda}{2}\right)^{s} \lambda^{\alpha} \frac{1}{s+1} \int_0^1 u^{s+1}(1-u)^{\alpha-1} e^{-\mu\eta(b,a,m)\lambda(1-u)} \mathrm{d}u \\
&= \left(\frac{\lambda}{2}\right)^{s} \lambda^{\alpha} \frac{1}{s+1} e^{-\mu\eta(b,a,m)\lambda} \int_0^1 u^{s+1}(1-u)^{\alpha-1} e^{\mu\eta(b,a,m)\lambda u} \mathrm{d}u \\
&= \frac{\lambda^{s+\alpha}\Gamma(s+2)\Gamma(\alpha)}{2^s(s+1)\Gamma(s+\alpha+2)e^{\mu\eta(b,a,m)\lambda}} \cdot {}_1F_1(s+2;s+\alpha+2;\mu\eta(b,a,m)\lambda).
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\int_0^1 \left(1-\frac{\lambda t}{2}\right)^{s} \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t \\
&\quad \leqslant \int_0^1 \left(2^{1-s}-\left(\frac{\lambda t}{2}\right)^{s}\right) \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t \\
&\quad = \frac{2^{2-s}}{\lambda}\left(\frac{\lambda}{2}\int_0^1 \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t\right) - \int_0^1 \left(\frac{\lambda t}{2}\right)^{s} \gamma_{\mu\eta(b,a,m)}(\alpha,\lambda(1-t))\mathrm{d}t \\
&\quad = \frac{2^{2-s}}{\lambda}\Delta_1(\alpha,\lambda,\mu) - \Delta_2(\alpha,\lambda,\mu).
\end{aligned}
$$

Thus, the proof is completed. □

COROLLARY 2.15. *If we take $\mu = 0$ in Theorem 2.6, then we have*

$$
\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^{\alpha}(b,a,m)}\left[\mathscr{I}^{\alpha}_{ma^+} f\left(ma+\frac{\lambda}{2}\eta(b,a,m)\right)\right.\right. \\
&\quad \left. + \mathscr{I}^{\alpha}_{(ma+\eta(b,a,m))^-} f\left(ma+\left(1-\frac{\lambda}{2}\right)\eta(b,a,m)\right)\right] \\
&\quad \left. - \frac{1}{2}\left[\lambda^{\alpha} f(ma) + \lambda^{\alpha} f(ma+\eta(b,a,m))\right]\right| \\
&\leqslant \frac{\alpha\eta(b,a,m)\lambda}{4}\left(\frac{\lambda^{\alpha}}{\alpha(\alpha+1)}\right)^{1-\frac{1}{q}} \\
&\quad \times \left\{\left[m\left(\frac{2^{1-s}\lambda^{\alpha}}{\alpha(\alpha+1)} - \frac{\lambda^{s+\alpha}}{2^s(s+1)}B(s+2,\alpha)\right)\left|f'(a)\right|^{q} + \frac{\lambda^{s+\alpha}}{2^s(s+1)}B(s+2,\alpha)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\right. \\
&\quad \left. + \left[m\frac{\lambda^{s+\alpha}}{2^s(s+1)}B(s+2,\alpha)\left|f'(a)\right|^{q} + \left(\frac{2^{1-s}\lambda^{\alpha}}{\alpha(\alpha+1)} - \frac{\lambda^{s+\alpha}}{2^s(s+1)}B(s+2,\alpha)\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\right\},
\end{aligned}
$$

where,

$$
B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\mathrm{d}u.
$$

In particular, if we take $\alpha = 1 = s$*, then we get*

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\frac{\lambda}{2}\eta(b,a,m)} f(t)\mathrm{d}t + \int_{ma+(1-\frac{\lambda}{2})\eta(b,a,m)}^{ma+\eta(b,a,m)} f(t)\mathrm{d}t\right]\right.\\
&\qquad\left. - \frac{\lambda}{2}\left[f(ma)+f\big(ma+\eta(b,a,m)\big)\right]\right|\\
&\quad\leqslant \frac{\eta(b,a,m)\lambda}{4}\left(\frac{\lambda}{2}\right)^{1-\frac{1}{q}}\left[\left(m\frac{\lambda(6-\lambda)}{12}\left|f'(a)\right|^q+\frac{\lambda^2}{12}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(m\frac{\lambda^2}{12}\left|f'(a)\right|^q+\frac{\lambda(6-\lambda)}{12}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{2.13}$$

COROLLARY 2.16. *Suppose that all the hypothesis of Corollary 2.15 are satisfied, we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{f(a)+f(b)}{2}\right|\\
&\quad\leqslant \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{12}\left|f'(a)\right|^q+\frac{5}{12}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\frac{5}{12}\left|f'(a)\right|^q+\frac{1}{12}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. If we take $\lambda = 1$ and $\eta(b,a) = b - ma$ with $m = 1$ in inequality (2.13), then we have the desired inequality. □

3. Examples

EXAMPLE 3.1. Considering the function $f(x) = \frac{1}{12}x^3$, for $x \in [0,1]$. Then $|f'| = \frac{1}{4}x^2$ is a generalized (s,m)-preinvex function with regard to $\eta(y,x,1) = \frac{4}{5}y - x$ for $s = 1$ and $m = 1$. If we take $a = 0$, $b = 1$, $\alpha = \frac{3}{2}$, $\mu = 1$ and $\lambda = 1$, then all assumptions in Theorem 2.4 are satisfied.

Clearly, the left-hand side part of (2.10) is:

$$\begin{aligned}
&\left|\mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b)\right|\\
&\quad=\left|\frac{5\sqrt{10}}{96}\left[e^{-\frac{4}{5}}\int_0^{\frac{2}{5}}\left(\frac{2}{5}-t\right)^{\frac{1}{2}}e^{2t}t^3\mathrm{d}t + e^{\frac{4}{5}}\int_{\frac{2}{5}}^{\frac{4}{5}}\left(t-\frac{2}{5}\right)^{\frac{1}{2}}e^{-2t}t^3\mathrm{d}t\right]\right.\\
&\qquad\left.-\frac{8}{375}\int_0^1 t^{\frac{1}{2}}e^{-\frac{4}{5}t}\mathrm{d}t\right|\\
&\quad\approx 0.004273.
\end{aligned}$$

The right-hand side part of (2.10) is:

$$\begin{aligned}
&\frac{m|f'(a)|+|f'(b)|}{2^s}\eta(b,a,m)\Delta_1(\alpha,\lambda,\mu)\\
&=\frac{1}{20}\left[\left(\frac{5}{4}\right)^{\frac{3}{2}}\int_0^{\frac{4}{5}}t^{\frac{1}{2}}e^{-t}\mathrm{d}t-\left(\frac{5}{4}\right)^{\frac{5}{2}}\int_0^{\frac{4}{5}}t^{\frac{3}{2}}e^{-t}\mathrm{d}t\right]\\
&=\frac{1}{16}e^{-\frac{4}{5}}-\frac{7\sqrt{5}}{256}\gamma\left(\frac{3}{2},\frac{4}{5}\right)\\
&\approx 0.009627.
\end{aligned}$$

It is clear that $0.004273<0.009627$, which demonstrates the result described in Theorem 2.4.

REMARK 3.1. *Case* 1: Assume that the parameter μ is not a fixed constant in Example 3.1. For instance, $\mu\in[1,3]$, based on Theorem 2.4, we get the result for the parameter μ as below:

$$\begin{aligned}
&-\frac{\sqrt{5}}{32}\left(1-\frac{15}{8\mu}\right)\gamma_\mu\left(\frac{3}{2},\frac{4}{5}\right)-\frac{e^{-\frac{4}{5}\mu}}{16}\\
&\leqslant\frac{5\sqrt{10}}{96}\left[e^{-\frac{4}{5}\mu}\int_0^{\frac{2}{5}}\left(\frac{2}{5}-t\right)^{\frac{1}{2}}e^{2\mu t}t^3\mathrm{d}t+e^{\frac{4}{5}\mu}\int_{\frac{2}{5}}^{\frac{4}{5}}\left(t-\frac{2}{5}\right)^{\frac{1}{2}}e^{-2\mu t}t^3\mathrm{d}t\right]-\frac{8}{375}\int_0^1 t^{\frac{1}{2}}e^{-\frac{4}{5}\mu t}\mathrm{d}t\\
&\leqslant\frac{\sqrt{5}}{32}\left(1-\frac{15}{8\mu}\right)\gamma_\mu\left(\frac{3}{2},\frac{4}{5}\right)+\frac{e^{-\frac{4}{5}\mu}}{16}.
\end{aligned}\tag{3.1}$$

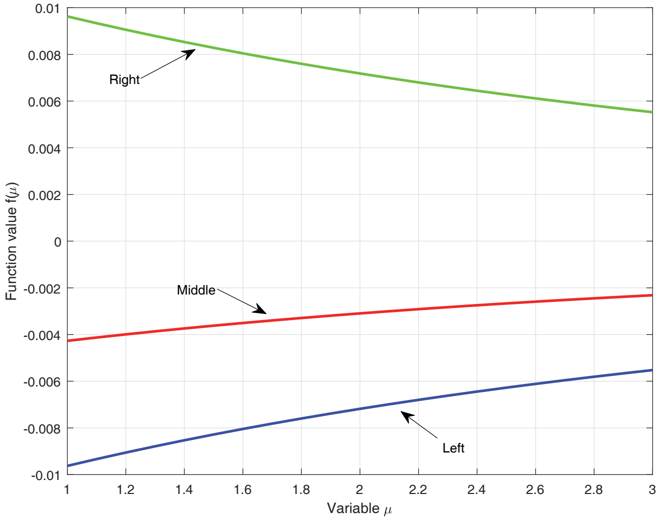

Figure 3.1: *Graphical representation of Example 3.1 for the variable* $\alpha=\frac{3}{2}$

The three functions given by the double inequalities on the left, middle and right sides (3.1) are plotted in Fig. 3.1 against $\mu \in [1,3]$. The graphs of the functions prove the validity of dual inequalities.

Case 2: Suppose that the parameter α is not a fixed constant in Example 3.1. For instance, $\alpha \in [1,3]$, based on Theorem 2.4, we get the result for the parameter α as below:

$$\begin{aligned}
&-\frac{5^{\alpha-1}}{4^{\alpha+1}}\left(1-\frac{5\alpha}{4}\right)\gamma\left(\alpha,\frac{4}{5}\right)-\frac{e^{-\frac{4}{5}}}{16}\\
&\leqslant \frac{5^{\alpha}}{3\cdot 2^{\alpha+3}}\left[e^{-\frac{4}{5}}\int_0^{\frac{2}{5}}\left(\frac{2}{5}-t\right)^{\alpha-1}e^{2t}t^3\mathrm{d}t+e^{\frac{4}{5}}\int_{\frac{2}{5}}^{\frac{4}{5}}\left(t-\frac{2}{5}\right)^{\alpha-1}e^{-2t}t^3\mathrm{d}t\right]\\
&\quad-\frac{8}{375}\int_0^1 t^{\frac{1}{2}}e^{-\frac{4}{5}t}\mathrm{d}t\\
&\leqslant \frac{5^{\alpha-1}}{4^{\alpha+1}}\left(1-\frac{5\alpha}{4}\right)\gamma\left(\alpha,\frac{4}{5}\right)+\frac{e^{-\frac{4}{5}}}{16}.
\end{aligned} \tag{3.2}$$

The visualization results of the three functions given by the double inequalities on the left, middle and right sides (3.2) are plotted in Fig. 3.2 against $\alpha \in [1,3]$. As can be seen from Fig. 3.2, the results given in Theorem 2.4 are always true if the parameters $\mu = 1$ and $\alpha \in [1,3]$ are given any value.

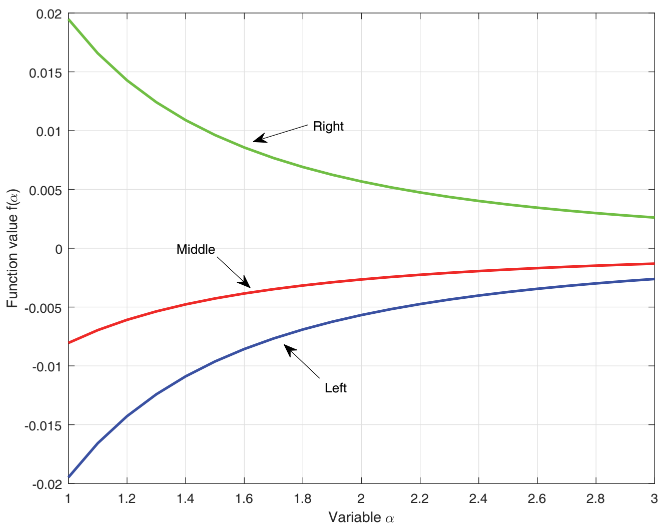

Figure 3.2: *Graphical representation of Example* 3.1 *for Variable* $\mu = 1$ *and* $\alpha \in [1,3]$

Case 3: Assume that the parameters α and μ are not two fixed constants in Example 3.1. For instance, $\alpha, \mu \in [1,3]$, based on Theorem 2.4, we get the result for the parameters α and μ as below:

$$
\begin{aligned}
&-\frac{5^{\alpha-1}}{4^{\alpha+1}}\left(1-\frac{5\alpha}{4\mu}\right)\gamma_\mu\left(\alpha,\frac{4}{5}\right)-\frac{e^{-\frac{4}{5}\mu}}{16\mu}\\
&\quad\leqslant\frac{5^\alpha}{3\cdot 2^{\alpha+3}}\left[e^{-\frac{4}{5}\mu}\int_0^{\frac{2}{5}}\left(\frac{2}{5}-t\right)^{\alpha-1}e^{2\mu t}t^3\mathrm{d}t+e^{\frac{4}{5}\mu}\int_{\frac{2}{5}}^{\frac{4}{5}}\left(t-\frac{2}{5}\right)^{\alpha-1}e^{-2\mu t}t^3\mathrm{d}t\right]\\
&\qquad-\frac{8}{375}\int_0^1 t^{\alpha-1}e^{-\frac{4}{5}\mu t}\mathrm{d}t\\
&\quad\leqslant\frac{5^{\alpha-1}}{4^{\alpha+1}}\left(1-\frac{5\alpha}{4\mu}\right)\gamma_\mu\left(\alpha,\frac{4}{5}\right)+\frac{e^{-\frac{4}{5}\mu}}{16\mu}.
\end{aligned}
$$

Let us discretize the region of $[1,3]\times[1,3]$. From the visual perspective of graphics, Fig. 3.3 vividly describes the result exhibited in Example 3.1.

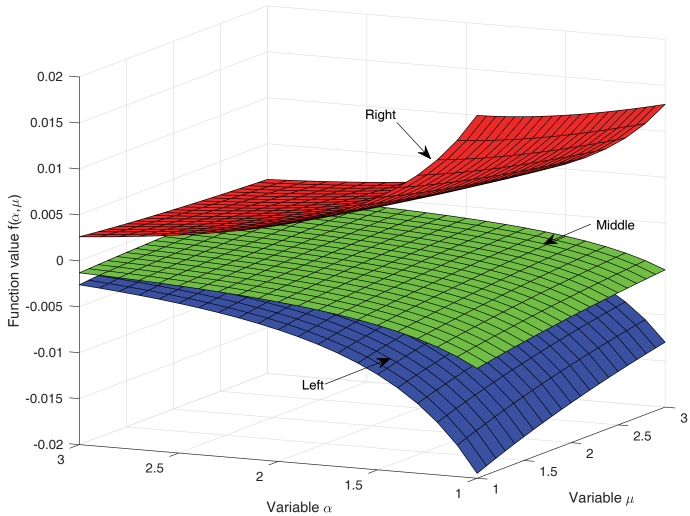

Figure 3.3: *Graphical representation of Example 3.1 for Three-dimensional*

EXAMPLE 3.2. Considering the function $f(x)=-\frac{2}{\ln 2}\cdot 0.5^{\frac{x}{2}}$, for $x\in[0,1]$. Then $|f'(x)|^2=0.5^x$ is a generalized (s,m)-preinvex function with regard to $\eta(y,x,1)=2y-x$ for $s=1$ and $m=1$. If we take $a=0$, $b=1$, $\alpha=\frac{3}{2}$, $\mu=1$ and $\lambda=1$, then all assumptions in Theorem 2.5 are satisfied.

Clearly, the left-hand side part of (2.11) is:

$$
\begin{aligned}
&\left|\mathscr{T}_f(\alpha,\mu,\lambda;\eta,m,a,b)\right|\\
&\quad\leqslant\left|\frac{1}{\ln 2}\left[e^{-2}\int_0^1(1-t)^{\frac{1}{2}}e^{2t}0.5^{\frac{t}{2}}\mathrm{d}t+e^2\int_1^2(t-1)^{\frac{1}{2}}e^{-2t}0.5^{\frac{t}{2}}\mathrm{d}t\right]+\frac{3}{\ln 4}\int_0^1 t^{\frac{1}{2}}e^{-2t}\mathrm{d}t\right|\\
&\quad\approx 0.0207.
\end{aligned}
$$

The right-hand side part of (2.11) is:

$$\begin{aligned}
&\frac{\eta(b,a,m)}{2}\mathscr{L}^{\frac{1}{2}}(\alpha,\lambda,\mu)\left\{\left(\frac{m}{s+1}\left[1-\left(1-\frac{\lambda}{2}\right)^{s+1}\right]\left|f'(a)\right|^2+\frac{1}{s+1}\left(\frac{\lambda}{2}\right)^{s+1}\left|f'(b)\right|^2\right)^{\frac{1}{2}}\right.\\
&\quad\left.+\left(\frac{m}{s+1}\left(\frac{\lambda}{2}\right)^{s+1}\left|f'(a)\right|^2+\frac{1}{s+1}\left[1-\left(1-\frac{\lambda}{2}\right)^{s+1}\right]\left|f'(b)\right|^2\right)^{\frac{1}{2}}\right\}\\
&=\left[\frac{1}{8\mu(2\alpha-1)}\left(\frac{1}{2\alpha}-\frac{1}{4^{2\alpha}}\int_0^4 t^{2\alpha-1}e^{-\mu t}\mathrm{d}t\right)\right]^{\frac{1}{2}}\cdot\left(\frac{\sqrt{7}+\sqrt{5}}{4}\right)\\
&=\left[\frac{1}{16}\left(\frac{1}{3}-\frac{1}{64}\left(10e^4-2\right)\right)\right]^{\frac{1}{2}}\cdot\left(\frac{\sqrt{7}+\sqrt{5}}{4}\right)\\
&\approx 0.1697.
\end{aligned}$$

It is clear that $0.0207 < 0.1697$, which demonstrates the result described in Theorem 2.5.

REMARK 3.2. *Case* 1: Suppose that the parameter μ is not a fixed constant in Example 3.2. For instance, $\mu \in [1,3]$, based on Theorem 2.5, we get the result for the parameter μ as below:

$$\begin{aligned}
&-\left[\frac{1}{16\mu}\left(\frac{1}{3}-\frac{1}{64}\gamma_\mu(3,4)\right)\right]^{\frac{1}{2}}\cdot\left(\frac{\sqrt{7}+\sqrt{5}}{4}\right)\\
&\leqslant\left|\frac{1}{\ln 2}\left[e^{-2\mu}\int_0^1(1-t)^{\frac{1}{2}}e^{2\mu t}0.5^{\frac{t}{2}}\mathrm{d}t+e^{2\mu}\int_1^2(t-1)^{\frac{1}{2}}e^{-2\mu t}0.5^{\frac{t}{2}}\mathrm{d}t\right]\right.\\
&\qquad\left.+\frac{3}{\ln 4}\int_0^1 t^{\frac{1}{2}}e^{-2\mu t}\mathrm{d}t\right|\\
&\leqslant\left[\frac{1}{16\mu}\left(\frac{1}{3}-\frac{1}{64}\gamma_\mu(3,4)\right)\right]^{\frac{1}{2}}\cdot\left(\frac{\sqrt{7}+\sqrt{5}}{4}\right),
\end{aligned}\tag{3.3}$$

where,

$$\gamma_\mu(3,4)=\frac{2}{\mu^3}-\left[\frac{16}{\mu}+\frac{8}{\mu^2}+\frac{2}{\mu^3}\right]e^{-4\mu}.$$

The three functions given by the double inequalities on the left, middle and right sides (3.3) are plotted in Fig. 3.4 against $\mu \in [1,3]$. The graphs of the functions prove the validity of dual inequalities.

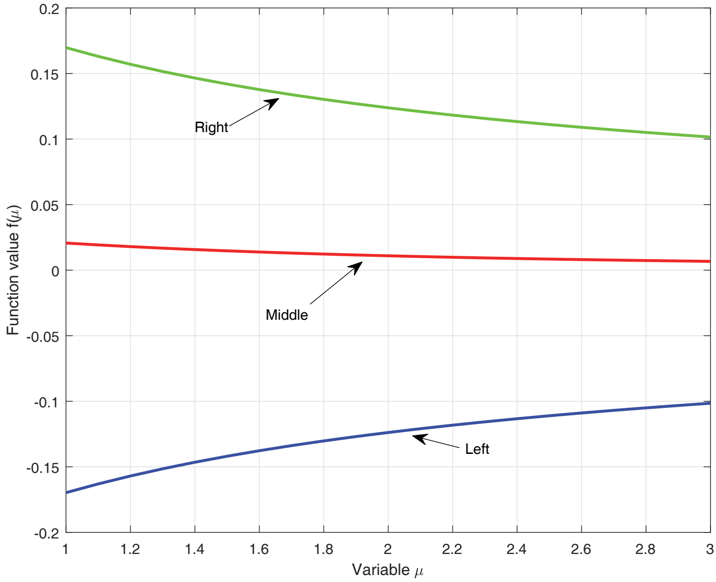

Figure 3.4: *Graphical representation of Example 3.2 for the variable* $\alpha = \frac{3}{2}$

Case 2: Assume that the parameter α is not a fixed constant in Example 3.2. For instance, $\alpha \in [1,3]$, based on Theorem 2.5, we get the result for the parameter α as below:

$$\begin{aligned}
&-\left[\frac{1}{8(2\alpha-1)}\left(\frac{1}{2\alpha}-\frac{1}{4^{2\alpha}}\gamma(2\alpha,4)\right)\right]^{\frac{1}{2}}\cdot\left(\frac{\sqrt{7}+\sqrt{5}}{4}\right)\\
&\leqslant\left|\frac{1}{\ln 2}\left[e^{-2}\int_0^1(1-t)^{\alpha-1}e^{2t}0.5^{\frac{t}{2}}\mathrm{d}t+e^2\int_1^2(t-1)^{\alpha-1}e^{-2t}0.5^{\frac{t}{2}}\mathrm{d}t\right]\right.\\
&\quad\left.+\frac{3}{\ln 4}\int_0^1 t^{\alpha-1}e^{-2t}\mathrm{d}t\right|\\
&\leqslant\left[\frac{1}{8(2\alpha-1)}\left(\frac{1}{2\alpha}-\frac{1}{4^{2\alpha}}\gamma(2\alpha,4)\right)\right]^{\frac{1}{2}}\cdot\left(\frac{\sqrt{7}+\sqrt{5}}{4}\right).
\end{aligned} \tag{3.4}$$

The visualization results of the three functions given by the double inequalities on the left, middle and right sides (3.4) are plotted in Fig. 3.5 against $\alpha \in [1,3]$. As can be seen from Fig. 3.5, the results given in Theorem 2.5 are always true if the parameters $\mu = 1$ and $\alpha \in [1,3]$ are given any value.

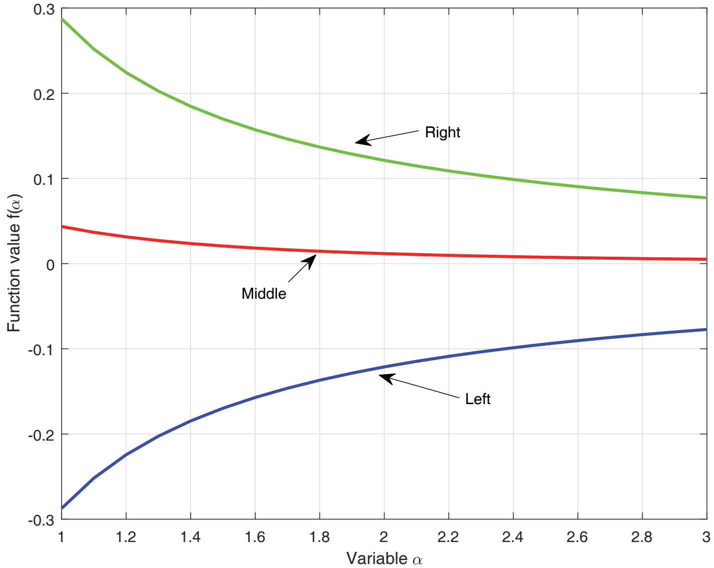

Figure 3.5: *Graphical representation of Example 3.2 for Variable* $\mu = 1$ *and* $\alpha \in [1,3]$

Case 3: Suppose that the parameters α and μ are not two fixed constants in Example 3.2. For instance, $\alpha, \mu \in [1,3]$, based on Theorem 2.5, we get the result for the parameters α and μ as below:

$$\begin{aligned}
&-\left[\frac{1}{8\mu(2\alpha-1)}\left(\frac{1}{2\alpha}-\frac{1}{4^{2\alpha}}\gamma_\mu(2\alpha,4)\right)\right]^{\frac{1}{2}}\cdot\left(\frac{\sqrt{7}+\sqrt{5}}{4}\right)\\
&\leqslant\left|\frac{1}{\ln 2}\left[e^{-2\mu}\int_0^1(1-t)^{\alpha-1}e^{2\mu t}0.5^{\frac{t}{2}}\mathrm{d}t+e^{2\mu}\int_1^2(t-1)^{\alpha-1}e^{-2\mu t}0.5^{\frac{t}{2}}\mathrm{d}t\right]\right.\\
&\qquad\left.+\frac{3}{\ln 4}\int_0^1 t^{\alpha-1}e^{-2\mu t}\mathrm{d}t\right|\\
&\leqslant\left[\frac{1}{8\mu(2\alpha-1)}\left(\frac{1}{2\alpha}-\frac{1}{4^{2\alpha}}\gamma_\mu(2\alpha,4)\right)\right]^{\frac{1}{2}}\cdot\left(\frac{\sqrt{7}+\sqrt{5}}{4}\right).
\end{aligned}$$

Let us discretize the region of $[1,3]\times[1,3]$. From the visual perspective of graphics, Fig. 3.6 vividly describes the result exhibited in Example 3.2.

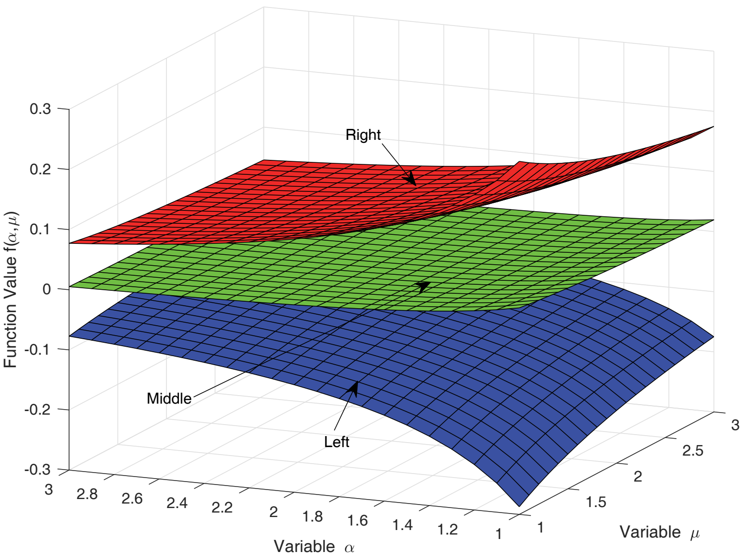

Figure 3.6: *Graphical representation of Example 3.2 for Three-dimensional*

4. Conclusion

In this study, we first introduce the conception of (μ,η)-incomplete gamma functions. Then we establish a general fractional-type identity with parameters. Using the identity and the new notation, we derive certain parameterized inequalities of the Hermite–Hadamard type pertaining to the tempered fractional integrals. Moreover, some examples are presented to identify the correctness of the results. The obtained results in this work substantially generalize certain previous findings in the literature of Hermite–Hadamard-type inequalities.

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