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ON SOME INEQUALITIES RELATED TO HEINZ MEANS FOR UNITARILY INVARIANT NORMS

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Abstract. In this paper, we improve and generalize some existing inequalities for unitarily invariant norms by using the convexity of the function $f(v)=\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|$ on the interval $[0,1]$.

1. Introduction

Let a and b be nonnegative real numbers. The Heinz mean is defined as

$$H_v(a,b)=\frac{a^{v}b^{1-v}+a^{1-v}b^{v}}{2},\quad 0\leqslant v\leqslant 1.$$

It is well-known that the Heinz mean interpolates between the geometric mean and the arithmetic mean:

$$\sqrt{ab}=H_{\frac{1}{2}}(a,b)\leqslant H_v(a,b)\leqslant H_0(a,b)=H_1(a,b)=\frac{a+b}{2},\quad 0\leqslant v\leqslant 1. \tag{1.1}$$

The matrix version of (1.1) was proved by Bhatia and Davis [3] says that

$$2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|\leqslant\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|\leqslant\|AX+XB\|,\quad 0\leqslant v\leqslant 1, \tag{1.2}$$

where $\|\cdot\|$ is every unitarily invariant norm.

In this paper, we always suppose that A,B and X are operators on a complex separable Hilbert space such that A and B are positive. For every unitarily invariant norm, setting $f(v)=\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|$, then the function $f(v)$ is convex on $[0,1]$, and $f(\frac{1}{2})\leqslant f(v)\leqslant f(0)=f(1)$. In fact, the operator version of (1.1) holds.

Kittaneh [4] gave a refinement of the second inequality in (1.2) as follows:

$$\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|\leqslant 4r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|+(1-2r_0)\|AX+XB\|, \tag{1.3}$$

where $r_0=\min\{v,1-v\}$, $0\leqslant v\leqslant 1$.

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Bhatia [2] introduced a refinement of (1.3) as follows:

$$\left\|\frac{A^{v}XB^{1-v}+A^{1-v}XB^{v}}{2}\right\| \leqslant \left\|(1-4v(1-v))\frac{AX+XB}{2}+4v(1-v)A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|, \quad 0\leqslant v\leqslant 1. \tag{1.4}$$

Unfortunately, the inequality holds only in the trivial cases of v equal to $0, \frac{1}{2}$ and 1.

Zou [7] proved that inequality (1.4) holds for Hilbert-Schmidt norm. For more information on the Heinz inequalities for unitarily invariant norms, the reader is referred to recent papers [1, 5, 6, 8], and references therein.

Let A,B and X be $n\times n$ complex matrices, if A,B are positive semidefinite, Zou and He [9] proved the following inequality

$$\int_0^1 \left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|dv \leqslant \frac{\|AX+XB\|}{2}+\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|. \tag{1.5}$$

In this paper, we will improve and generalize the inequalities (1.3) and (1.5) using the convexity of $f(v)$.

2. Main results

The following result is a refinement of the inequality (1.3).

THEOREM 1. *Let A,B and X be operators such that A and B are positive. Then for $0\leqslant v\leqslant 1$, $0<\mu<1$ and for every unitarily invariant norm*

$$\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\| \leqslant \left(1-\frac{v}{\mu}\right)\|AX+XB\|+\frac{v}{\mu}\left\|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right\|, \quad 0\leqslant v\leqslant \mu \tag{2.1}$$

and

$$\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\| \leqslant \left(1-\frac{1-v}{1-\mu}\right)\|AX+XB\|+\frac{1-v}{1-\mu}\left\|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right\|, \quad \mu<v\leqslant 1. \tag{2.2}$$

Proof. For $v=0,\ \mu,1$, inequalities (2.1) and (2.2) are obvious. For $0<v<\mu$, since $f(v)=\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\|$ is convex on $[0,1]$, it follows by a slope argument that

$$\frac{f(v)-f(0)}{v-0}\leqslant\frac{f(\mu)-f(0)}{\mu-0},$$

and so

$$f(v)\leqslant\left(1-\frac{v}{\mu}\right)f(0)+\frac{v}{\mu}f(\mu),$$

that is

$$\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\| \leqslant \left(1-\frac{v}{\mu}\right)\|AX+XB\|+\frac{v}{\mu}\left\|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right\|.$$

For $\mu < v < 1$, similarly, we have

$$\frac{f(v)-f(\mu)}{v-\mu} \leqslant \frac{f(1)-f(\mu)}{1-\mu},$$

and so

$$f(v) \leqslant \left(1-\frac{1-v}{1-\mu}\right)f(1)+\frac{1-v}{1-\mu}f(\mu),$$

that is

$$\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\| \leqslant \left(1-\frac{1-v}{1-\mu}\right)\|AX+XB\|+\frac{1-v}{1-\mu}\left\|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right\|.$$

This completes the proof. $\square$

COROLLARY 1. *Let A,B and X be operators such that A and B are positive. Then for $0 \leqslant v \leqslant 1$, $0<\mu<1$ and for every unitarily invariant norm*

$$\begin{aligned}\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\| &\leqslant (1-r_0)\|AX+XB\|+r_0\left\|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right\| \\ &\leqslant \|AX+XB\|,\end{aligned}$$

where $r_0=\begin{cases}\frac{v}{\mu}, 0\leqslant v\leqslant \mu,\\ \frac{1-v}{1-\mu}, \mu<v\leqslant 1.\end{cases}$

REMARK 1. Taking $\mu=\frac{1}{2}$ in Theorem 1, we can obtain inequality (1.3).

REMARK 2. Theorem 1 is better than inequality (1.3).
For $0<\mu\leqslant\frac{1}{2}$, by (2.1) and (1.3), we have

$$\begin{aligned}&\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\| \\ &\leqslant \left(1-\frac{v}{\mu}\right)\|AX+XB\|+\frac{v}{\mu}\left\|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right\| \\ &\leqslant \frac{\mu-v}{\mu}\|AX+XB\|+\frac{v}{\mu}\left[(1-2\mu)\|AX+XB\|+4\mu\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|\right] \\ &=(1-2v)\|AX+XB\|+4v\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|.\end{aligned}$$

For $\frac{1}{2}<\mu<1$, by (2.2) and (1.3), we have

$$\begin{aligned}&\left\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\right\| \\ &\leqslant \left(1-\frac{1-v}{1-\mu}\right)\|AX+XB\|+\frac{1-v}{1-\mu}\left\|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\right\| \\ &\leqslant \frac{v-\mu}{1-\mu}\|AX+XB\|+\frac{1-v}{1-\mu}\left[(2\mu-1)\|AX+XB\|+4(1-\mu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|\right] \\ &=(2v-1)\|AX+XB\|+4(1-v)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|.\end{aligned}$$

In the following, we give a lemma which will turn out to be useful in the proof of our result.

LEMMA 1. *Let $f(\mu)$ be convex on $[0,1]$, then*

$$f(\mu) \leqslant (1-2\mu)f(0) + 2\mu f\left(\frac{1}{2}\right), \ \ 0 \leqslant \mu \leqslant \frac{1}{2}$$

and

$$f(\mu) \leqslant (2\mu-1)f(1) + 2(1-\mu)f\left(\frac{1}{2}\right), \quad \frac{1}{2} < \mu \leqslant 1.$$

Proof. For $0 \leqslant \mu \leqslant \frac{1}{2}$, let $\mu = (1-\lambda)\cdot 0 + \lambda \cdot \frac{1}{2}$, by the convexity of f, it follows that

$$f(\mu) = f\left((1-\lambda)\cdot 0 + \lambda \cdot \frac{1}{2}\right) \leqslant (1-\lambda)f(0) + \lambda f\left(\frac{1}{2}\right) = (1-2\mu)f(0) + 2\mu f\left(\frac{1}{2}\right).$$

For $\frac{1}{2} \leqslant \mu \leqslant 1$, let $\mu = (1-\lambda)\cdot \frac{1}{2} + \lambda \cdot 1$, similarly, we have

$$f(\mu) = f\left((1-\lambda)\cdot \frac{1}{2} + \lambda \cdot 1\right) \leqslant (1-\lambda)f\left(\frac{1}{2}\right) + \lambda f(1) = (2\mu-1)f(1) + 2(1-\mu)f\left(\frac{1}{2}\right).$$

This completes the proof. □

THEOREM 2. *Let A,B and X be operators such that A and B are positive. Then for every unitarily invariant norm*

$$\begin{aligned}\int_0^1 \left\|A^v X B^{1-v} + A^{1-v} X B^v\right\| dv &\leqslant \frac{\|AX + XB\|}{2} + \left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\| \\ &\leqslant (1-2\mu_0)\|AX+XB\| + 4\mu_0 \left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\|, \\ &\qquad \mu \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\int_0^1 \left\|A^v X B^{1-v} + A^{1-v} X B^v\right\| dv &\leqslant (1-2\mu_0)\|AX+XB\| + 4\mu_0 \left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\| \\ &\leqslant \frac{\|AX + XB\|}{2} + \left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\|, \ \ \mu \in \left[\frac{1}{4}, \frac{3}{4}\right],\end{aligned} \tag{2.4}$$

where $\mu_0 = \min\{\mu, 1-\mu\}$, $0 \leqslant v \leqslant 1$.

Proof. Since $f(v) = \left\|A^v X B^{1-v} + A^{1-v} X B^v\right\|$ is continuous on $[0,1]$, it follows that

$$\int_0^1 f(v)dv = f(\mu).$$

By Lemma 1, we have

$$\begin{aligned}\int_0^1 f(v)dv &= \int_0^{\frac{1}{2}} f(v)dv + \int_{\frac{1}{2}}^1 f(v)dv \\ &\leqslant \int_0^{\frac{1}{2}} \left[(1-2v)f(0) + 2vf\left(\frac{1}{2}\right)\right] dv + \int_{\frac{1}{2}}^1 \left[(2v-1)f(1) + 2(1-v)f\left(\frac{1}{2}\right)\right] dv \\ &\leqslant \frac{f(0) + f(\frac{1}{2})}{2}.\end{aligned} \tag{2.5}$$

If $0 \leqslant \mu \leqslant \frac{1}{4}$, by Lemma 1, we obtain

$$\int_0^1 f(v)dv = f(\mu) \leqslant (1-2\mu)f(0) + 2\mu f\left(\frac{1}{2}\right). \tag{2.6}$$

By a small calculation, we have

$$(1-2\mu)f(0) + 2\mu f\left(\frac{1}{2}\right) - \frac{f(0)+f(\frac{1}{2})}{2} = \left(\frac{1}{2} - 2\mu\right)\left(f(0) - f\left(\frac{1}{2}\right)\right) \geqslant 0. \tag{2.7}$$

It follows from (2.5), (2.6) and (2.7) that

$$\int_0^1 f(v)dv \leqslant \frac{f(0)+f(\frac{1}{2})}{2} \leqslant (1-2\mu)f(0) + 2\mu f\left(\frac{1}{2}\right),$$

that is

$$\begin{aligned}\int_0^1 \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|dv &\leqslant \frac{\|AX+XB\|}{2} + \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| \\ &\leqslant (1-2\mu)\|AX+XB\| + 4\mu\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|.\end{aligned}$$

If $\frac{1}{4} \leqslant \mu \leqslant \frac{1}{2}$, by Lemma 1, we obtain

$$\int_0^1 f(v)dv = f(\mu) \leqslant (1-2\mu)f(0) + 2\mu f\left(\frac{1}{2}\right). \tag{2.8}$$

By a small calculation, we have

$$(1-2\mu)f(0) + 2\mu f\left(\frac{1}{2}\right) - \frac{f(0)+f(\frac{1}{2})}{2} = \left(\frac{1}{2} - 2\mu\right)\left(f(0) - f\left(\frac{1}{2}\right)\right) \leqslant 0. \tag{2.9}$$

It follows from (2.5), (2.8) and (2.9) that

$$\int_0^1 f(v)dv \leqslant (1-2\mu)f(0) + 2\mu f\left(\frac{1}{2}\right) \leqslant \frac{f(0)+f(\frac{1}{2})}{2},$$

that is

$$\begin{aligned}\int_0^1 \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|dv &\leqslant (1-2\mu)\|AX+XB\| + 4\mu\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| \\ &\leqslant \frac{\|AX+XB\|}{2} + \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|.\end{aligned}$$

For $\frac{1}{2} \leqslant \mu \leqslant \frac{3}{4}$. Similarly, we have

$$\begin{aligned}\int_0^1 \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|dv &\leqslant (2\mu-1)\|AX+XB\| + 4(1-\mu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| \\ &\leqslant \frac{\|AX+XB\|}{2} + \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|.\end{aligned}$$

For $\frac{3}{4} \leqslant \mu \leqslant 1$. Similarly, we have

$$\int_0^1 \left\|A^v X B^{1-v} + A^{1-v} X B^v\right\| dv \leqslant \frac{\|AX + XB\|}{2} + \left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\|$$
$$\leqslant (2\mu - 1)\|AX + XB\| + 4(1-\mu)\left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\|.$$

This completes the proof. □

REMARK 3. If $\mu \in [\frac{1}{4}, \frac{3}{4}]$, (2.4) is sharper than (1.5).

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