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POST–QUANTUM OSTROWSKI TYPE INTEGRAL INEQUALITIES FOR TWICE (p,q)–DIFFERENTIABLE FUNCTIONS

WAEWTA LUANGBOON, KAMSING NONLAOPON*, JESSADA TARIBOON,
SORTIRIS K. NTOUYAS AND HÜSEYIN BUDAK

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Abstract. In this paper, we establish a new (p,q)-integral identity using twice (p,q)-differentiable functions. Then, we use this result to derive some new post-quantum Ostrowski type integral inequalities for twice (p,q)-differentiable functions. The newly established results are also proven to be generalizations of some existing results in the area of integral inequalities.

1. Introduction

Integral inequalities are a very necessary tool in the study of applied and pure mathematics. Ostrowski type integral inequalities, one of the integral inequalities, have been studied by many authors. They have been frequently employed in statistics, quadrature, stochastic, probability and optimization theory, integral operator theory, and information. The classical integral inequality for the differentiable function is as follows:

THEOREM 1. [45] *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b) and $\|f'\|_\infty=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leqslant\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty$$

for all $x\in[a,b]$.

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* Corresponding author.

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In recent years, many researchers have focused on the Ostrowski type integral inequalities and their applications, see [7, 18, 19, 20, 21, 25, 26, 38, 39, 51, 52] and the references cited therein for more details. Specifically, many researchers have worked on the Ostrowski type integral inequalities and their applications using quantum calculus, some results can be found in [1, 4, 5, 6, 15, 17, 23, 37, 44] and the references cited therein.

Quantum calculus, also known as q-calculus, is the study of calculus without limits. In q-calculus, we obtain q-analoques of mathematical objects that can be recaptured by taking $q \to 1$. The concept was revealed by renowned mathematician Euler (1707–1783), who introduced q parameter in Newton's work on infinite series. In 1910, Jackson [33] defined q_a-integral and q_a-derivative on the interval $(0,\infty)$ extending the concept of Euler. In 1966, Al-Salam [8] introduced fractional q_a-integral and fractional q_a-derivative. The topic of q-calculus has been received outstanding attention from many researchers because it has numerous applications in various fields of physics and mathematics, for example, the theory of relativity, combinatorics, hypergeometric functions, orthogonal polynomials, mechanics, and number theory, see [10, 12, 24, 28, 29, 30, 31, 32, 35, 46, 49] and the references cited therein for more details.

In 2013, Tariboon and Ntouyas [55] presented the q_a-integral and the q_a-derivative on finite intervals and addressed numerous problems on q_a-analogues of classical inequalities. Recently, in 2020, Bermudo et al. [13] presented q^b-integral and q^b-derivative on finite intervals and also proved some of their basic properties. Currently, these topics of q-calculus have been studied in various integral inequalities such as Hanh, Hermite-Hadamard, Hermite-Hadamard-like, Newton, Simpson, Fejér, and Ostrowski type integral inequalities, see [3, 11, 14, 16, 34, 48, 60] and the references cited therein for more details.

The q-calculus generalization is called (p,q)-calculus, also known as post-quantum calculus. The (p,q)-calculus has two independent parameters that are p-number and q-number. Apparently, the q-calculus cannot be directly obtained by substituting q by q/p in q-calculus, but it can be directly obtained by taking $p = 1$ in (p,q)-calculus. Then, the classical inequalities can be gained by taking $q \to 1$. The concept of $(p,q)_a$-integral and $(p,q)_a$-derivative on the interval $(0,\infty)$ was first presented by Chakrabarti and Jagannathan [22] in 1991. Later on, the concept of $(p,q)_a$-integral and $(p,q)_a$-derivative on finite intervals was presented by Tunç, and Göv [57, 58] in 2016. Recently, the concept of the $(p,q)^b$-integral and $(p,q)^b$-derivative on the finite intervals has been presented by Vivas-Cortez et al. [59] in 2021. Currently, the topic of (p,q)-calculus has been receiving outstanding attention from many researchers, some new results can be found in [9, 27, 36, 40, 41, 43, 47, 50, 53] and the references cited therein.

In 2021, Ali et al. [3] introduced quantum Ostrowski type integral inequalities for twice q-differentiable functions. By taking $q \to 1$, they obtain classical results on some Ostrowski type integral inequalities for functions, whose second derivatives are h-convex functions [42]. Inspired by the above-mentioned reports, we establish some new post-quantum Ostrowski type integral inequalities for twice (p,q)-differentiable functions extend and generalize the results given in previous reports.

The rest of the paper is organized as follows: In Section 2, we provide some basic knowledge and definitions of (p,q)-calculus. In Section 3, post-quantum Ostrowski type integral inequalities for twice (p,q)-differentiable functions are presented. In Section 4, we summarize our results.

2. Preliminaries

In this section, we discuss some basic knowledge and definitions of (p,q)-calculus which will be used in our work. Throughout this paper, we assume that $0 < q < p \leqslant 1$ are constants and $[a,b] \subseteq \mathbb{R}$ is an interval with $a < b$. The (p,q)-number of n is given by

$$[n]_{p,q} = \frac{p^n - q^n}{p-q} = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1}, \quad n \in \mathbb{N},$$

which is a generalization of the q-analogue or q-number of n such that

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-2} + q^{n-1}, \quad n \in \mathbb{N},$$

see [35] for more details.

DEFINITION 1. [57, 58] For a continuous function $f : [a,b] \to \mathbb{R}$, the $(p,q)_a$-derivative on $[a,b]$ of function f at t is defined by

$$\begin{aligned} {}_aD_{p,q}f(t) &= \frac{f(pt + (1-p)a) - f(qt + (1-q)a)}{(p-q)(t-a)}, \ t \neq a, \\ {}_aD_{p,q}f(a) &= \lim_{t \to a} {}_aD_{p,q}f(t). \end{aligned} \tag{1}$$

The function f is called $(p,q)_a$-differentiable function on $[a,b]$ if ${}_aD_{p,q}f(t)$ exists for all $t \in [a, (b-a)/p + a]$.

Note that if $p = 1$ and ${}_aD_{1,q}f(t) = {}_aD_qf(t)$ in (1), then (1) reduces to

$$\begin{aligned} {}_aD_q\varphi(t) &= \frac{f(t) - f(qt + (1-q)a)}{(1-q)(t-a)}, \ t \neq a, \\ {}_aD_qf(a) &= \lim_{t \to a} {}_aD_qf(t), \end{aligned} \tag{2}$$

which is the well-known q_a-derivative of function f on $[a,b]$, see [54, 56] for more details.

Moreover, if $a = 0$ and ${}_0D_qf(t) = D_qf(t)$ in (2), then (2) reduces to

$$\begin{aligned} D_qf(t) &= \frac{f(t) - f(qt)}{(1-q)t}, \ t \neq a, \\ D_qf(a) &= \lim_{t \to 0} D_qf(t), \end{aligned}$$

which is the well-known q-derivative of function f on $[0,b]$, also called q-Jackson derivative, see [35] for more details.

EXAMPLE 1. Define function $f:[a,b]\to\mathbb{R}$ by $f(t)=t^2+C$, where C is constant. Applying Definition 1 for $t\neq a$, we have

$$\begin{aligned}
{}_aD_{p,q}(t^2+C) &= \frac{\left[(pt+(1-p)a)^2+C\right]-\left[(qt+(1-q)a)^2+C\right]}{(p-q)(t-a)}\\
&= \frac{(p+q)t^2+2at[1-(p+q)]+a^2[(p+q)-2]}{(t-a)}\\
&= \frac{(p+q)(t-a)^2+2a(t-a)}{(t-a)}\\
&= [2]_{p,q}(t-a)+2a.
\end{aligned}$$

DEFINITION 2. [59] For a continuous function $f:[a,b]\to\mathbb{R}$, the $(p,q)^b$-derivative on $[a,b]$ of function f at t is defined by

$$\begin{aligned}
{}^bD_{p,q}f(t) &= \frac{f(qt+(1-q)b)-f(pt+(1-p)b)}{(p-q)(b-t)}, \quad t\neq b,\\
{}^bD_{p,q}f(b) &= \lim_{t\to b} {}^bD_{p,q}f(t).
\end{aligned} \tag{3}$$

The function f is called $(p,q)^b$-differentiable function on $[a,b]$ if ${}^bD_{p,q}f(t)$ exists for all $t\in[b-(b-a)/p,b]$.

Note that if $p=1$ and ${}^bD_{1,q}f(t)={}^bD_qf(t)$ in (3), then (3) reduces to

$$\begin{aligned}
{}^bD_qf(t) &= \frac{f(qt+(1-q)b)-f(t)}{(1-q)(b-t)}, \quad t\neq b,\\
{}^bD_qf(b) &= \lim_{t\to b} {}^bD_qf(t),
\end{aligned}$$

which is the well-known q^b-derivative of function f on $[a,b]$, see [2, 13] for more details.

EXAMPLE 2. Define function $f:[a,b]\to\mathbb{R}$ by $f(t)=t^2+C$, where C is constant. Applying Definition 2 for $t\neq b$, we have

$$\begin{aligned}
{}^bD_{p,q}(t^2+C) &= \frac{\left[(qt+(1-q)b)^2+C\right]-\left[(pt+(1-p)b)^2+C\right]}{(p-q)(b-t)}\\
&= \frac{-(p+q)t^2+2bt[(p+q)-1]+b^2[2-(p+q)]}{(b-t)}\\
&= \frac{-(p+q)(b-t)^2+2b(b-t)}{(b-t)}\\
&= [2]_{p,q}(t-b)+2b.
\end{aligned}$$

DEFINITION 3. [57] For a continuous function $f:[a,b]\to\mathbb{R}$, the $(p,q)_a$-integral on $[a,b]$ of function f is defined by

$$\int_a^x f(t)\ {}_ad_{p,q}t=(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right) \tag{4}$$

for $x\in[a,b]$.

The function f is called $(p,q)_a$-integrable function on $[a,b]$ if $\int_a^x f(t)\ {}_ad_{p,q}t$ exists.

EXAMPLE 3. Define function $f:[a,b]\to\mathbb{R}$ by $f(t)=At+B$, where A and B are constants. Applying Definition 3, we have

$$\begin{aligned}\int_a^b f(t)\ {}_ad_{p,q}t&=\int_a^b (At+B)\ {}_ad_{p,q}t\\&=A(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\\&\quad+B(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\\&=\frac{A(b-a)(b-a(1-p-q))}{[2]_{p,q}}+B(b-a).\end{aligned}$$

DEFINITION 4. [59] For a continuous function $f:[a,b]\to\mathbb{R}$, the $(p,q)^b$-integral on $[a,b]$ of function f is defined by

$$\int_x^b f(t)\ {}^bd_{p,q}t=(p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)b\right) \tag{5}$$

for $x\in[a,b]$.

The function f is called $(p,q)^b$-integrable function on $[a,b]$ if $\int_a^b f(t)\ {}^bd_{p,q}t$ exists.

EXAMPLE 4. Define function $f:[a,b]\to\mathbb{R}$ by $f(t)=At+B$, where A and B are constants. Applying Definition 4, we have

$$\begin{aligned}\int_a^b f(t)\ {}^bd_{p,q}t&=\int_a^b (At+B)\ {}^bd_{p,q}t\\&=A(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}a+\left(1-\frac{q^n}{p^{n+1}}\right)b\right)\\&\quad+B(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\\&=\frac{A(b-a)(a-b(1-p-q))}{[2]_{p,q}}+B(b-a).\end{aligned}$$

LEMMA 1. [57] *For $\alpha \in \mathbb{R} \setminus \{-1\}$, the following inequality holds:*

$$\int_a^b (t-a)^\alpha \ {}_a d_{p,q}t = \frac{(b-a)^{\alpha+1}}{[\alpha+1]_{p,q}}. \tag{6}$$

THEOREM 2. [58] *Let $f,g:[a,b] \to \mathbb{R}$ be continuous functions and $r>1$ with $1/s+1/r=1$, then*

$$\int_a^b |f(t)g(t)| \ {}_a d_{p,q}t \leqslant \left(\int_a^b |f(t)|^s \ {}_a d_{p,q}t\right)^{1/s} \left(\int_a^b |g(t)|^r \ {}_a d_{p,q}t\right)^{1/r}. \tag{7}$$

3. Main results

In this section, we give some new estimates of post-quantum Ostrowski type integral inequalities for twice (p,q)-differentiable functions. We define $J_1 = [b-p(b-x),b]$ and $J_2 = [a,a+p(x-a)]$. The (p,q)-integral identity is as follows:

THEOREM 3. *If $f:[a,b] \to \mathbb{R}$ is a twice (p,q)-differentiable function such that ${}^bD^2_{p,q}f$ and ${}_aD^2_{p,q}f$ are continuous and integrable functions on J_1 and J_2, respectively. Then*

$$\begin{aligned} {}^b_aL_{p,q}(x) = (x-a)^2(b-x)^2 \Bigg[&(a-x)\int_0^1 t^2 {}_aD^2_{p,q}f(tx+(1-t)a)\ d_{p,q}t \\ &+(x-b)\int_0^1 t^2\ {}^bD^2_{p,q}f(tx+(1-t)b)\ d_{p,q}t \Bigg], \end{aligned} \tag{8}$$

where

$$\begin{aligned} &{}^b_aL_{p,q}(x) \\ &= \frac{(x-a)(b-x)}{pq^3(p-q)}\Big[(x-a)pqf(qx+(1-q)b)+(b-x)pqf(qx+(1-q)a) \\ &\quad -(x-a)(q^2+pq-p^2)f(px+(1-p)b)-(b-x)(q^2+pq-p^2)f(px+(1-p)a)\Big] \\ &\quad -\frac{[2]_{p,q}}{p^3q^3}\left[(x-a)^2\int_{p^2x+(1-p^2)b}^b f(t)\ {}^bd_{p,q}t+(b-x)^2\int_a^{p^2x+(1-p^2)a} f(t)\ {}_ad_{p,q}t\right]. \end{aligned}$$

Proof. Using Definition 1, we have

$$\begin{aligned} &{}_aD^2_{p,q}f(tb+(1-t)a) \\ &= {}_aD_{p,q}(\ {}_aD_{p,q}f(tb+(1-t)a)) \\ &= {}_aD_{p,q}\left(\frac{f(ptb+(1-pt)a)-f(qtb+(1-qt)a)}{(p-q)(b-a)t}\right) \end{aligned}$$

$$= \frac{1}{(p-q)(b-a)t}\left[\frac{f(p^2tb+(1-p^2t)a)-f(pqtb+(1-pqt)a)}{pt(p-q)(b-a)}\right.$$
$$\left.-\frac{f(pqtb+(1-pqt)a)-f(q^2tb+(1-q^2t)a)}{qt(p-q)(b-a)}\right]$$
$$= \frac{qf(p^2tb+(1-p^2t)a)-[2]_{p,q}f(pqtb+(1-pqt)a)+pf(q^2tb+(1-q^2t)a)}{pqt^2(p-q)^2(b-a)^2}. \quad (9)$$

Applying (9) and Definition 3, we obtain

$$\int_0^1 t^2\,{}_aD^2_{p,q}f(tx+(1-t)a)\;d_{p,q}t$$
$$= \int_0^1 \frac{qf(p^2tx+(1-p^2t)a)-[2]_{p,q}f(pqtx+(1-pqt)a)+pf(q^2tx+(1-q^2t)a)}{pq(p-q)^2(x-a)^2}\;d_{p,q}t$$
$$= \frac{q(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(p^2\frac{q^n}{p^{n+1}}x+\left(1-p^2\frac{q^n}{p^{n+1}}\right)a\right)}{pq(p-q)^2(x-a)^3}$$
$$-\frac{[2]_{p,q}(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+1}}f\left(p\frac{q^{n+1}}{p^{n+1}}x+\left(1-p\frac{q^{n+1}}{p^{n+1}}\right)a\right)}{pq^2(p-q)^2(x-a)^3}$$
$$+\frac{p(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^{n+2}}{p^{n+1}}f\left(\frac{q^{n+2}}{p^{n+1}}x+\left(1-\frac{q^{n+2}}{p^{n+1}}\right)a\right)}{pq^3(p-q)^2(x-a)^3}$$
$$= \frac{(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(p^2\frac{q^n}{p^{n+1}}x+\left(1-p^2\frac{q^n}{p^{n+1}}\right)a\right)}{p(p-q)^2(x-a)^3}$$
$$-\frac{p[2]_{p,q}(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+2}}f\left(p^2\frac{q^{n+1}}{p^{n+2}}x+\left(1-p^2\frac{q^{n+1}}{p^{n+2}}\right)a\right)}{pq^2(p-q)^2(x-a)^3}$$
$$+\frac{p^3(p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^{n+2}}{p^{n+3}}f\left(p^2\frac{q^{n+2}}{p^{n+3}}x+\left(1-p^2\frac{q^{n+2}}{p^{n+3}}\right)a\right)}{pq^3(p-q)^2(x-a)^3}$$
$$= \frac{[2]_{p,q}}{p^3q^3(x-a)^3}\int_a^{p^2x+(1-p^2)a} f(t)\;{}_ad_{p,q}t+\frac{(q^2+pq-p^2)f(px+(1-p)a)}{pq^3(p-q)(x-a)^2}$$
$$-\frac{f(qx+(1-q)a)}{q^2(p-q)(x-a)^2}. \quad (10)$$

Using Definition 2, we have

$${}^bD^2_{p,q}f(ta+(1-t)b) = {}^bD_{p,q}\left({}^bD_{p,q}f(ta+(1-t)b)\right)$$
$$= {}^bD_{p,q}\left(\frac{f(qta+(1-qt)b)-f(pta+(1-pt)b)}{(p-q)(b-a)t}\right)$$

$$
\begin{aligned}
&= \frac{1}{(p-q)(b-a)t}\left[\frac{f(q^2ta+(1-q^2t)b)-f(pqta+(1-pqt)b)}{qt(p-q)(b-a)}\right.\\
&\quad \left.-\frac{f(pqta+(1-pqt)b)-f(p^2ta+(1-p^2t)b)}{pt(p-q)(b-a)}\right]\\
&= \frac{pf(q^2ta+(1-q^2t)b)-[2]_{p,q}f(pqta+(1-pqt)b)+qf(p^2ta+(1-p^2t)b)}{pqt^2(p-q)^2(b-a)^2}.
\end{aligned} \tag{11}
$$

Applying (11) and Definition 4, we obtain

$$
\begin{aligned}
&\int_0^1 t^2\, {}^bD^2_{p,q} f(tx+(1-t)b)\, d_{p,q}t\\
&= \int_0^1 \frac{pf(q^2tx+(1-q^2t)b)-[2]_{p,q}f(pqtx+(1-pqt)b)+qf(p^2tx+(1-p^2t)b)}{pq(p-q)^2(b-x)^2}\, d_{p,q}t\\
&= \frac{p(p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^{n+2}}{p^{n+1}}f\left(\frac{q^{n+2}}{p^{n+1}}x+\left(1-\frac{q^{n+2}}{p^{n+1}}\right)b\right)}{pq^3(p-q)^2(b-x)^3}\\
&\quad -\frac{[2]_{p,q}(p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+1}}f\left(p\frac{q^{n+1}}{p^{n+1}}x+\left(1-p\frac{q^{n+1}}{p^{n+1}}\right)b\right)}{pq^2(p-q)^2(b-x)^3}\\
&\quad +\frac{q(p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n+1}}f\left(p^2\frac{q^{n+2}}{p^{n+1}}x+\left(1-p^2\frac{q^{n+2}}{p^{n+1}}\right)b\right)}{pq(p-q)^2(b-x)^3}\\
&= \frac{p^3(p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^{n+2}}{p^{n+3}}f\left(p^2\frac{q^{n+2}}{p^{n+3}}x+\left(1-p^2\frac{q^{n+2}}{p^{n+3}}\right)b\right)}{pq^3(p-q)^2(b-x)^3}\\
&\quad -\frac{p[2]_{p,q}(p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+2}}f\left(p^2\frac{q^{n+1}}{p^{n+1}}x+\left(1-p^2\frac{q^{n+1}}{p^{n+2}}\right)b\right)}{pq^2(p-q)^2(b-x)^3}\\
&\quad +\frac{q(p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n+1}}f\left(p^2\frac{q^{n+2}}{p^{n+1}}x+\left(1-p^2\frac{q^{n+2}}{p^{n+1}}\right)b\right)}{pq(p-q)^2(b-x)^3}\\
&= \frac{[2]_{p,q}}{p^3q^3(b-x)^3}\int_{p^2x+(1-p^2)b}^{b} f(t)\, {}^bd_{p,q}t+\frac{(q^2+pq-p^2)f(px+(1-p)b)}{pq^3(p-q)(b-x)^2}\\
&\quad -\frac{f(qx+(1-q)b)}{q^2(p-q)(b-x)^2}.
\end{aligned} \tag{12}
$$

By multiplying (10) and (12) by $(x-a)^2(b-x)^2(a-x)$ and $(x-a)^2(b-x)^2(x-b)$, respectively, and adding the resultant inequalities, we obtain the required identity (8). Therefore, the proof is completed. □

REMARK 1. If $p=1$ in (8), then we have the following identity:

$$\begin{aligned} {}_a^bL_q(x) = (x-a)^2(b-x)^2 \Big[&(a-x)\int_0^1 t^2 {}_aD_q^2 f(tx+(1-t)a)\, d_qt \\ &+(x-b)\int_0^1 t^2\, {}^bD_q^2 f(tx+(1-t)b)\, d_qt \Big], \end{aligned}$$

where

$$\begin{aligned} {}_a^bL_q(x) = &\frac{(x-a)(b-x)}{q^3(p-q)}\left[(x-a)qf(qx+(1-q)b)+(b-x)qf(qx+(1-q)a)\right. \\ &\left.-(q^2+q-1)(b-a)f(x)\right] \\ &-\frac{[2]_q}{q^3}\left[(x-a)^2\int_x^b f(t)\, {}^bd_qt+(b-x)^2\int_a^x f(t)\, {}_ad_qt\right], \end{aligned}$$

which appeared in [3].

THEOREM 4. *Let $f:[a,b]\to\mathbb{R}$ be a twice (p,q)-differentiable function such that ${}^bD_{p,q}^2 f$ and ${}_aD_{p,q}^2 f$ are continuous and integrable functions on J_1 and J_2, respectively. If $|{}^bD_{p,q}^2 f|$ and $|{}_aD_{p,q}^2 f|$ are convex functions, then*

$$\begin{aligned} \left|{}_a^bL_{p,q}(x)\right| \leqslant\ &(x-a)^2(b-x)^2 \\ &\times\left[(x-a)\left(\frac{1}{[4]_{p,q}}\left|{}_aD_{p,q}^2 f(x)\right|+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}_aD_{p,q}^2 f(a)\right|\right)\right. \\ &\left.+(b-x)\left(\frac{1}{[4]_{p,q}}\left|{}^bD_{p,q}^2 f(x)\right|+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}^bD_{p,q}^2 f(b)\right|\right)\right]. \end{aligned} \tag{13}$$

Proof. Taking modulus of 8, applying the convexity of $|{}^bD_{p,q}^2 f|$ and $|{}_aD_{p,q}^2 f|$, and by using Lemma 1, we obtain

$$\begin{aligned} \left|{}_a^bL_{p,q}(x)\right| &\leqslant (x-a)^2(b-x)^2\left[(x-a)\int_0^1 t^2\left|{}_aD_{p,q}^2 f(tx+(1-t)a)\right|\, d_{p,q}t\right. \\ &\quad\left.+(b-x)\int_0^1 t^2\left|{}^bD_{p,q}^2 f(tx+(1-t)b)\right|\, d_{p,q}t\right] \\ &\leqslant (x-a)^2(b-x)^2\left[(x-a)\int_0^1 t^2\left(t\left|{}_aD_{p,q}^2 f(x)\right|+(1-t)\left|{}_aD_{p,q}^2 f(a)\right|\right)\, d_{p,q}t\right. \\ &\quad\left.+(b-x)\int_0^1 t^2\left(t\left|{}^bD_{p,q}^2 f(x)\right|+(1-t)\left|{}^bD_{p,q}^2 f(b)\right|\right)\, d_{p,q}t\right] \\ &= (x-a)^2(b-x)^2\left[(x-a)\left(\frac{1}{[4]_{p,q}}\left|{}_aD_{p,q}^2 f(x)\right|+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}_aD_{p,q}^2 f(a)\right|\right)\right. \\ &\quad\left.+(b-x)\left(\frac{1}{[4]_{p,q}}\left|{}^bD_{p,q}^2 f(x)\right|+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}^bD_{p,q}^2 f(b)\right|\right)\right], \end{aligned}$$

which completes the proof. □

COROLLARY 1. *With the assumptions of Theorem 4, if* $|{}^{b}D_{p,q}^{2}f|$ *and* $|{}_{a}D_{p,q}^{2}f| \leqslant M$*, then the following inequality holds:*

$$\left|{}_{a}^{b}L_{p,q}(x)\right| \leqslant \frac{M(x-a)^2(b-x)^2(b-a)}{[3]_{p,q}}, \tag{14}$$

where M *is constant.*

REMARK 2. If $p=1$ in (13), then we have the following inequality:

$$\begin{aligned}\left|{}_{a}^{b}L_{q}(x)\right| \leqslant (x-a)^2(b-x)^2 \Bigg[&(x-a)\left(\frac{1}{[4]_q}\left|{}_{a}D_{q}^{2}f(x)\right| + \frac{q^3}{[3]_q[4]_q}\left|{}_{a}D_{q}^{2}f(a)\right|\right)\\ &+(b-x)\left(\frac{1}{[4]_q}\left|{}^{b}D_{q}^{2}f(x)\right| + \frac{q^3}{[3]_q[4]_q}\left|{}^{b}D_{q}^{2}f(b)\right|\right)\Bigg],\end{aligned}$$

which appeared in [3].

REMARK 3. If $p=1$ in (14), then we have the following inequality:

$$\left|{}_{a}^{b}L_{q}(x)\right| \leqslant \frac{M(x-a)^2(b-x)^2(b-a)}{[3]_{q}}, \tag{15}$$

which appeared in [3]. Moreover, if $q \to 1$ and $x=(a+b)/2$ in (15), then we obtain the following inequality:

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{M(b-a)^2}{24},$$

which appeared in [42] and it can be found in [39].

THEOREM 5. *Let* $f:[a,b]\to\mathbb{R}$ *be a twice* (p,q)*-differentiable function on* (a,b) *such that* ${}^{b}D_{p,q}^{2}f$ *and* ${}_{a}D_{p,q}^{2}f$ *are continuous and integrable functions on* J_1 *and* J_2*, respectively. If* $|{}^{b}D_{p,q}^{2}f|^r$ *and* $|{}_{a}D_{p,q}^{2}f|^r$ *are convex functions for* $r \geqslant 1$*, then*

$$\begin{aligned}\left|{}_{a}^{b}L_{p,q}(x)\right| \leqslant\ & (x-a)^2(b-x)^2\left(\frac{1}{[3]_{p,q}}\right)^{1-1/r}\\ &\times\Bigg[(x-a)\left(\frac{1}{[4]_{p,q}}\left|{}_{a}D_{p,q}^{2}f(x)\right|^r + \frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}_{a}D_{p,q}^{2}f(a)\right|^r\right)^{1/r}\\ &+(b-x)\left(\frac{1}{[4]_{p,q}}\left|{}^{b}D_{p,q}^{2}f(x)\right|^r + \frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}^{b}D_{p,q}^{2}f(b)\right|^r\right)^{1/r}\Bigg].\end{aligned} \tag{16}$$

Proof. Applying Theorem 3 and the power mean inequality, we obtain

$$\begin{aligned}
\left|{}_a^bL_{p,q}(x)\right| &\leqslant (x-a)^2(b-x)^2\left[(x-a)\int_0^1 t^2\left|{}_aD^2_{p,q}f(tx+(1-t)a)\right|\,d_{p,q}t\right.\\
&\quad\left.+(b-x)\int_0^1 t^2\left|{}^bD^2_{p,q}f(tx+(1-t)b)\right|\,d_{p,q}t\right]\\
&\leqslant (x-a)^2(b-x)^2\\
&\quad\times\left[(x-a)\left(\int_0^1 t^2\,d_{p,q}t\right)^{1-1/r}\left(\int_0^1 t^2\left|{}_aD^2_{p,q}f(tx+(1-t)a)\right|^r\,d_{p,q}t\right)^{1/r}\right.\\
&\quad\left.+(b-x)\left(\int_0^1 t^2\,d_{p,q}t\right)^{1-1/r}\left(\int_0^1 t^2\left|{}^bD^2_{p,q}f(tx+(1-t)b)\right|^r\,d_{p,q}t\right)^{1/r}\right].
\end{aligned}$$

Using Lemma 1 and applying the convexity of $|{}^bD^2_{p,q}f|^r$ and $|{}_aD^2_{p,q}f|^r$, we have

$$\begin{aligned}
&\left|{}_a^bL_{p,q}(x)\right|\\
&\leqslant (x-a)^2(b-x)^2\left[(x-a)\left(\int_0^1 t^2\,d_{p,q}t\right)^{1-1/r}\right.\\
&\quad\times\left(\int_0^1 t^2\left(t\left|{}_aD^2_{p,q}f(x)\right|^r+(1-t)\left|{}_aD^2_{p,q}f(a)\right|^r\right)\,d_{p,q}t\right)^{1/r}\\
&\quad\left.+(b-x)\left(\int_0^1 t^2\,d_{p,q}t\right)^{1-1/r}\left(\int_0^1 t^2\left(t\left|{}^bD^2_{p,q}f(x)\right|^r+(1-t)\left|{}^bD^2_{p,q}f(b)\right|^r\right)\,d_{p,q}t\right)^{1/r}\right]\\
&= (x-a)^2(b-x)^2\left[(x-a)\left(\frac{1}{[3]_{p,q}}\right)^{1-1/r}\right.\\
&\quad\times\left(\frac{1}{[4]_{p,q}}\left|{}_aD^2_{p,q}f(x)\right|^r+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}_aD^2_{p,q}f(a)\right|^r\right)^{1/r}\\
&\quad\left.+(b-x)\left(\frac{1}{[3]_{p,q}}\right)^{1-1/r}\left(\frac{1}{[4]_{p,q}}\left|{}^bD^2_{p,q}f(x)\right|^r+\frac{[4]_{p,q}-[3]_{p,q}}{[3]_{p,q}[4]_{p,q}}\left|{}^bD^2_{p,q}f(b)\right|^r\right)^{1/r}\right],
\end{aligned}$$

which completes the proof. □

REMARK 4. If $p=1$ in (16), then we have the following inequality:

$$\begin{aligned}
\left|{}_a^bL_q(x)\right| &\leqslant (x-a)^2(b-x)^2\left(\frac{1}{[3]_{p,q}}\right)^{1-1/r}\\
&\quad\times\left[(x-a)\left(\frac{1}{[4]_q}\left|{}_aD^2_qf(x)\right|^r+\frac{q^3}{[3]_q[4]_q}\left|{}_aD^2_qf(a)\right|^r\right)^{1/r}\right.\\
&\quad\left.+(b-x)\left(\frac{1}{[4]_q}\left|{}^bD^2_qf(x)\right|^r+\frac{q^3}{[3]_q[4]_q}\left|{}^bD^2_qf(b)\right|^r\right)^{1/r}\right],
\end{aligned}$$

which appeared in [3].

THEOREM 6. *Let $f:[a,b]\to\mathbb{R}$ be a twice (p,q)-differentiable function such that ${}^{b}D^{2}_{p,q}f$ and ${}_{a}D^{2}_{p,q}f$ are continuous and integrable functions on J_1 and J_2, respectively. If $|{}^{b}D^{2}_{p,q}f|^r$ and $|{}_{a}D^{2}_{p,q}f|^r$ are convex functions for $r>1$ and $1/s+1/r=1$, then*

$$\left|{}^{b}_{a}L_{p,q}(x)\right| \leqslant (x-a)^2(b-x)^2\left(\frac{1}{[2s+1]_{p,q}}\right)^{1/s} \times\left[(x-a)\left(\frac{\left|{}_{a}D^{2}_{q}f(x)\right|^r+(p+q-1)\left|{}_{a}D^{2}_{q}f(a)\right|^r}{[2]_{p,q}}\right)^{1/r} +(b-x)\left(\frac{\left|{}^{b}D^{2}_{q}f(x)\right|^r+(p+q-1)\left|{}^{b}D^{2}_{q}f(b)\right|^r}{[2]_{p,q}}\right)^{1/r}\right]. \tag{17}$$

Proof. Applying Theorem 3 and the Hölder's inequality, we have

$$\begin{aligned}\left|{}^{b}_{a}L_{p,q}(x)\right| &\leqslant (x-a)^2(b-x)^2\left[(x-a)\int_0^1 t^2\left|{}_{a}D^{2}_{p,q}f(tx+(1-t)a)\right|\,d_{p,q}t\right.\\&\quad\left.+(b-x)\int_0^1 t^2\left|{}^{b}D^{2}_{p,q}f(tx+(1-t)b)\right|\,d_{p,q}t\right]\\&\leqslant (x-a)^2(b-x)^2\\&\quad\times\left[(x-a)\left(\int_0^1 t^{2s}\,d_{p,q}t\right)^{1/s}\left(\int_0^1\left|{}_{a}D^{2}_{p,q}f(tx+(1-t)a)\right|^r\,d_{p,q}t\right)^{1/r}\right.\\&\quad\left.+(b-x)\left(\int_0^1 t^{2s}\,d_{p,q}t\right)^{1/s}\left(\int_0^1\left|{}^{b}D^{2}_{p,q}f(tx+(1-t)b)\right|^r\,d_{p,q}t\right)^{1/r}\right].\end{aligned}$$

Using Lemma 1 and applying the convexity of $|{}^{b}D^{2}_{p,q}f|^r$ and $|{}_{a}D^{2}_{p,q}f|^r$, we obtain

$$\begin{aligned}\left|{}^{b}_{a}L_{p,q}(x)\right| &\leqslant (x-a)^2(b-x)^2\left[(x-a)\left(\int_0^1 t^{2s}\,d_{p,q}t\right)^{1/s}\right.\\&\quad\times\left(\int_0^1\left(t\left|{}_{a}D^{2}_{p,q}f(x)\right|^r+(1-t)\left|{}_{a}D^{2}_{p,q}f(a)\right|^r\right)\,d_{p,q}t\right)^{1/r}\\&\quad+(b-x)\left(\int_0^1 t^{2s}\,d_{p,q}t\right)^{1/s}\\&\quad\left.\times\left(\int_0^1\left(t\left|{}^{b}D^{2}_{p,q}f(x)\right|^r+(1-t)\left|{}^{b}D^{2}_{p,q}f(b)\right|^r\right)\,d_{p,q}t\right)^{1/r}\right]\end{aligned}$$

$$= (x-a)^2(b-x)^2$$
$$\times \left[(x-a)\left(\frac{1}{[2s+1]_{p,q}}\right)^{1/s} \left(\frac{\left|\,{}_aD_q^2 f(x)\right|^r + (p+q-1)\left|\,{}_aD_q^2 f(a)\right|^r}{[2]_{p,q}}\right)^{1/r} \right.$$
$$\left. + (b-x)\left(\frac{1}{[2s+1]_{p,q}}\right)^{1/s} \left(\frac{\left|\,{}^bD_q^2 f(x)\right|^r + (p+q-1)\left|\,{}^bD_q^2 f(b)\right|^r}{[2]_{p,q}}\right)^{1/r} \right],$$

which completes the proof. □

REMARK 5. If $p = 1$ in (17), then we have the following inequality:

$$\left|{}_a^bL_q(x)\right|$$
$$\leqslant (x-a)^2(b-x)^2\left(\frac{1}{[2s+1]_q}\right)^{1/s}\left[(x-a)\left(\frac{\left|\,{}_aD_q^2 f(x)\right|^r + q\left|\,{}_aD_q^2 f(a)\right|^r}{[2]_q}\right)^{1/r}\right.$$
$$\left. +(b-x)\left(\frac{\left|\,{}^bD_q^2 f(x)\right|^r + q\left|\,{}^bD_q^2 f(b)\right|^r}{[2]_q}\right)^{1/r}\right],$$

which appeared in [3].

4. Conclusions

In this work, we established a new (p,q)-integral identity using the second $(p,q)_a$- and $(p,q)^b$-derivatives. Then, we used this result to derive some new post-quantum Ostrowski type integral inequalities for twice (p,q)-differentiable functions. The main results in this study were proven to be generalizations of some previously proved results of quantum Ostrowski type integral inequalities for twice q-differentiable functions. Researchers can obtain similar inequalities in future works by using (p,q)-fractional calculus.

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Waewta Luangboon
Department of Mathematics
Faculty of Science, Khon Kaen University
Khon Kaen 40002, Thailand
e-mail: waewta_l@kkumail.com

Kamsing Nonlaopon
Department of Mathematics
Faculty of Science, Khon Kaen University
Khon Kaen 40002, Thailand
e-mail: nkamsi@kku.ac.th

Jessada Tariboon
Department of Mathematics, Faculty of Applied Science
King Mongkut's University of Technology North Bangkok
Bangkok 10800, Thailand
e-mail: jessada.t@sci.kmutnb.ac.th

Sortiris K. Ntouyas
Department of Mathematics
University of Ioannina
451 10 Ioannina, Greece
and
Nonlinear Analysis and Applied Mathematics (NAAM) – Research Group
Department of Mathematics
Faculty of Science, King Abdulaziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: sntouyas@uoi.gr

Hüseyin Budak
Department of Mathematics
Faculty of Science and Arts, Düzce University
Düzce 81620, Turkey
e-mail: hsyn.budak@gmail.com