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NORM INEQUALITIES RELATED TO HEINZ AND LOGARITHMIC MEANS

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Abstract. In this paper, we got some refinements of the norm inequalities related to the Heinz mean and logarithmic mean.

1. Introduction

There are several means that interpolate between the geometric and arithmetic means. For instance, the Heinz mean $H_t(a,b)$, defined by

$$H_t(a,b) = \frac{a^{1-t}b^t + a^t b^{1-t}}{2} \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

In 1993, Bhatia-Davis [2] obtained that if A,B and X are $n \times n$ matrices with A,B positive semidefinite, then for every unitarily invariant norm $|||\cdot|||$,

$$\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| \leqslant \frac{1}{2}|||A^{1-t}XB^t + A^tXB^{1-t}||| \leqslant \frac{1}{2}|||AX+XB|||. \tag{1.1}$$

The logarithmic mean $L(a,b)$, defined by

$$L(a,b) = \frac{a-b}{\log a - \log b} = \int_0^1 a^t b^{1-t}\,dt,$$

also interpolates the geometric and arithmetic means. In 1999, Hiai-Kosaki [5] proved the following inequality

$$\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|\right|\right| \leqslant \left|\left|\left|\int_0^1 A^vXB^{1-v}\,dv\right|\right|\right| \leqslant \frac{1}{2}|||AX+XB|||. \tag{1.2}$$

Moreover, in 2006, Drissi [4] proved that the following Heinz-logarithmic inequality

$$|||A^{1-t}XB^t + A^tXB^{1-t}||| \leqslant 2\left|\left|\left|\int_0^1 A^vXB^{1-v}\,dv\right|\right|\right| \tag{1.3}$$

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holds for $\frac{1}{4} \leqslant t \leqslant \frac{3}{4}$.

A complex-valued function φ on $\mathbb{R}$ is said to be positive definite if the matrix $[\varphi(x_i - x_j)]$ is positive semidefinite for all choices of real numbers $x_1, x_2, \ldots, x_n$, and $n = 1, 2, \ldots$. Let $M(a,b)$ and $N(a,b)$ be two symmetric homogeneous means on $(0,\infty) \times (0,\infty)$. M is said to strongly dominate N, denoted by $M \ll N$, if and only if the matrix

$$\left[\frac{M(\lambda_i, \lambda_j)}{N(\lambda_i, \lambda_j)}\right]_{i,j=1,\ldots,n}$$

is positive semidefinite for any size n and $\lambda_1, \ldots, \lambda_n > 0$. Drissi [4] proved that for $a, b \geqslant 0$, $H_t(a,b) \ll L(a,b)$ if and only if $\frac{1}{4} \leqslant t \leqslant \frac{3}{4}$. In general, the inequality $M \ll N$ is stronger than the Löwner's order inequality $M \leqslant N$.

For more operator or norm inequalities related to the Heinz mean and logarithmic mean we refer the readers to [8, 9, 7] and the references therein.

2. Main results

LEMMA 2.1. *For* $\sinh x$ *and* $\cosh x$*, we have*

(i) If $|\beta| > |\alpha| > 0$*, then the function* $\frac{\cosh \alpha x}{\cosh \beta x}$ *is positive definite.*

(ii) If $|\beta| > |\alpha| > 0$ *with* α, β *the same sign, then* $\frac{\sinh \alpha x}{\sinh \beta x}$ *is positive definite.*

(iii) If $\beta > 0$ *and* $|\alpha| < \beta/2$*, then* $\frac{\beta x \cosh \alpha x}{\sinh \beta x}$ *is positive definite.*

Proof. We follow a similar argument as in Chapter 5 of [1]. From the product representations in p. 147–148 of [1],

$$\frac{sinhx}{x} = \prod_{k=1}^{\infty}\left(1 + \frac{x^2}{k^2\pi^2}\right), \qquad \cosh x = \prod_{k=0}^{\infty}\left(1 + \frac{4x^2}{(2k+1)^2\pi^2}\right), \tag{2.1}$$

we have

$$\frac{\sinh \alpha x}{\sinh \beta x} = \frac{\alpha}{\beta}\prod_{k=1}^{\infty}\frac{1+\alpha^2x^2/k^2\pi^2}{1+\beta^2x^2/k^2\pi^2}, \qquad \frac{\cosh \alpha x}{\cosh \beta x} = \prod_{k=0}^{\infty}\frac{1+4\alpha^2x^2/(2k+1)^2\pi^2}{1+4\beta^2x^2/(2k+1)^2\pi^2}. \tag{2.2}$$

Each factor in the product is of the form

$$\frac{1+a^2x^2}{1+b^2x^2} = \frac{a^2}{b^2} + \frac{1-a^2/b^2}{1+b^2x^2}, \quad b^2 > a^2.$$

Since $1/(1+b^2x^2)$ is positive definite [1, 5.2.7], it follows that the function $\cosh \alpha x / \cosh \beta x$ is positive definite for $|\beta| > |\alpha| > 0$, and $\sinh \alpha x / \sinh \beta x$ is positive definite for $|\beta| > |\alpha| > 0$ with α, β the same sign.

Since

$$\frac{\beta x \cosh \alpha x}{\sinh \beta x} = \frac{\frac{\beta}{2}x}{\sinh \frac{\beta}{2}x} \cdot \frac{\cosh \alpha x}{\cosh \frac{\beta}{2}x}$$

and $x/\sinh x$ is positive definite [1, 5.2.9], it follows from the above argument that when $\frac{\beta}{2} > |\alpha|$ the function $\frac{\beta x \cosh \alpha x}{\sinh \beta x}$ is positive definite. □

Now, we define

$$L_s(a,b) = \frac{a^{1-s}b^s - a^s b^{1-s}}{(1-2s)(\log a - \log b)} = \frac{1}{1-2s}\int_s^{1-s} a^v b^{1-v} dv$$

for $a,b > 0$, and $0 \leqslant s < \frac{1}{2}$. When $s = 0$, it is the logarithmic mean. So we can call it the generalized logarithmic mean. And we also have $\lim_{s\to\frac{1}{2}} L_s(a,b) = a^{\frac{1}{2}} b^{\frac{1}{2}}$.

THEOREM 2.2. *For Heinz mean and the generalized logarithmic mean, we have*

(i) If $0 \leqslant s < 1/2$, *and* $|1-2t| < \frac{1-2s}{2}$, *then* $H_t(a,b) \ll L_s(a,b)$.

(ii) If $|1-2t| < |1-2s|$, *then* $H_t(a,b) \ll H_s(a,b)$.

(iii) If $0 \leqslant s < t < 1/2$, *or* $1 \geqslant s > t > 1/2$, *then* $L_t(a,b) \ll L_s(a,b)$.

Proof. By definition, $H_t(a,b) \ll L_s(a,b)$ if

$$[y_{i,j}] = \left[\frac{H_t(\lambda_i,\lambda_j)}{L_s(\lambda_i,\lambda_j)}\right]_{i,j=1,\ldots,n}$$

is positive semidefinite. Set $\lambda_i = e^{x_i}$ and $\lambda_j = e^{x_j}$, with $x_i, x_j \in \mathbb{R}$. Then

$$y_{i,j} = (1-2s)\frac{(\frac{x_i-x_j}{2})(e^{(1-2t)\frac{x_i-x_j}{2}} + e^{(1-2t)\frac{x_j-x_i}{2}})}{e^{(1-2s)\frac{x_i-x_j}{2}} - e^{(1-2s)\frac{x_j-x_i}{2}}}$$

Thus the matrix $[y_{i,j}]$ is congruent to one with entries

$$\frac{\beta(\frac{x_i-x_j}{2})\cosh(\alpha(\frac{x_i-x_j}{2}))}{\sinh(\beta(\frac{x_i-x_j}{2}))},$$

where $\alpha = 1-2t$, $\beta = 1-2s$. Hence $[y_{i,j}]$ is positive semidefinite if and only if the function $\frac{\beta x \cosh \alpha x}{\sinh \beta x}$ is positive definite, which by lemma 2.1 is correct.

Similarly, we have $H_t(a,b) \ll H_s(a,b)$ if $\frac{\cosh \alpha x}{\cosh \beta x}$ is positive definite, and $L_t(a,b) \ll L_s(a,b)$ if $\frac{\sinh \alpha x}{\sinh \beta x}$ is positive definite. □

THEOREM 2.3. *Let A,B be any positive matrices. Then for any matrix X and for* $0 \leqslant s < 1/2$ *and* $|1-2t| < (1-2s)/2$, *we have*

$$|||A^{1-t}XB^t + A^t XB^{1-t}||| \leqslant \frac{2}{1-2s}\left|\left|\left|\int_s^{1-s} A^v X B^{1-v} dv\right|\right|\right|. \tag{2.3}$$

Proof. Firstly, we assume $A=B$. Since $|||\cdot|||$ is unitarily invariant, we may suppose A is diagonal with entries $\lambda_1,\ldots,\lambda_n$. Then we have

$$A^{1-t}XA^t+A^tXA^{1-t}=Y\circ\left(\int_s^{1-s}A^vXA^{1-v}dv\right),$$

where Y is the matrix with entries

$$y_{i,j}=\frac{\lambda_i^t\lambda_j^{1-t}+\lambda_i^{1-t}\lambda_j^t}{\frac{\lambda_i^{1-s}\lambda_j^s-\lambda_i^s\lambda_j^{1-s}}{\log\lambda_i-\log\lambda_j}}=\frac{2H_t(\lambda_i,\lambda_j)}{(1-2s)L_s(\lambda_i,\lambda_j)}.$$

A well-known result related to the Schur multiplier norm [6, Theorem 5.5.18, 5.5.19] says that if Y is any positive semidefinite matrix, then for all matrix X,

$$|||Y\circ X|||\leqslant\max_i y_{ii}|||X||| \tag{2.4}$$

for every unitarily invariant norm. By Theorem 2.2, Y is positive semidefinite. Applying (2.4), we have

$$|||A^{1-t}XA^t+A^tXA^{1-t}|||\leqslant\frac{2}{1-2s}\left|\left|\left|\int_s^{1-s}A^vXA^{1-v}dv\right|\right|\right|. \tag{2.5}$$

Now replacing A and X in the inequality (2.5) by the 2 by 2 matrices $\begin{pmatrix}A&0\\0&B\end{pmatrix}$ and $\begin{pmatrix}0&X\\0&0\end{pmatrix}$. This gives the desired inequality (2.3). □

When $s=0$, we get Drissi's result (1.3). Moreover, when $s=0, t=1/2$, we get the first inequality of (1.2).

THEOREM 2.4. *Let A,B be any positive matrices. Then for any matrix X and for $0\leqslant s<1/2$, we have*

$$\left|\left|\left|\int_s^{1-s}A^vXB^{1-v}dv\right|\right|\right|\leqslant\frac{1-2s}{2}|||A^{1-s}XB^s+A^sXB^{1-s}|||. \tag{2.6}$$

Proof. Suppose A is diagonal with entries $\lambda_1,\ldots,\lambda_n$. Then we have

$$\int_s^{1-s}A^vXA^{1-v}dv=Y\circ(A^{1-s}XA^s+A^sXA^{1-s}),$$

where Y is the matrix with entries

$$y_{i,j}=\frac{\frac{\lambda_i^{1-s}\lambda_j^s-\lambda_i^s\lambda_j^{1-s}}{\log\lambda_i-\log\lambda_j}}{\lambda_i^s\lambda_j^{1-s}+\lambda_i^{1-s}\lambda_j^s}=\frac{(1-2s)L_s(\lambda_i,\lambda_j)}{2H_s(\lambda_i,\lambda_j)}.$$

By a similar argument as in the proof of Theorem 2.2, we know that Y is positive definite if and only if

$$\frac{\beta}{2}\frac{\sinh\beta x}{\beta x\cosh\beta x}=\frac{\beta}{2}\frac{\tanh\beta x}{\beta x}$$

is positive definite for $\beta=1-2s>0$, because $\tanh x/x$ is positive definite (see Bhatia [1, 5.2.11]). Thus Applying (2.4) we have

$$\left|\left|\left|\int_s^{1-s}A^vXA^{1-v}dv\right|\right|\right|\leqslant\frac{1-2s}{2}|||A^{1-s}XA^s+A^sXA^{1-s}|||. \tag{2.7}$$

Hence the desired result follows. □

When $s=0$, we get the second inequality of (1.2).

THEOREM 2.5. *Let A,B be any positive matrices. Then for any matrix X and for $|1-2t|<|1-2s|$ with $1-2t,1-2s$ the same sign, we have*

$$|||A^{1-t}XB^t-A^tXB^{1-t}|||\leqslant\left|\frac{1-2t}{1-2s}\right||||A^{1-s}XB^s-A^sXB^{1-s}|||. \tag{2.8}$$

Proof. Suppose A is diagonal with entries $\lambda_1,\dots,\lambda_n$. Then we have

$$A^{1-t}XA^t-A^tXA^{1-t}=Y\circ(A^{1-s}XA^s-A^sXA^{1-s}),$$

where Y is the matrix with entries

$$y_{i,j}=\frac{\lambda_i^{1-t}\lambda_j^t-\lambda_i^t\lambda_j^{1-t}}{\lambda_i^{1-s}\lambda_j^s-\lambda_i^s\lambda_j^{1-s}}.$$

Put $\lambda_i=e^{x_i}$ and $\lambda_j=e^{x_j}$, with $x_i,x_j\in\mathbb{R}$. Then Y is congruent to the matrix with entries

$$\frac{\sinh(\alpha\frac{x_i-x_j}{2})}{\sinh(\beta\frac{x_i-x_j}{2})},$$

where $\alpha=1-2t,\beta=1-2s$. Since $(\sinh\alpha x)/(\sinh\beta x)$ is positive for $|\beta|>|\alpha|>0$ with α,β the same sign, it follows that Y is positive definite. Applying the inequality (2.4), we have

$$|||A^{1-t}XA^t-A^tXA^{1-t}|||\leqslant\left|\frac{1-2t}{1-2s}\right||||A^{1-s}XA^s-A^sXA^{1-s}|||. \tag{2.9}$$

Hence the desired result follows. □

Set $s=0$ and $s=1$, and combine the conclusions, we have the following inequality proved by Bhatia-Davis [3], COROLLARY 2.6. *Let A,B be any positive matrices. Then for any matrix X and for $0 \leqslant t \leqslant 1$ we have*

$$|||A^{1-t}XB^t - A^tXB^{1-t}||| \leqslant |1-2t| |||AX - XB|||. \tag{2.10}$$

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