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SOME REFINEMENTS OF YOUNG TYPE INEQUALITIES

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Abstract. In this paper, we study some improvements of Young type inequalities. We obtain some reverse improvements of Young inequality. Among other results, we show a refined Young inequality. For $v \in [0,1]$ and $a,b > 0$, then

$$a\nabla_v b \leqslant M_v^R(h)a\sharp_v b,$$

where $M_v(t) = 1 + v(1-v)\frac{(t-1)^2}{t}$ and $h = \frac{b}{a}$. And we also obtain a new Young type inequality. Furthermore, corresponding operator inequalities are also established.

1. Introduction

The weighted arithmetic-geometric mean inequality, which is also called Young inequality, states that

$$(1-v)a + vb \geqslant a^{1-v}b^v$$

for $a,b \geqslant 0$ and $v \in [0,1]$.

The Heinz mean is defined by

$$H_v(a,b) = \frac{a^{1-v}b^v + a^v b^{1-v}}{2}$$

for $a,b \geqslant 0$ and $v \in [0,1]$.

Recently, a number of refinements for Young inequality are studied. The refinement in [4] of Young inequality was proved by Kittaneh and Manasrah, which can be stated in the following form

$$va + (1-v)b \leqslant a^v b^{1-v} + R(\sqrt{a} - \sqrt{b})^2, \tag{1}$$

where $R = \max\{v, 1-v\}$.

In [9, 5], Zuo-Shi-Fujii and Liao-Wu-Zhao obtained respectively the refinement and reverse refinement of Young inequality with Kantorovich constant $K(h) = \frac{(h+1)^2}{4h}$, $(h > 0)$. If $v \in [0,1]$ and $t > 0$, then

$$K^r(t) \leqslant \frac{(1-v) + vt}{t^v} \leqslant K^R(t), \tag{2}$$

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where $r = \min\{v, 1-v\}$ and $R = \max\{v, 1-v\}$.

Dragomir established the following refinement of Young inequality in [1]

$$\frac{(1-v)+vt}{t^v} \leqslant exp\Big(v(1-v)\frac{(t-1)^2}{t}\Big)$$

for $t > 0$ and $v \in [0,1]$.

Furuichi focused on the refinement for Young inequality by Dragomir and got the following lemma.

LEMMA 1.1. [2] *For $t > 0$ and $v \in [0,1]$,*

$$\frac{(1-v)+vt}{t^v} \leqslant 1 + v(1-v)\frac{(t-1)^2}{t}.$$

Recently, Ghazanfari, Malekinejad and Talebi in [3] gave a new inequality, which can be stated that if $a,b \geqslant 0$ and $v \in (0,1]$, then

$$(1-v^2+v^3)a + (1-v^2)b \leqslant v^{v-2}a^v b^{1-v} + (\sqrt{a}-\sqrt{b})^2. \tag{3}$$

In [6], Ren proved a new Young type inequality

$$(1-v^{N+1}+v^{N+2})a + (1-v^2)b \leqslant v^{vN-(N+1)}a^v b^{1-v} + (\sqrt{a}-\sqrt{b})^2, \tag{4}$$

where $v \in (0,1]$, $N \in \mathbb{N}$ and $a,b \geqslant 0$. It's obvious that (3) is a special case of inequality (4) for $N = 1$, which implies that (4) is a generalization of (3).

In 2020, Yang and Li [8] studied an improvement of inequality (3), they obtained the following inequality,

$$(1-v^{N_1+1}+v^{N_1+2})a + (1-v^{N_2+2})b \leqslant v^{-(1-v)N_1-vN_2-1}a^v b^{1-v} + (\sqrt{a}-\sqrt{b})^2, \tag{5}$$

where $v \in (0,1]$, $N_1, N_2 \in \mathbb{N}$ and $a,b \geqslant 0$. It's obvious that (3) is a special case of inequality (5) for $N_1 = 1$, $N_2 = 0$, which implies that (5) is a generalization of (3). And they also gave an inequality for $N_1 = N_2$. As follows

$$(1-v^{N+1}+v^{N+2})a + (1-v^{N+2})b \leqslant v^{-N-1}a^v b^{1-v} + (\sqrt{a}-\sqrt{b})^2.$$

In this paper, our main task is to study improvements of Young type inequalities. In Theorem 2.1, we obtain a tighter upper bound than that of inequality (2). Theorem 2.12 is a new Young type inequality, which is another generalization of inequality (3). In section 3, modified inequalities are used to establish corresponding operator inequalities.

2. Main results

In this section, we present the numerical inequalities needed to prove the operator versions. This section is divided into two subsections. In the first subsection, we refine Young inequality based on the result of Furuichi. In the other subsection, we show a new Young type inequality.

2.1. Refined reverse Young inequalities

We now state and prove the main theorem, which will be frequently used in this paper.

THEOREM 2.1. *Suppose* $v \in [0,1]$ *and* $t > 0$, *then*

$$\frac{(1-v)+vt}{t^v} \leqslant \left[1+v(1-v)\frac{(t-1)^2}{t}\right]^R, \tag{6}$$

where $R = \max\{v, 1-v\}$.

Proof. Consider the function

$$f(t) = \log[(1-v)+vt] - v\log t - R\log\left[1+v(1-v)\frac{(t-1)^2}{t}\right].$$

$$\begin{aligned}
f'(t) &= \frac{v}{(1-v)+vt} - \frac{v}{t} - \frac{Rv(1-v)(1-\frac{1}{t^2})}{1+v(1-v)\frac{(t-1)^2}{t}} \\
&= \frac{v(1-v)(t-1)}{[(1-v)+vt]t} - \frac{Rv(1-v)(1-\frac{1}{t^2})}{1+v(1-v)\frac{(t-1)^2}{t}} \\
&= \frac{v(1-v)(t-1)[1+v(1-v)\frac{(t-1)^2}{t}] - Rv(1-v)(1-\frac{1}{t^2})[(1-v)+vt]t}{[(1-v)+vt]t[1+v(1-v)\frac{(t-1)^2}{t}]} \\
&= \frac{v(1-v)(t-1)\{1+v(1-v)(t+\frac{1}{t}-2) - R(\frac{1}{t}+\frac{1}{t^2})[(1-v)+vt]t\}}{[(1-v)+vt]t[1+v(1-v)\frac{(t-1)^2}{t}]}.
\end{aligned}$$

Let

$$g(t) = 1+v(1-v)\Big(t+\frac{1}{t}-2\Big) - R\Big(1+\frac{1}{t}\Big)[(1-v)+vt].$$

(i) If $v \in [0, \frac{1}{2}]$, then $R = 1-v$ and

$$g(t) = v(2v-1) + \frac{(1-v)(2v-1)}{t} \leqslant 0.$$

(ii) If $v \in [\frac{1}{2}, 1]$, then $R = v$ and

$$g(t) = v(1-2v)t + (1-2v)(1-v) \leqslant 0.$$

Hence, $g(t) \leqslant 0$ for $v \in [0,1]$. If $t \in (0,1]$, $f'(t) \geqslant 0$, $f(t)$ is an increasing function for $t \in (0,1]$. And if $t \in [1,\infty)$, $f'(t) \leqslant 0$, $f(t)$ is a decreasing function for $t \in [1,\infty)$. When $t=1$, $f(t)$ attains the maximum value, $f(1)=0$. Therefore,

$$\frac{(1-v)+vt}{t^v} \leqslant \left[1+v(1-v)\frac{(t-1)^2}{t}\right]^R. \quad \square$$

REMARK 2.2. It is easy to see that $1+v(1-v)\frac{(t-1)^2}{t} \leqslant 1+\frac{(t-1)^2}{4t} = K(t)$. As we noted in Introduction, the inequalities (2) are known. Therefore, when $t>0$ and $v \in [0,1]$, inequality (6) of Theorem 2.1 gives a tighter upper bound than that of inequality (2).

The following remark states some characterizations of $1+v(1-v)\frac{(t-1)^2}{t}$, which are similar to that of Kantorovich constant $K(t)$.

REMARK 2.3. Let $M_v(t) = 1+v(1-v)\frac{(t-1)^2}{t}$ for $t>0$ and $v \in [0,1]$. By simple calculation, we have

$$\begin{aligned} M_v(t) &= 1+v(1-v)\frac{(t-1)^2}{t} \\ &= 1+v(1-v)\frac{(1-t)^2}{t} \\ &= 1+v(1-v)\frac{(\frac{1}{t}-1)^2}{\frac{1}{t}} \\ &= M_v\Big(\frac{1}{t}\Big). \\ M_v'(t) &= v(1-v)\Big(1-\frac{1}{t^2}\Big), \end{aligned}$$

then if $t \in (0,1]$, $M_v'(t) \leqslant 0$ and if $t \in [1,\infty)$, $M_v'(t) \geqslant 0$. Therefore, $M_v(t)$ has the following properties:

(1) $M_v(t) = M_v(\frac{1}{t})$.

(2) $M_v(t)$ is decreasing for $t \in (0,1]$ and $M_v(t)$ is increasing for $t \in [1,\infty)$. And $M_v(1) = 1$.

(3) $M_v(t) = M_{1-v}(t)$.

Therefore, we have

$$a\nabla_v b \leqslant M_v^R(h)\, a\sharp_v b \tag{7}$$

for $a,b>0$ and $v \in [0,1]$. According to inequality (7), we can also get geometric-harmonic mean inequality and arithmetic-Heinz mean inequality, which is respectively an improvement of previous corresponding inequality.

COROLLARY 2.4. *Suppose* $a,b>0$ *and* $v \in [0,1]$ *, then*

$$a\sharp_v b \leqslant M_v^R(h)\, a!_v b,$$

where $R = \max\{v, 1-v\}$ *and* $h = \frac{b}{a}$.

Proof. Using inequality (7) and replacing a with $\frac{1}{a}$ and b with $\frac{1}{b}$, we can get the desired inequality. □

COROLLARY 2.5. *Suppose $a,b>0$ and $v\in[0,1]$, then*

$$a\nabla b \leqslant M_v^R(h)H_v(a,b),$$

where $R=\max\{v,1-v\}$ and $h=\frac{b}{a}$.

Proof. Applying inequality (7) and Remark 2.3, we can get the desired inequality. □

The following theorem can be obtained by iteration of inequality (7).

THEOREM 2.6. *Suppose $a,b>0$, $v\in[0,1]$ and $h=\sqrt{\frac{b}{a}}$, then*

$$(1-v)a+vb-r(\sqrt{a}-\sqrt{b})^2 \leqslant M_{r'}^{R'}(h)a^{1-v}b^v, \tag{8}$$

where $M_v(h)=1+\frac{v(1-v)(h-1)^2}{h}$, $r=\min\{v,1-v\}$, $r'=\min\{2r,1-2r\}$ and $R'=\max\{2r,1-2r\}$.

Proof. (i) If $v\in[0,\frac{1}{2}]$, using inequality (7) and Remark 2.3, by simple calculation, we can get

$$\begin{aligned}(1-v)a+vb-v(\sqrt{a}-\sqrt{b})^2 =&(1-2v)a+2v\sqrt{ab}\\ \leqslant& M_{2v}^{R'}(h)a^{1-v}b^v=M_{1-2v}^{R'}(h)a^{1-v}b^v.\end{aligned}$$

(ii) If $v\in[\frac{1}{2},1]$, using inequality (7) and Remark 2.3, by simple calculation, we have

$$\begin{aligned}(1-v)a+vb-(1-v)(\sqrt{a}-\sqrt{b})^2 =&(2v-1)b+2(1-v)\sqrt{ab}\\ \leqslant& M_{2v-1}^{R'}(h)a^{1-v}b^v=M_{2-2v}^{R'}(h)a^{1-v}b^v.\end{aligned}$$

To sum up, we get

$$(1-v)a+vb-r(\sqrt{a}-\sqrt{b})^2 \leqslant M_{r'}^{R'}(h)a^{1-v}b^v. \quad \square$$

It is the most common method of reverse improvement of Young inequality that geometric mean adds one term is larger than arithmetic mean. Can the form of inequality (8) be extended to n terms? To get the result we want, the following lemma is needed.

LEMMA 2.7. [7] *Let $a,b>0$ and $v\in[0,1]$. Given $N\in\mathbb{N}$, consider the integers $k_j=[2^{j-1}v]$ and $r_j=[2^jv]$, $j=1,2,\ldots,N$. Then*

$$\begin{aligned}&(1-v)a+vb-\sum_{j=1}^{N}\left((-1)^{r_j}2^{j-1}v+(-1)^{r_j+1}\left[\frac{r_j+1}{2}\right]\right)\\ &\times\left(\sqrt[2^j]{a^{2^{j-1}-k_j}b^{k_j}}-\sqrt[2^j]{b^{k_j+1}a^{2^{j-1}-k_j-1}}\right)^2\\ =&([2^Nv]+1-2^Nv)\sqrt[2^N]{a^{2^N-[2^Nv]}b^{[2^Nv]}}+(2^Nv-[2^Nv])\sqrt[2^N]{b^{[2^Nv]+1}a^{2^N-[2^Nv]-1}}.\end{aligned}$$

THEOREM 2.8. *Let* $a,b>0$ *and* $v\in[0,1]$. *Consider the integers* $k_j=[2^{j-1}v]$ *and* $r_j=[2^jv]$ *for* $j=1,2,\dots,N,\ N\in\mathbb{N}$. *Suppose* $a_N=[2^Nv]+1-2^Nv$ *and* $h_N=\sqrt[2^N]{\frac{a}{b}}$, *then*

$$\begin{aligned}&\Big[1+a_N(1-a_N)\frac{(h_N-1)^2}{h_N}\Big]^{R_N}a^{1-v}b^v\\&+\sum_{j=1}^{N}\Big((-1)^{r_j}2^{j-1}v+(-1)^{r_j+1}\Big[\frac{r_j+1}{2}\Big]\Big)\big(\sqrt[2^j]{a^{2^{j-1}-k_j}b^{k_j}}-\sqrt[2^j]{b^{k_j+1}a^{2^{j-1}-k_j-1}}\big)^2\\&\geqslant(1-v)a+vb.\end{aligned}\tag{9}$$

where $R_N=\max\{a_N,1-a_N\}$.

Proof. Applying Lemma 2.7 and inequality (7), we have

$$\begin{aligned}&(1-v)a+vb-\sum_{j=1}^{N}\Big((-1)^{r_j}2^{j-1}v+(-1)^{r_j+1}\Big[\frac{r_j+1}{2}\Big]\Big)\\&\quad\times\big(\sqrt[2^j]{a^{2^{j-1}-k_j}b^{k_j}}-\sqrt[2^j]{b^{k_j+1}a^{2^{j-1}-k_j-1}}\big)^2\\&=([2^Nv]+1-2^Nv)\sqrt[2^N]{a^{2^N-[2^Nv]}b^{[2^Nv]}}+(2^Nv-[2^Nv])\sqrt[2^N]{b^{[2^Nv]+1}a^{2^N-[2^Nv]-1}}\\&\leqslant\Big[1+a_N(1-a_N)\frac{(h_N-1)^2}{h_N}\Big]^{R_N}\big(\sqrt[2^N]{a^{2^N-[2^Nv]}b^{[2^Nv]}}\big)^{[2^Nv]+1-2^Nv}\\&\quad\times\big(\sqrt[2^N]{b^{[2^Nv]+1}a^{2^N-[2^Nv]-1}}\big)^{2^Nv-[2^Nv]}\\&=\Big[1+a_N(1-a_N)\frac{(h_N-1)^2}{h_N}\Big]^{R_N}a^{1-v}b^v.\end{aligned}$$

Then, we get the desired inequality. □

Next, we will discuss the relationship between Young inequality and generalized exponential function. Throughout this subsection, we use the generalized exponential function defined by $exp_r(x)=(1+rx)^{\frac{1}{r}}$ for $x>0$ and $-1\leqslant r\leqslant 1$ with $r\neq 0$ under the assumption that $1+rx\geqslant 0$. To achieve further result, we need the following lemma (Lemma 2.9).

LEMMA 2.9. [2] *The function* $exp_r(x)$ *defined for* $x>0$ *and* $0<r\leqslant 1$ *or* $0\leqslant x\leqslant 1$ *and* $-1\leqslant r<0$, *is monotone decreasing in* r.

LEMMA 2.10. [2] *For* $t>0,\ 0\leqslant v\leqslant 1$ *and* $0<r\leqslant 1$, *then*

$$\frac{(1-v)+vt}{t^v}\leqslant exp_r\Big(v(1-v)\frac{(t-1)^2}{t}\Big).$$

COROLLARY 2.11. *For* $t>0,\ 0\leqslant v\leqslant 1$ *and* $0<r\leqslant 1$, *then*

$$\frac{(1-v)+vt}{t^v}\leqslant\Big(exp_r(v(1-v)\frac{(t-1)^2}{t})\Big)^R,$$

where $R = \max\{v, 1-v\}$.

Proof. Using Lemma 2.9 and Theorem 2.1, we have

$$\begin{aligned}(exp_r(v(1-v)\frac{(t-1)^2}{t}))^R &\geqslant \left(exp_1\left(v(1-v)\frac{(t-1)^2}{t}\right)\right)^R \\ &= \left[1+v(1-v)\frac{(t-1)^2}{t}\right]^R \\ &\geqslant \frac{(1-v)+vt}{t^v}. \quad \square\end{aligned}$$

It is not difficult to see that Corollary 2.11 is weaker than Theorem 2.1, while Corollary 2.11 is stronger than Lemma 2.10.

2.2. Young type inequality

In this part, we will further generalize inequality (3) based on inequality (4) and (5). We try to turn inequality (5) with two variables (N_1, N_2) into a new inequality with one variable. The result is as follows.

THEOREM 2.12. *Suppose that* $a, b \geqslant 0$, $N \in \mathbb{N}$ *and* $v \in (0,1]$, *then*

$$(1-v^{N+1}+v^{N+2})a+(1-v^{N+1})b \leqslant v^{-N-1+v}a^v b^{1-v}+(\sqrt{a}-\sqrt{b})^2. \tag{10}$$

Proof. By simple calculation, we have

$$\begin{aligned}&v^{-N-1+v}a^v b^{1-v}+(\sqrt{a}-\sqrt{b})^2-(1-v^{N+1}+v^{N+2})a-(1-v^{N+1})b \\ =&v^{-N-1+v}a^v b^{1-v}-2\sqrt{ab}+(1-v)v^{N+1}a+vv^N b \\ \geqslant&v^{-N-1+v}a^v b^{1-v}-2\sqrt{ab}+(v^{N+1}a)^{1-v}(v^N b)^v \\ =&v^{-N-1+v}a^v b^{1-v}-2\sqrt{ab}+v^{N+1-v}a^{1-v}b^v \\ =&(v^{\frac{-N-1+v}{2}}a^{\frac{v}{2}}b^{\frac{1-v}{2}}-v^{\frac{N+1-v}{2}}a^{\frac{1-v}{2}}b^{\frac{v}{2}})^2 \\ \geqslant&0.\end{aligned}$$

Therefore

$$(1-v^{N+1}+v^{N+2})a+(1-v^{N+1})b \leqslant v^{-N-1+v}a^v b^{1-v}+(\sqrt{a}-\sqrt{b})^2. \quad \square$$

REMARK 2.13. It's obvious that (3) is a special case of inequality (10) for $N = 1$, which implies that (10) is a generalization of (3). And for any N, it's not difficult to find that both the left hand side and the right hand side in inequality (10) are greater than or equal to the corresponding sides in inequalities (1) and (3) respectively, which indicates that inequality (10) can be regarded as a new Young type inequality.

The following theorem presents a multiplicative refinement of inequality (10).

THEOREM 2.14. *Suppose that* $a,b \geqslant 0$, $N \in \mathbb{N}$ *and* $v \in (0,1]$, *then*

$$(1-v^{N+1}+v^{N+2})a+(1-v^{N+1})b \leqslant K^{-r}(h)v^{-N-1+v}a^{v}b^{1-v}+(\sqrt{a}-\sqrt{b})^{2}, \quad (11)$$

where $r=\min\{v,1-v\}$ *and* $h=\frac{va}{b}$.

Proof. By simple calculation and using the first inequality of (2), we have

$$\begin{aligned}
&K^{-r}(h)v^{-N-1+v}a^{v}b^{1-v}+(\sqrt{a}-\sqrt{b})^{2}-(1-v^{N+1}+v^{N+2})a-(1-v^{N+1})b\\
=&K^{-r}(h)v^{-N-1+v}a^{v}b^{1-v}-2\sqrt{ab}+(1-v)v^{N+1}a+vv^{N}b\\
\geqslant&K^{-r}(h)v^{-N-1+v}a^{v}b^{1-v}-2\sqrt{ab}+K^{r}(h)(v^{N+1}a)^{1-v}(v^{N}b)^{v}\\
=&K^{-r}(h)v^{-N-1+v}a^{v}b^{1-v}-2\sqrt{ab}+K^{r}(h)v^{N+1-v}a^{1-v}b^{v}\\
=&(K^{-\frac{r}{2}}(h)v^{\frac{-N-1+v}{2}}a^{\frac{v}{2}}b^{\frac{1-v}{2}}-K^{\frac{r}{2}}(h)v^{\frac{N+1-v}{2}}a^{\frac{1-v}{2}}b^{\frac{v}{2}})^{2}\\
\geqslant&0.
\end{aligned}$$

Then

$$(1-v^{N+1}+v^{N+2})a+(1-v^{N+1})b \leqslant K^{-r}(h)v^{-N-1+v}a^{v}b^{1-v}+(\sqrt{a}-\sqrt{b})^{2}. \quad \square$$

3. Operator inequalities

Let $B(H)$ denote the C^{*}-algebra of all bounded linear operators on a complex Hilbert space H. A self-adjoint operator $A \in B(H)$ is called positive, and we write $A \geqslant 0$ if $\langle Ax,x\rangle \geqslant 0$ for all $x \in H$. The set of all positive operators is denoted by $B^{+}(H)$. The set of all invertible operators in $B^{+}(H)$ is denoted by $B^{++}(H)$. We say $A \geqslant B$ if $A-B \geqslant 0$.

Let $A,B \in B^{+}(H)$ and $v \in [0,1]$. The weighted operator arithmetic mean and geometric mean of A and B are respectively defined as

$$A\nabla_{v}B=(1-v)A+vB, \quad A\sharp_{v}B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{v}A^{\frac{1}{2}}.$$

When $v=\frac{1}{2}$, $A\nabla_{\frac{1}{2}}B$ and $A\sharp_{\frac{1}{2}}B$ are called respectively operator arithmetic mean and operator geometric mean, which are denoted by $A\nabla B$ and $A\sharp B$.

LEMMA 3.1. *Let* $A \in B(H)$ *be self-adjoint. If* f *and* g *are both continuous functions with* $f(t) \geqslant g(t)$ *for* $t \in Sp(A)$ *(where* $Sp(A)$ *denotes the spectrum of operator* A*), then* $f(A) \geqslant g(A)$.

In this section by applying Lemma 3.1 and inequalities in section 2, we have the following operator inequalities.

COROLLARY 3.2. *If* $A,B \in B(H)$, *satisfy* $0 < mI \leqslant A,B \leqslant MI$, *then*

$$A\nabla_{v}B \leqslant M_{v}^{R}(h)A\sharp_{v}B,$$

where $M_{v}(t)=1+v(1-v)\frac{(t-1)^{2}}{t}$, $R=\max\{v,1-v\}$ *and* $h=\frac{M}{m}$.

Proof. Using inequality (6) and Remark 2.3, we have

$$1\nabla_v t \leqslant M_v^R(t) 1\sharp_v t,$$

where $t \in [\frac{m}{M}, \frac{M}{m}]$. $\frac{M}{m} \geqslant 1$, $M_v(t)$ is increasing for $t \geqslant 1$, then $M_v(t) \leqslant M_v(h)$ for $t \in [1, \frac{M}{m}]$, where $h = \frac{M}{m}$; $\frac{m}{M} \leqslant 1$, $M_v(t)$ is decreasing for $0 < t \leqslant 1$, then $M_v(t) \leqslant M_v(\frac{1}{h})$ for $t \in [\frac{m}{M}, 1]$. Notice that $M_v(\frac{1}{h}) = M_v(h)$, so

$$(1-v) + vt \leqslant M_v^R(h) t^v.$$

Let $X = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ and $Sp(X) \in [\frac{m}{M}, \frac{M}{m}]$, we have

$$(1-v)I + vX \leqslant M_v^R(h) X^v.$$

Multiplying both sides of the above inequality by $A^{\frac{1}{2}}$, the desired inequality is obtained. □

COROLLARY 3.3. *If $A, B \in B(H)$, satisfy $0 < mI \leqslant A, B \leqslant MI$ and $v \in [0,1]$. Consider the integers $k_j = [2^{j-1}v]$ and $r_j = [2^j v]$, $j = 1, 2, \ldots, N$, $N \in \mathbb{N}$. Let $a_N = 1 + [2^N v] - 2^N v$, then*

$$\begin{aligned}
&\Big[1 + a_N(1-a_N)\frac{(h_N-1)^2}{h_N}\Big]^{R_N} A\sharp_v B \\
&+ \sum_{j=1}^{N} \Big((-1)^{r_j} 2^{j-1} v + (-1)^{r_j+1}\Big[\frac{r_j+1}{2}\Big]\Big)\big(A\sharp_{\frac{k_j}{2^{j-1}}} B + A\sharp_{\frac{k_j+1}{2^{j-1}}} B - 2A\sharp_{\frac{2k_j+1}{2^j}} B\big) \\
\geqslant & A\nabla_v B,
\end{aligned}$$

where $R_N = \max\{a_N, 1-a_N\}$ and $h_N = \sqrt[2^N]{\frac{M}{m}}$.

COROLLARY 3.4. *If $A, B \in B(H)$, satisfy $0 < mI \leqslant A \leqslant m'I < vM'I \leqslant B \leqslant vMI$ or $0 < vmI \leqslant B \leqslant vm'I < M'I \leqslant A \leqslant MI$, $v \in (0,1]$ and $N \in \mathbb{N}$, then*

$$(1 - v^{N+1} + v^{N+2})A + (1 - v^{N+1})B \leqslant K^{-r}(h) v^{-N-1+v} A\sharp_v B + 2(A\nabla B - A\sharp B),$$

where $r = \min\{v, 1-v\}$ and $h = \frac{m'}{M'}$.

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