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COMPACTNESS OF EXTREMALS FOR SINGULAR TRUDINGER–MOSER INEQUALITIES ON THE WHOLE EUCLIDEAN SPACE

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Abstract. Let $W^{1,n}(\mathbb{R}^n)$ be the standard Sobolev space. For any $\tau > 0$, $0 < \beta < 1$, Li and Yang [16] proved the existence of extremals for a singular Trudinger-Moser inequality. Namely, the supremum

$$\sup_{u \in W^{1,n}(\mathbb{R}^n),\, \int_{\mathbb{R}^n}(|\nabla u|^n + \tau |u|^n)dx \leqslant 1} \int_{\mathbb{R}^n} \frac{\Phi(n, \alpha_n(1-\beta)|u|^{\frac{n}{n-1}})}{|x|^{n\beta}} dx$$

can be attained by some function $u_\beta \in W^{1,n}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}(|\nabla u|^n + \tau|u|^n)dx = 1$. Here $\Phi(n,t) = e^t - \sum_{j=0}^{n-2} t^j/j!$, and $\alpha_n = n\omega_{n-1}^{1/(n-1)}$ with ω_{n-1} being the surface area of the $(n-1)$-dimensional unit sphere. In this note, we consider the compactness of the function family $\{u_\beta\}_{0<\beta<1}$ and prove that up to a subsequence, u_β converges to some function u_0 in $C^1(\mathbb{R}^n)$ when $\beta \to 0$. Moreover, u_0 is an extremal function of the supremum

$$\sup_{u \in W^{1,n}(\mathbb{R}^n),\, \int_{\mathbb{R}^n}(|\nabla u|^n + \tau |u|^n)dx \leqslant 1} \int_{\mathbb{R}^n} \Phi(n, \alpha_n |u|^{\frac{n}{n-1}}) dx.$$

Let us explain the result in geometry. Denote $\omega_0(x) = \sum_{j=1}^n dx_j{}^2$ and $\omega_\beta(x) = |x|^{-2\beta}\omega_0(x)$ as the standard and conical metrics on $\mathbb{R}^n$. Then the extremal family $\{u_\beta\}_{0<\beta<1}$ of the following singular Trudinger-Moser functionals

$$\int_{\mathbb{R}^n} \Phi(n, \alpha_n(1-\beta)|u|^{\frac{n}{n-1}}) dv_{\omega_\beta}$$

is compactness as $\beta \to 0$. This extends earlier result of Wang and Yang [33] and complements that of Li and Yang [16].

1. Introduction and main result

Let Ω be a smooth domain in $\mathbb{R}^n$, $W_0^{1,n}(\Omega)$ be the usual Sobole space, that is, the completion of $C_0^\infty(\Omega)$ equipped with the norm

$$\|u\|_{W_0^{1,n}(\Omega)} = \left(\int_\Omega |\nabla u|^n dx \right)^{\frac{1}{n}}.$$

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Denote $\alpha_n = n\omega_{n-1}^{1/(n-1)}$, where ω_{n-1} is the surface area of the $(n-1)$-dimensional unit sphere. Then, the classical Trudinger-Moser inequality [21, 25, 26, 32, 38] asserts

$$\sup_{u\in W_0^{1,n}(\Omega),\|\nabla u\|_n\leqslant 1}\int_\Omega e^{\alpha|u|^{\frac{n}{n-1}}}dx<\infty \tag{1}$$

for any $\alpha\leqslant\alpha_n$; moreover, the inequality is sharp in the sense that supremum in (1) will be infinity if $\alpha>\alpha_n$. Here and in the sequel, $\|\cdot\|_s$ denotes the usual L^s-norm with respect to the Lebesgue measure.

An important question about (1) is whether extremal function exists or not. The first result for the attainability was due to Carleson and Chang [6] when Ω is a unit disk in $\mathbb{R}^n$. Then Struwe [29] proved the same existence when the domain is close to a disc in a measure sense. These results were extended by Flucher [11] for any bounded smooth domain in $\mathbb{R}^2$ and by Lin [19] to bounded smooth domain in $\mathbb{R}^n$.

When $|\Omega|=+\infty$, the Trudinger-Moser inequality (1) is not available. It was extended for unbounded domains by Cao [5], do Ó [10], Panda [24], Ruf [27], Li and Ruf [18]. Precisely, there holds for all $\alpha\leqslant\alpha_n$,

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\,\int_{\mathbb{R}^n}(|\nabla u|^n+|u|^n)dx\leqslant 1}\int_{\mathbb{R}^n}\Phi(n,\alpha|u|^{\frac{n}{n-1}})dx<\infty, \tag{2}$$

where

$$\Phi(n,t)=e^t-\sum_{j=0}^{n-2}\frac{t^j}{j!}. \tag{3}$$

The existence of extremal function for (2) was proved by Ruf [27] and Ishiwata [12] for $n=2$ and $\alpha_0\leqslant\alpha\leqslant 4\pi$ for some constant $\alpha_0>0$, by Ishiwata [12] for $n\geqslant 3$ and $0<\alpha<\alpha_n$, by Li and Ruf [18] for $n\geqslant 3$ and $\alpha=\alpha_n$.

Another meaningful extension of (1) is to establish Trudinger-Moser type inequalities in the presence of singular potentials. By a rearrangement argument, Adimurthi and Sandeep [1] proved that for any $0<\beta<1$,

$$\sup_{u\in W_0^{1,n}(\Omega),\|\nabla u\|_n\leqslant 1}\int_\Omega \frac{e^{\alpha_n(1-\beta)|u|^{\frac{n}{n-1}}}}{|x|^{n\beta}}dx<\infty. \tag{4}$$

Obviously, (4) reduces to (1) when $\beta=0$. The existence of extremal function for (4) was proved by Csato and Roy [7], Yang and Zhu [37], Iula and Mancini [13] in dimension two. Then, Csato, Roy and Nguyen [8] studied the existence of extremal function for the general case $n\geqslant 3$. Applying rearrangement argument and Young inequality, Adimurthi and Yang [3] extended (4) to the entire $\mathbb{R}^n$ which can be described as follows

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\,\int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)dx\leqslant 1}\int_{\mathbb{R}^n}\frac{\Phi(n,\alpha_n(1-\beta)|u|^{\frac{n}{n-1}})}{|x|^{n\beta}}dx<\infty \tag{5}$$

for constants $\tau>0$, $0\leqslant\beta<1$. The author and Yang [16] proved the existence of extremals for (5) by using blow-up analysis. Very recently, Wang and Yang [33] studied the compactness of extremals $\{u_\beta\}_{0<\beta<1}$ for (4) in dimension two and proved that u_β converges to some function u^* in $C^1(\overline{\Omega})$ as $\beta \to 0$. Moreover, u^* is an extremal of the following supremum

$$\sup_{u\in W_0^{1,2}(\Omega),\|\nabla u\|_2\leqslant 1} \int_\Omega e^{4\pi u^2} dx.$$

For more works related to Trudinger-Moser inequality, we refer the reader to [14, 15, 20, 22, 23, 30, 34, 36] and the references therein.

Denote $\omega_0(x) = dx_1{}^2 + dx_2{}^2 + \cdots + dx_n{}^2$ be the Euclidean metric, $\omega_\beta(x) = |x|^{-2\beta}\omega_0$ be the conical metric on $\mathbb{R}^n$ for $0<\beta<1$. Then $dv_{\omega_\beta} = |x|^{-n\beta}dx$ and $\omega_\beta \to \omega_0$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$. Motivated by [16], a natural question is whether or not a maximizer sequence $\{u_\beta\}_{0<\beta<1}$ converges when the conical metric ω_β converges to the Euclidean metric ω_0 in $C^2_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$.

Define a singular Trudinger-Moser functional $TM_\beta : W^{1,n}(\mathbb{R}^n,\omega_\beta) \mapsto \mathbb{R}$ by

$$TM_\beta(u) = \int_{\mathbb{R}^n} \Phi(n, \alpha_n(1-\beta)|u|^{\frac{n}{n-1}})dv_{\omega_\beta}.$$

Then we rephrase the result in [16] as below: for any $\tau>0$, $0<\beta<1$, there exists some nonnegative decreasing radially symmetric function $u_\beta \in W^{1,n}(\mathbb{R}^n)\cap C^0(\mathbb{R}^n)\cap C^1(\mathbb{R}^n\setminus\{0\})$ satisfying $\int_{\mathbb{R}^n}(|\nabla u|^n + \tau|u|^n)dx = 1$ and

$$TM_\beta(u_\beta) = \sup_{u\in W^{1,n}(\mathbb{R}^n),\ \int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)dx\leqslant 1} TM_\beta(u). \tag{6}$$

In this paper, we establish the compactness of extremals analogous to the one obtained in [33] in the case of the entire Euclidean space $\mathbb{R}^n$ $(n>2)$. The main result reads as follows.

THEOREM 1. *Let $\omega_0 = dx_1{}^2 + dx_2{}^2 + \cdots + dx_n{}^2$ be the Euclidean metric, $\omega_\beta = |x|^{-2\beta}\omega_0$ be the conical metric for any $0<\beta<1$. Assume u_β be a sequence of maximizers for the supremum in (6). Then up to a subsequence, there exists some function u_0 satisfying $u_\beta \to u_0$ in $C^1(\mathbb{R}^n)$ and u_0 is an extremal function of the supremum*

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\ \int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)dx\leqslant 1} \int_{\mathbb{R}^n} \Phi(n,\alpha_n|u|^{\frac{n}{n-1}})dx. \tag{7}$$

Following Li-Ruf [18] and thereby following [17], we prove Theorem 1 via the standard blowing up analysis procedure. See also [2, 6, 9] and the references therein.

Let us give the outline of the proof of Theorem 1. Let u_β be the extremals for the supremum in (6). According to [16], we see that u_β is nonnegative decreasing and radially symmetric. Besides, it is a solution of the equation (8) below. Assume u_β is not bounded in $\mathbb{R}^n$, then we have

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)dx\leqslant 1} \int_{\mathbb{R}^n} \Phi(n,\alpha_n|u|^{\frac{n}{n-1}})dx \leqslant \frac{\omega_{n-1}}{n} e^{\alpha_n A_0 + \sum_{k=1}^{n-1}\frac{1}{k}},$$

where A_0 is given as in (49). On the other hand, we can also get

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)dx\leqslant 1}\int_{\mathbb{R}^n}\Phi(n,\alpha_n|u|^{\frac{n}{n-1}})dx > \frac{\omega_{n-1}}{n}e^{\alpha_n A_0+\Sigma_{k=1}^{n-1}\frac{1}{k}}.$$

The contradiction implies that u_β must be uniformly bounded. Then applying the elliptic estimates (see [28, 31]) to (8), we get the desire result immediately.

Throughout this note, the norm of $W^{1,n}(\mathbb{R}^n)$ is defined by $\|u\|^n_{W^{1,n}(\mathbb{R}^n)}=\int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)dx$. $\mathbb{B}_r$ denotes the ball in $\mathbb{R}^n$ with the radius r centered at the origin. The constant C may be different from line to line. And we pass to subsequence freely.

2. The proof of Theorem 1

We prove Theorem 1 and divide the proof into several subsections.

2.1. The Euler-Lagrange equation of u_β

For simplicity, denote the n-Laplacian $\Delta_n u=\mathrm{div}(|\nabla u|^{n-2}\nabla u)$ for any $u\in W^{1,n}(\mathbb{R}^n)$. By simple calculation, one has

$$\frac{d}{dt}\Phi(n,t)=\Phi(n-1,t),$$

where $\Phi(n,t)$ is defined as in (3). For $0<\beta<1$, we write

$$STM_\beta=\sup_{u\in W^{1,n}(\mathbb{R}^n),\|u\|_{W^{1,n}(\mathbb{R}^n)}\leqslant 1}\int_{\mathbb{R}^n}\frac{\Phi(n,\alpha_n(1-\beta)|u|^{\frac{n}{n-1}})}{|x|^{n\beta}}dx.$$

Thus (6) is equivalent to

$$\int_{\mathbb{R}^n}\frac{\Phi(n,\alpha_n(1-\beta)|u_\beta|^{\frac{n}{n-1}})}{|x|^{n\beta}}dx=STM_\beta.$$

According to [16], u_β is nonnegative decreasing radially and symmetric. We now show the Euler-Lagrange equation of u_β. In a distrubutional sence, u_β satisfies the following Euler-Lagrange equation

$$\begin{cases}-\Delta_n u_\beta+\tau u_\beta^{n-1}=\dfrac{u_\beta^{\frac{1}{n-1}}}{\lambda_\beta}\dfrac{\Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}} \quad \text{in} \quad \mathbb{R}^n,\\ u_\beta>0 \quad \text{in} \quad \mathbb{R}^n,\\ \|u_\beta\|_{W^{1,n}(\mathbb{R}^n)}=1,\\ \lambda_\beta=\int_{\mathbb{R}^n}|x|^{-n\beta}u_\beta^{\frac{n}{n-1}}\Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})dx.\end{cases} \tag{8}$$

Since u_β is bounded in $W^{1,n}(\mathbb{R}^n)$, we can find some function u_0 such that up to a subsequence as $\beta\to 0$, $u_\beta\rightharpoonup u_0$ weakly in $W^{1,n}(\mathbb{R}^n)$, $u_\beta\to u_0$ strongly in $L^s_{\mathrm{loc}}(\mathbb{R}^n)$

for any $s>1$, $u_\beta \to u_0$ almost everywhere in $\mathbb{R}^n$. Also, u_0 is nonnegative decreasing radially symmetric in $\mathbb{R}^n$ and $\|u_0\|_{W^{1,n}(\mathbb{R}^n)} \leqslant \limsup_{\beta\to 0}\|u_\beta\|_{W^{1,n}(\mathbb{R}^n)} = 1$.

To continue our investigation, we shall show

$$\lim_{\beta\to 0}\int_{\mathbb{R}^n} \frac{\Phi(n,\alpha_n(1-\beta)|u|^{\frac{n}{n-1}})}{|x|^{n\beta}}dx = \int_{\mathbb{R}^n}\Phi(n,\alpha_n|u|^{\frac{n}{n-1}})dx \tag{9}$$

for $u\in W^{1,n}(\mathbb{R}^n)$ with $\|u\|_{W^{1,n}(\mathbb{R}^n)}\leqslant 1$.

In fact, we first note that for $x\in\mathbb{B}_1$,

$$0\leqslant \frac{\Phi(n,\alpha_n(1-\beta)|u|^{\frac{n}{n-1}})}{|x|^{n\beta}} \leqslant \frac{\Phi(n,\alpha_n|u|^{\frac{n}{n-1}})}{|x|^{\frac{n}{2}}}$$

and $\frac{\Phi(n,\alpha_n|u|^{\frac{n}{n-1}})}{|x|^{\frac{n}{2}}} \in L^1(\mathbb{B}_1)$ for any $u\in W^{1,n}(\mathbb{R}^n)$. By Lebesgue dominated convergence theorem, we obtain

$$\lim_{\beta\to 0}\int_{\mathbb{B}_1} \frac{\Phi(n,\alpha_n(1-\beta)|u|^{\frac{n}{n-1}})}{|x|^{n\beta}}dx = \int_{\mathbb{B}_1}\Phi(n,\alpha_n|u|^{\frac{n}{n-1}})dx. \tag{10}$$

On the other hand, if $x\in\mathbb{R}^n\setminus\mathbb{B}_1$, then we have

$$0\leqslant \frac{\Phi(n,\alpha_n(1-\beta)|u|^{\frac{n}{n-1}})}{|x|^{n\beta}} \leqslant \Phi(n,\alpha_n|u|^{\frac{n}{n-1}}).$$

Again, employing Lebesgue dominated convergence theorem, we deduce

$$\lim_{\beta\to 0}\int_{\mathbb{R}^n\setminus\mathbb{B}_1} \frac{\Phi(n,\alpha_n(1-\beta)|u|^{\frac{n}{n-1}})}{|x|^{n\beta}}dx = \int_{\mathbb{R}^n\setminus\mathbb{B}_1}\Phi(n,\alpha_n|u|^{\frac{n}{n-1}})dx. \tag{11}$$

Combining (10) and (11), we conclude (9) holds. Moreover, it is not difficult to see

$$\lim_{\beta\to 0}\int_{\mathbb{R}^n} \frac{\Phi(n,\alpha_n(1-\beta)|u|^{\frac{n}{n-1}})}{|x|^{n\beta}}dx \leqslant \liminf_{\beta\to 0} STM_\beta = \liminf_{\beta\to 0}\int_{\mathbb{R}^n} \frac{\Phi(n,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}}dx. \tag{12}$$

In view of (8), an important problem is whether λ_β has a positive lower bound or not. For this purpose, we have the following lemma.

LEMMA 2. *There holds*

$$\liminf_{\beta\to 0}\lambda_\beta > 0.$$

Proof. Since $t\Phi(n-1,t)\geqslant \Phi(n,t)$ for $t\geqslant 0$, we get

$$\begin{aligned}\lambda_\beta &= \int_{\mathbb{R}^n} u_\beta^{\frac{n}{n-1}} \frac{\Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}}dx \\ &\geqslant \frac{1}{\alpha_n(1-\beta)}\int_{\mathbb{R}^n} \frac{\Phi(n,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}}dx.\end{aligned}$$

Sending the limit $\beta \to 0$ gives

$$\liminf_{\beta\to 0} \lambda_\beta \geqslant \frac{1}{\alpha_n} \sup_{u\in W^{1,n}(\mathbb{R}^n), \|u\|_{W^{1,n}(\mathbb{R}^n)}\leqslant 1} \int_{\mathbb{R}^n} \Phi(n, \alpha_n |u|^{\frac{n}{n-1}}) > 0,$$

thanks to (9) and (12). □

Denote

$$c_\beta = u_\beta(0) = \max_{\mathbb{R}^n} u_\beta \quad \text{(maximum point)}$$

There are the following two possibilities to analysis: $\{c_\beta\}_{0<\beta<1}$ is a bound sequence or up to a subsequence $c_\beta \to +\infty$ as $\beta \to 0$. We are now in a position to exclude the blow-up phenomenon.

2.2. Blow-up analysis

In this subsection, we shall apply the blow-up method to describe asymptotic behavior of u_β as $\beta \to 0$. It is useful to have the following lemma; it implies the concentration phenomenon.

LEMMA 3. *It holds $|\nabla u_\beta|^n dx \rightharpoonup \delta_0$ in the sense of measure, where δ_0 denotes the usual Dirac measure giving unit mass to the point 0. Consequently, $u_0 \equiv 0$. Moreover, $u_\beta \to 0$ strongly in $L^q(\mathbb{R}^n)$ for all $q \geqslant n$.*

Proof. Let us recall an elementary inequality. Namely,

$$(\Phi(n,t))^s \leqslant \Phi(n, st) \tag{13}$$

for $s \geqslant 1$, $t \geqslant 0$, which is due to Yang ([35], Lemma 2.1).

At first, we show that $|\nabla u_\beta|^n dx \rightharpoonup \delta_0$ in the sense of measure. Suppose not. There exists $r_0 > 0$ such that

$$\limsup_{\beta\to 0} \int_{\mathbb{B}_{r_0}} |\nabla u_\beta|^n dx \leqslant \mu < 1.$$

Let $\widetilde{u}_\beta(x) = u_\beta(x) - u_\beta(r_0)$ for $x \in \mathbb{B}_{r_0}$. Note that u_β is decreasing radially symmetric. Then we have $\widetilde{u}_\beta \in W_0^{1,n}(\mathbb{B}_{r_0})$ and $\|\nabla \widetilde{u}_\beta\|_{L^n(\mathbb{B}_{r_0})} \leqslant \mu < 1$. In addition, since $u_\beta^n(r_0)|\mathbb{B}_{r_0}| \leqslant \int_{\mathbb{B}_{r_0}} u_\beta^n dx \leqslant 1/\tau$, we find

$$u_\beta^n(r_0) \leqslant \frac{n}{\omega_{n-1}\tau r_0^n}. \tag{14}$$

Set

$$f_\beta(x) = \frac{1}{\lambda_\beta} u_\beta^{\frac{1}{n-1}} \frac{\Phi(n-1, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}}.$$

We claim that, for some $p > 1$, there exists a constant C such that

$$\int_{\mathbb{B}_{r_0}} f_\beta^p(x) dx \leqslant C. \tag{15}$$

Indeed, using (13) and Hölder inequality, we get

$$\begin{aligned}\int_{\mathbb{B}_{r_0}} f_\beta^p(x)dx &\leqslant \frac{1}{\lambda_\beta^p}\int_{\mathbb{B}_{r_0}} \frac{u_\beta^{\frac{p}{n-1}}\Phi(n-1,\alpha_n(1-\beta)pu_\beta^{\frac{n}{n-1}})}{|x|^{n\beta p}}dx \\ &\leqslant \frac{1}{\lambda_\beta^p}\left(\int_{\mathbb{B}_{r_0}}\frac{u_\beta^{\frac{pp_1}{n-1}}}{|x|^{n\beta p}}dx\right)^{\frac{1}{p_1}}\left(\int_{\mathbb{B}_{r_0}}\frac{\Phi(n-1,\alpha_n(1-\beta)pp_2u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta p}}dx\right)^{\frac{1}{p_2}} \\ &\leqslant \frac{1}{\lambda_\beta^p}\left(\int_{\mathbb{B}_{r_0}}\frac{u_\beta^{\frac{pp_1}{n-1}}}{|x|^{n\beta p}}dx\right)^{\frac{1}{p_1}}\left(\int_{\mathbb{B}_{r_0}}\frac{e^{\alpha_n(1-\beta)pp_2u_\beta^{\frac{n}{n-1}}}}{|x|^{n\beta p}}dx\right)^{\frac{1}{p_2}}, \end{aligned} \tag{16}$$

where $1/p_1+1/p_2=1$. Assume $\varepsilon>0$, there is a constant C depending on n and ε such that

$$u_\beta^{\frac{n}{n-1}}(x)\leqslant(1+\varepsilon)\widetilde{u}_\beta^{\frac{n}{n-1}}(x)+Cu_\beta^{\frac{n}{n-1}}(r_0) \tag{17}$$

for all $x\in\mathbb{B}_{r_0}$. We can choose $p>1$, $p_2>1$ sufficiently close to 1 and $\varepsilon>0$ sufficiently small such that $(1-\beta)pp_2(1+\varepsilon)\|\nabla\widetilde{u}_\beta\|_{L^n(\mathbb{B}_{r_0})}^{\frac{n}{n-1}}+\beta p<1$. Combining (4), (14) and (17), we have

$$\int_{\mathbb{B}_{r_0}}\frac{e^{\alpha_n(1-\beta)pp_2u_\beta^{\frac{n}{n-1}}}}{|x|^{n\beta p}}dx\leqslant C. \tag{18}$$

Recall that $u_\beta\to u_0$ in $L^s(\mathbb{B}_{r_0})$ for any $s>1$. Choosing positive numbers q_1 and q_2 with $1/q_1+1/q_2=1$ such that $1<q_1<1/\beta p$, we have by Hölder inequality

$$\int_{\mathbb{B}_{r_0}}\frac{u_\beta^{\frac{pp_1}{n-1}}}{|x|^{n\beta p}}dx\leqslant\left(\int_{\mathbb{B}_{r_0}}\frac{1}{|x|^{n\beta pq_1}}dx\right)^{\frac{1}{q_1}}\left(\int_{\mathbb{B}_{r_0}}u_\beta^{\frac{pp_1q_2}{n-1}}dx\right)^{\frac{1}{q_2}}\leqslant C. \tag{19}$$

Substitution of (18) and (19) into (16) leads to (15). Notice that u_ε is bounded in $L^q(\mathbb{B}_{r_0})$ for all $q>0$. And from Lemma 2 and (15), we get $\Delta_n u_\beta$ is bounded in $L^p(\mathbb{B}_{r_0})$ for some $p>1$. Thus elliptic estimate ([28], Theorems 6 and 8) implies that u_β is uniformly bounded in $\mathbb{B}_{r_0/2}$. This contradicts that $c_\beta\to+\infty$ as $\beta\to 0$. Therefore, we arrive at $|\nabla u_\beta|^n dx\rightharpoonup\delta_0$ as $\beta\to 0$.

We are now in a position to prove $u_0\equiv 0$. Suppose the assertion were false. It follows from $\|u_\beta\|_{W^{1,n}(\mathbb{R}^n)}=1$ and $|\nabla u_\beta|^n dx\rightharpoonup\delta_0$ that $\|u_\beta\|_{L^n(\mathbb{R}^n)}=o_\beta(1)$. Then we have

$$\int_{\mathbb{R}^n}u_0^n dx\leqslant\limsup_{\beta\to 0}\int_{\mathbb{R}^n}u_\beta^n dx=0$$

and so duduce $u_0\equiv 0$. Choose $L>0$ such that $u_\beta<1$ for $|x|>L$. Then one has for any $q\geqslant n$,

$$\int_{\mathbb{R}^n}u_\beta^q dx=\int_{|x|>L}u_\beta^q dx+\int_{|x|\leqslant L}u_\beta^q dx\leqslant\int_{\mathbb{R}^n}u_\beta^n dx+o_\beta(1)=o_\beta(1).$$

And the proof of the lemma is completed. □

Set

$$r_\beta = \lambda_\beta^{\frac{1}{n}} c_\beta^{-\frac{1}{n-1}} e^{-\frac{\alpha_n(1-\beta)}{n} c_\beta^{\frac{n}{n-1}}}.$$

Then we have the following:

LEMMA 4. *For any* $\sigma \in (0,\alpha_n)$, *there holds* $r_\beta e^{\frac{\sigma(1-\beta)}{n} c_\beta^{\frac{n}{n-1}}} \to 0$ *as* $\beta \to 0$. *In particular,* $r_\beta \to 0$ *as* $\beta \to 0$.

Proof. For $\sigma \in (0,\alpha_n)$, we have after using the definition of r_β, that

$$\begin{aligned} r_\beta^n e^{\sigma(1-\beta)c_\beta^{\frac{n}{n-1}}} &= c_\beta^{-\frac{n}{n-1}} e^{-\alpha_n(1-\frac{\sigma}{\alpha_n})(1-\beta)c_\beta^{\frac{n}{n-1}}} \int_{\mathbb{R}^n} \frac{u_\beta^{\frac{n}{n-1}} \Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}} dx \\ &= c_\beta^{-\frac{n}{n-1}} e^{-\alpha_n(1-\frac{\sigma}{\alpha_n})(1-\beta)c_\beta^{\frac{n}{n-1}}} \int_{\mathbb{R}^n\setminus\mathbb{B}_R} \frac{u_\beta^{\frac{n}{n-1}} \Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}} dx \\ &\quad + c_\beta^{-\frac{n}{n-1}} e^{-\alpha_n(1-\frac{\sigma}{\alpha_n})(1-\beta)c_\beta^{\frac{n}{n-1}}} \int_{\mathbb{B}_R} \frac{u_\beta^{\frac{n}{n-1}} \Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}} dx. \end{aligned} \tag{20}$$

We employ the radial lemma [4] to estimate

$$\begin{aligned} \int_{\mathbb{R}^n\setminus\mathbb{B}_R} \frac{u_\beta^{\frac{n}{n-1}} \Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}} dx &= \sum_{j=n-2}^{\infty} \frac{\alpha_n^j(1-\beta)^j}{j!} \int_{\mathbb{R}^n\setminus\mathbb{B}_R} \frac{u_\beta^{\frac{n}{n-1}(1+j)}}{|x|^{n\beta}} dx \\ &\leqslant C(R). \end{aligned} \tag{21}$$

Since $-\alpha_n(1-\frac{\sigma}{\alpha_n})(1-\beta)c_\beta^{\frac{n}{n-1}} \leqslant -\alpha_n(1-\frac{\sigma}{\alpha_n})(1-\beta)u_\beta^{\frac{n}{n-1}}$ for $x \in \mathbb{R}^n$, we have by Hölder inequality that

$$\begin{aligned} &\int_{\mathbb{B}_R} e^{-\alpha_n(1-\frac{\sigma}{\alpha_n})(1-\beta)c_\beta^{\frac{n}{n-1}}} \frac{u_\beta^{\frac{n}{n-1}} \Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}} dx \\ &\leqslant \int_{\mathbb{B}_R} \frac{u_\beta^{\frac{n}{n-1}} e^{\sigma(1-\beta)u_\beta^{\frac{n}{n-1}}}}{|x|^{n\beta}} dx \\ &\leqslant \left(\int_{\mathbb{B}_R} \frac{e^{\sigma p(1-\beta)u_\beta^{\frac{n}{n-1}}}}{|x|^{np\beta}} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{B}_R} u_\beta^{\frac{nq}{n-1}} dx \right)^{\frac{1}{q}}, \end{aligned}$$

where $p>1$ sufficiently close to 1 and $1/p+1/q=1$. Observe that $\sigma\in(0,\alpha_n)$. An obvious analog of (18) is

$$\int_{\mathbb{B}_R}\frac{e^{\sigma p(1-\beta)u_\beta^{\frac{n}{n-1}}}}{|x|^{np\beta}}dx\leqslant C(R,p).$$

Since $u_\beta\to 0$ strongly in $L^s(\mathbb{B}_R)$ for any $s>1$, we have

$$\int_{\mathbb{B}_R}e^{-\alpha_n(1-\frac{\sigma}{\alpha_n})(1-\beta)c_\beta^{\frac{n}{n-1}}}\frac{u_\beta^{\frac{n}{n-1}}\Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}}dx=o_\beta(1). \tag{22}$$

Substituting (21) and (22) into (20) and recalling $c_\beta\to+\infty$ as $\beta\to 0$, we conclude

$$\lim_{\beta\to 0}r_\beta^n e^{\sigma(1-\beta)c_\beta^{\frac{n}{n-1}}}=0.$$

In particular, $r_\beta=o_\beta(1)$ and this completes the proof of the lemma. □

In order to continue blow-up analysis, we define

$$\phi_\beta=c_\beta^{-1}u_\beta(r_\beta^{\frac{1}{1-\beta}}x)$$

and

$$\psi_\beta=c_\beta^{\frac{1}{n-1}}(u_\beta(r_\beta^{\frac{1}{1-\beta}}x)-c_\beta).$$

We investigate the asymptotic behavior of u_β as $\beta\to 0$ near zero, namely

LEMMA 5. *Up to a subsequence, $\phi_\beta\to 1$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\psi_\beta\to\psi_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $\beta\to 0$, where*

$$\psi_0=-\frac{n-1}{\alpha_n}\log\left(1+\frac{\alpha_n}{n^{\frac{n}{n-1}}}|x|^{\frac{n}{n-1}}\right)$$

and

$$\int_{\mathbb{R}^n}e^{\alpha_n\frac{n}{n-1}\psi_0}dx=1.$$

Proof. A direct calculation gives that

$$-\Delta_n\phi_\beta(x)=-\tau r_\beta^{\frac{n}{1-\beta}}\phi_\beta^{n-1}+\frac{\phi_\beta^{\frac{1}{n-1}}\Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}}(r_\beta^{\frac{1}{1-\beta}}x))}{|x|^{n\beta}c_\beta^n e^{\alpha_n(1-\beta)c_\beta^{\frac{n}{n-1}}}}. \tag{23}$$

Since $|\phi_\beta|\leqslant 1$, $r_\beta=o_\beta(1)$ and $c_\beta\to+\infty$, we obtain by using the elliptic estimate ([28, 31]) to (23) that $\phi_\beta\to\phi_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ with ϕ_0 satisfies $-\Delta_n\phi_0(x)=0$ in $\mathbb{R}^n$. Then the Liouville-type theorem for n-harmonic function implies that $\phi_0\equiv 1$.

Also it can be easily checked that $\psi_\beta(x)$ is a distribution solution of equation

$$-\Delta_n\psi_\beta(x) = -\tau c_\beta^n r_\beta^{\frac{n}{1-\beta}}\phi_\beta^{n-1} + \frac{\phi_\beta^{\frac{1}{n-1}}\Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}}(r_\beta^{\frac{1}{1-\beta}}x))}{|x|^{n\beta}e^{\alpha_n(1-\beta)c_\beta^{\frac{n}{n-1}}}}. \tag{24}$$

Note that $\psi_\beta \leqslant 0 = \max_{\mathbb{R}^n}\psi_\beta$. We have by elliptic estimates to (24) that $\psi_\beta \to \psi_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Since ψ_β is decreasingly symmetric on $\mathbb{R}^n$, it follows that ψ_0 is decreasingly symmetric. To derive the equation of ψ_0, we first estimate

$$0 \leqslant \frac{\sum_{j=0}^{n-3}\frac{(\alpha_n(1-\beta))^j u_\beta^{\frac{nj}{n-1}}(r_\beta^{\frac{1}{1-\beta}}x)}{j!}}{|x|^{n\beta}e^{\alpha_n(1-\beta)c_\beta^{\frac{n}{n-1}}}} \leqslant \frac{\sum_{j=0}^{n-3}\frac{(\alpha_n(1-\beta))^j c_\beta^{\frac{nj}{n-1}}}{j!}}{|x|^{n\beta}e^{\alpha_n(1-\beta)c_\beta^{\frac{n}{n-1}}}} = o_\beta(1).$$

By the mean value theorem, we also have

$$\begin{aligned} u_\beta^{\frac{n}{n-1}}(r_\beta^{\frac{1}{1-\beta}}x) - c_\beta^{\frac{n}{n-1}} &= \frac{n}{n-1}\vartheta_\beta^{\frac{1}{n-1}}(u_\beta(r_\beta^{\frac{1}{1-\beta}}x) - c_\beta) \\ &= \frac{n}{n-1}(\vartheta_\beta/c_\beta)^{\frac{1}{n-1}}\psi_\beta(x) \\ &= \frac{n}{n-1}\psi_0(x) + o_\beta(1), \end{aligned}$$

where ϑ_β lies between $u_\beta(r_\beta^{1/(1-\beta)}x)$ and c_β. We then obtain

$$\begin{cases} -\Delta_n\psi_0(x) = e^{\alpha_n\frac{n}{n-1}\psi_0(x)} \quad \text{in} \quad \mathbb{R}^n, \\ \psi_0(0) = \max_{\mathbb{R}^n}\psi_0 = 0. \end{cases} \tag{25}$$

We take $\psi_0(r)$ for $\psi_0(x)$ with $r = |x|$ and rewrite (25) as

$$\begin{cases} \left((-r\psi_0'(r))^{n-1}\right)' = r^{n-1}e^{\alpha_n\frac{n}{n-1}\psi_0(r)}, \\ \psi_0(0) = 0. \end{cases} \tag{26}$$

By a standard uniqueness result of ordinary differential equations (see for example [18]), the equation (26) is solved by

$$\psi_0(x) = -\frac{n-1}{\alpha_n}\log\left(1 + b_n|x|^{\frac{n}{n-1}}\right)$$

with $b_n = \alpha_n n^{-n/(n-1)}$. Thus we have

$$\begin{aligned} \int_{\mathbb{R}^n} e^{\alpha_n\frac{n}{n-1}\psi_0(x)}dx &= \omega_{n-1}\int_0^\infty \frac{r^{n-1}}{(1+b_n r^{\frac{n}{n-1}})^n}dr \\ &= \frac{\omega_{n-1}}{n}(n-1)\int_0^\infty \frac{t^{n-2}}{(1+b_n t)^n}dt. \end{aligned} \tag{27}$$

Writing $I_n = (n-1)\int_0^\infty \frac{t^{n-2}}{(1+b_n t)^n}dt$ and integrating by parts, we observe the following recurrence relation

$$I_n = \frac{n-2}{b_n}\int_0^\infty \frac{t^{n-3}}{(1+b_n t)^{n-1}}dt = \frac{1}{b_n}I_{n-1}.$$

Iteration gives

$$I_n = \frac{1}{b_n^{n-2}}I_2 = \frac{1}{b_n^{n-2}}\int_0^\infty \frac{1}{(1+b_n t)^2}dt = \frac{1}{b_n^{n-1}} = \frac{n^n}{\alpha_n^{n-1}}. \tag{28}$$

Combining (27) and (28), we arrive at

$$\int_{\mathbb{R}^n} e^{\alpha_n \frac{n}{n-1}\psi_0(x)}dx = 1,$$

and the lemma is proved. □

We proceed to analysis the convergence of u_β as $\beta \to 0$ away from zero. Similar to [16] and [18], define

$$u_{\beta,\iota} = \min\{u_\beta, \iota c_\beta\}$$

for $0 < \iota < 1$.

LEMMA 6. *There holds*

$$\lim_{\beta\to 0}\int_{\mathbb{R}^n} |\nabla u_{\beta,\iota}|^n dx = \iota.$$

Proof. For any $R > 0$, we multiply equation (8) by $u_{\beta,\iota}$ and integrate over $\mathbb{R}^n$ to get

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla u_{\beta,\iota}|^n dx &= -\tau\int_{\mathbb{R}^n} u_\beta^{n-1}u_{\beta,\iota}dx + \lambda_\beta^{-1}\int_{\mathbb{R}^n} |x|^{-n\beta}u_\beta^{\frac{1}{n-1}}u_{\beta,\iota}\Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})dx\\ &\geqslant \lambda_\beta^{-1}\int_{\mathbb{B}_{Rr_\beta^{1/(1-\beta)}}} |x|^{-n\beta}\iota c_\beta u_\beta^{\frac{1}{n-1}}e^{\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}}}dx + o_\beta(1)\\ &= \iota(1+o_\beta(1))\int_{\mathbb{B}_R} |y|^{-n\beta}e^{\alpha_n(1-\beta)(u_\beta^{\frac{n}{n-1}}(r_\beta^{\frac{1}{1-\beta}}y)-c_\beta^{\frac{n}{n-1}})}dy + o_\beta(1).\end{aligned}$$

Letting $\beta \to 0$, we have

$$\liminf_{\beta\to 0}\int_{\mathbb{R}^n} |\nabla u_{\beta,\iota}|^n dx \geqslant \iota\int_{\mathbb{B}_R} e^{\alpha_n\frac{n}{n-1}\psi_0(x)}dx.$$

Passing to the limit $R \to \infty$, we obtain

$$\liminf_{\beta\to 0}\int_{\mathbb{R}^n} |\nabla u_{\beta,\iota}|^n dx \geqslant \iota. \tag{29}$$

Similarly, inserting $(u_\beta - \iota c_\beta)^+$ as a test function in equation (8), we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla(u_\beta - \iota c_\beta)^+|^n dx \\
&= \lambda_\beta^{-1} \int_{\mathbb{R}^n} (u_\beta - \iota c_\beta)^+ u_\beta^{\frac{1}{n-1}} |x|^{-n\beta} \Phi(n-1, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}}) dx - \tau \int_{\mathbb{R}^n} u_\beta^{n-1} (u_\beta - \iota c_\beta)^+ dx \\
&\geqslant \lambda_\beta^{-1} \int_{\mathbb{B}_{Rr_\beta^{1/(1-\beta)}}} u_\beta^{\frac{1}{1-n}} (u_\beta - \iota c_\beta)^+ |x|^{-n\beta} e^{\alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}}} dx + o_\beta(1) \\
&= (1-\iota)(1+o_\beta(1)) \int_{\mathbb{B}_R} |y|^{-n\beta} e^{\alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}}(r_\beta^{\frac{1}{1-\beta}} y) - c_\beta^{\frac{n}{n-1}}} dy + o_\beta(1).
\end{aligned}$$

Letting $\beta \to 0$ first and then $R \to \infty$, we have

$$\liminf_{\beta \to 0} \int_{\mathbb{R}^n} |\nabla(u_\beta - \iota c_\beta)^+|^n dx \geqslant 1 - \iota. \tag{30}$$

Note that

$$\int_{\mathbb{R}^n} |\nabla u_{\beta,\iota}|^n dx + \int_{\mathbb{R}^n} |\nabla(u_\beta - \iota c_\beta)^+|^n dx = \int_{\mathbb{R}^n} |\nabla u_\beta|^n dx = 1 + o_\beta(1). \tag{31}$$

Combining(29), (30) and (31), we immediately get the desire result. □

Rely on Lemma 6 and one can derive the following lemma.

LEMMA 7. *There holds*

$$\lim_{R \to +\infty} \limsup_{\beta \to 0} \int_{\mathbb{B}_{Rr_\beta^{1/(1-\beta)}}} |x|^{-n\beta} \Phi(n, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}}(x)) dx = \limsup_{\beta \to 0} \frac{\lambda_\beta}{c_\beta^{\frac{n}{n-1}}}. \tag{32}$$

Proof. For any ι, $0 < \iota < 1$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} |x|^{-n\beta} \Phi(n, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}}(x)) &= \int_{u_\beta \leqslant \iota c_\beta} |x|^{-n\beta} \Phi(n, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}}) dx \\
&\quad + \int_{u_\beta > \iota c_\beta} |x|^{-n\beta} \Phi(n, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}}) dx \\
&=: I + II.
\end{aligned}$$

Recall that for all $t \geqslant 0$,

$$\Phi(n,t) = \Phi'(n,t) - \frac{t^{n-2}}{(n-2)!} = \Phi(n-1,t) - \frac{t^{n-2}}{(n-2)!},$$

which together with the mean value theorem implies that

$$
\begin{aligned}
I &\leqslant \int_{\mathbb{R}^n} |x|^{-n\beta}\Phi(n,\alpha_n(1-\beta)u_{\beta,\iota}^{\frac{n}{n-1}})dx \\
&\leqslant \alpha_n(1-\beta)\int_{\mathbb{R}^n} |x|^{-n\beta}\Phi(n-1,\vartheta)u_{\beta,\iota}^{\frac{n}{n-1}}dx \\
&\leqslant \alpha_n(1-\beta)\int_{\mathbb{R}^n} |x|^{-n\beta}\Phi(n-1,\alpha_n(1-\beta)u_{\beta,\iota}^{\frac{n}{n-1}})u_{\beta,\iota}^{\frac{n}{n-1}}dx \\
&\leqslant \alpha_n(1-\beta)\int_{\mathbb{R}^n} |x|^{-n\beta}\Phi(n,\alpha_n(1-\beta)u_{\beta,\iota}^{\frac{n}{n-1}})u_{\beta,\iota}^{\frac{n}{n-1}}dx + \frac{(\alpha_n(1-\beta))^{n-1}}{(n-2)!}\int_{\mathbb{R}^n}|x|^{-n\beta}u_{\beta,\iota}^{n}dx,
\end{aligned} \tag{33}
$$

where ϑ is between 0 and $\alpha_n(1-\beta)u_{\beta,\iota}^{\frac{n}{n-1}}$. For any $R>0$ and $q\geqslant 1$, we can choose $1<s<1/\beta$ with $1/s+1/t=1$. Then Hölder inequality together with the estimate $\|u_{\beta,\iota}\|_n \leqslant \|u_\beta\|_n = o_\beta(1)$ gives that

$$
\int_{\mathbb{B}_R}|x|^{-n\beta}u_{\beta,\iota}^{nq}dx \leqslant \left(\int_{\mathbb{B}_R}|x|^{-n\beta s}dx\right)^{\frac{1}{s}}\left(\int_{\mathbb{B}_R}u_{\beta,\iota}^{nqt}dx\right)^{\frac{1}{t}} = o_\beta(1). \tag{34}
$$

On the other hand, by the radial lemma [4], we have

$$
\int_{\mathbb{R}^n\setminus\mathbb{B}_R}|x|^{-n\beta}u_{\beta,\iota}^{nq}dx \leqslant \left(\frac{n}{\omega_{n-1}}\right)^{q}\|u_\beta\|_n^{nq}\int_{\mathbb{R}^n\setminus\mathbb{B}_R}|x|^{-n\beta-nq}dx = o_\beta(1). \tag{35}
$$

According to (34) and (35), we find

$$
\int_{\mathbb{R}^n}|x|^{-n\beta}u_{\beta,\iota}^{nq}dx = o_\beta(1) \tag{36}
$$

for any $q\geqslant 1$. By virtue of Lemma 6, we obtain

$$
\lim_{\beta\to 0}\int_{\mathbb{R}^n}(|\nabla u_{\beta,\iota}|^n+\tau|u_{\beta,\iota}|^n)dx = \iota < 1.
$$

Let $1<p<1/\iota$ and $1/p+1/p'=1$. Using (36), Hölder inequality, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n}|x|^{-n\beta}\Phi(n,\alpha_n(1-\beta)u_{\beta,\iota}^{\frac{n}{n-1}})u_{\beta,\iota}^{\frac{n}{n-1}}dx \\
&\qquad \leqslant \left(\int_{\mathbb{R}^n}|x|^{-n\beta}\Phi(n,\alpha_n(1-\beta)pu_{\beta,\iota}^{\frac{n}{n-1}})dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n}|x|^{-n\beta}u_{\beta,\iota}^{\frac{n}{n-1}p'}dx\right)^{\frac{1}{p'}} \\
&\qquad = o_\beta(1).
\end{aligned} \tag{37}
$$

Inserting (36) and (37) into (33), we get

$$
I = \int_{u_\beta\leqslant \iota c_\beta}|x|^{-n\beta}\Phi(n,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})dx = o_\beta(1). \tag{38}
$$

In addition, we estimate

$$\begin{aligned} II &\leqslant \frac{1}{\iota^{\frac{n}{n-1}} c_\beta^{\frac{n}{n-1}}} \int_{u_\beta > \iota c_\beta} |x|^{-n\beta} u_\beta^{\frac{n}{n-1}} \Phi(n, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}}) dx \\ &\leqslant \frac{1}{\iota^{\frac{n}{n-1}} c_\beta^{\frac{n}{n-1}}} \int_{u_\beta > \iota c_\beta} |x|^{-n\beta} u_\beta^{\frac{n}{n-1}} \Phi(n-1, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}}) dx + o_\beta(1) \\ &\leqslant \frac{\lambda_\beta}{\iota^{\frac{n}{n-1}} c_\beta^{\frac{n}{n-1}}} + o_\beta(1). \end{aligned} \tag{39}$$

It then follows from (38) and (39) that

$$\lim_{\beta\to 0} \int_{\mathbb{R}^n} |x|^{-n\beta} \Phi(n, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}}(x)) dx \leqslant \frac{1}{\iota^{\frac{n}{n-1}}} \liminf_{\beta\to 0} \frac{\lambda_\beta}{c_\beta^{\frac{n}{n-1}}}.$$

Taking the limit as $\iota \to 1$ we get

$$\lim_{\beta\to 0} \int_{\mathbb{R}^n} |x|^{-n\beta} \Phi(n, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}}(x)) dx \leqslant \liminf_{\beta\to 0} \frac{\lambda_\beta}{c_\beta^{\frac{n}{n-1}}}. \tag{40}$$

We now verify that

$$\begin{aligned} \frac{\lambda_\beta}{c_\beta^{\frac{n}{n-1}}} &= \int_{\mathbb{R}^n} \frac{u_\beta^{\frac{n}{n-1}}}{c_\beta^{\frac{n}{n-1}}} \frac{\Phi(n-1, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}} dx \\ &= \int_{\mathbb{R}^n} \frac{u_\beta^{\frac{n}{n-1}}}{c_\beta^{\frac{n}{n-1}}} \frac{\Phi(n, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}}) + \frac{\left(\alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}}\right)^{n-2}}{(n-2)!}}{|x|^{n\beta}} dx \\ &\leqslant \int_{\mathbb{R}^n} \frac{\Phi(n, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}} dx + o_\beta(1) \end{aligned}$$

and therefore

$$\limsup_{\beta\to 0} \frac{\lambda_\beta}{c_\beta^{\frac{n}{n-1}}} \leqslant \lim_{\beta\to 0} \int_{\mathbb{R}^n} \frac{\Phi(n, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}} dx. \tag{41}$$

By (40) and (41), we see that

$$\limsup_{\beta\to 0} \int_{\mathbb{R}^n} \frac{\Phi(n, \alpha_n(1-\beta) u_\beta^{\frac{n}{n-1}}(x))}{|x|^{n\beta}} dx = \limsup_{\beta\to 0} \frac{\lambda_\beta}{c_\beta^{\frac{n}{n-1}}}. \tag{42}$$

On the other hand, a direct calculation shows that

$$\begin{aligned}\int_{\mathbb{B}_{Rr_\beta^{1/(1-\beta)}}} \frac{\Phi(n,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}}dx &= \int_{\mathbb{B}_R} r_\beta^n |y|^{-n\beta} e^{\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}}(r_\beta^{\frac{1}{1-\beta}}y)}dy + o_\beta(R) \\ &= \frac{\lambda_\beta}{c_\beta^{\frac{n}{n-1}}}\left(\int_{\mathbb{B}_R} e^{\alpha_n\frac{n}{n-1}\psi_0(y)}dy + o_\beta(R)\right) + o_\beta(R) \\ &= \frac{\lambda_\beta}{c_\beta^{\frac{n}{n-1}}}(1+o_\beta(R)) + o_\beta(R),\end{aligned}$$

where $o_\beta(R) \to 0$ as $\beta \to 0$ for any fixed $R > 0$. Hence the desired result (32) follows by virtue of the above two equalities. □

From Lemma 7 just proved, we obtain successively the following corollary.

COROLLARY 8. *For any $\theta < \frac{n}{n-1}$, we have*

$$\lim_{\beta\to 0}\frac{\lambda_\beta}{c_\beta^\theta} = +\infty.$$

Proof. We suppose by contradiction that $\lambda_\beta / c_\beta^{\frac{n}{n-1}} \to 0$ as $\beta \to 0$. Using (9) and (12), we then get

$$\begin{aligned}\int_{\mathbb{R}^n}\Phi(n,\alpha_n|u|^{\frac{n}{n-1}})dx &\leqslant \sup_{u\in W^{1,n}(\mathbb{R}^n),\, \|u\|_{W^{1,n}(\mathbb{R}^n)}\leqslant 1}\int_{\mathbb{R}^n}\Phi(n,\alpha_n|u|^{\frac{n}{n-1}})dx \\ &\leqslant \liminf_{\beta\to 0}\int_{\mathbb{R}^n}|x|^{-n\beta}\Phi(n,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})dx \\ &\leqslant \limsup_{\beta\to 0}\frac{\lambda_\beta}{c_\beta^{\frac{n}{n-1}}} \\ &= 0\end{aligned}$$

for any fixed $u \in W^{1,n}(\mathbb{R}^n)$ with $\|u\|_{W^{1,n}(\mathbb{R}^n)} = 1$. This is impossible and we obtain the assertion. □

To continue our program, we shall now discuss the convergence of $c_\beta^{\frac{1}{n-1}}u_\beta$ under the assumption $c_\beta \to +\infty$.

LEMMA 9. *The family $c_\beta^{\frac{1}{n-1}}u_\beta \rightharpoonup G_0$ weakly in $W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$ for any $1 < q < n$, and $c_\beta^{\frac{1}{n-1}}u_\beta \to G_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n\setminus\{0\})$. Here G_0 is a Green's function and satisfies*

$$-\Delta_n G_0 + \tau G_0^{n-1} = \delta_0 \tag{43}$$

in a distributional sense, where δ_0 is the usual Dirac measure centered at 0.

Proof. For simplicity, denote

$$g_\beta(x) = \lambda_\beta^{-1}|x|^{-n\beta} c_\beta u_\beta^{\frac{1}{n-1}} \Phi(n-1, \alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}}).$$

We remark first of all that $g_\beta(x)$ converges weakly to δ_0. That is, for any $\eta(x) \in C_0^\infty(\mathbb{R}^n)$, we have

$$\lim_{\beta\to 0}\int_{\mathbb{R}^n} \eta(x) g_\beta(x)dx = \eta(0). \tag{44}$$

To see this, we split the integral

$$\int_{\mathbb{R}^n} \eta(x) g_\beta(x)dx = \int_{u_\beta \leqslant \iota c_\beta} \eta(x) g_\beta(x)dx + \int_{u_\beta > \iota c_\beta} \eta(x) g_\beta(x)dx. \tag{45}$$

In view of Corollary 8, we obtain

$$\begin{aligned}\int_{u_\beta \leqslant \iota c_\beta} \eta(x) g_\beta(x)dx &\leqslant \frac{c_\beta}{\lambda_\beta}\left(\sup_{\mathbb{R}^n}|\eta(x)|\right)\int_{u_\beta \leqslant \iota c_\beta} \frac{u_\beta^{\frac{1}{n-1}}\Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}}dx \\ &\leqslant \frac{c_\beta}{\lambda_\beta}\left(\sup_{\mathbb{R}^n}|\eta(x)|\right)\left(\int_{\mathbb{R}^n} \frac{u_{\beta,\iota}^{\frac{1}{n-1}}\Phi(n,\alpha_n(1-\beta)u_{\beta,\iota}^{\frac{n}{n-1}})}{|x|^{n\beta}}dx + o_\beta(1)\right) \\ &= o_\beta(1).\end{aligned} \tag{46}$$

By Lemma 4, one can easily see that $\mathbb{B}_{Rr_\beta^{1/(1-\beta)}} \subset \{u_\beta > \iota c_\beta\}$ for β sufficiently small. Then we have

$$\int_{\{u_\beta > \iota c_\beta\}\bigcap \mathbb{B}_{Rr_\beta^{1/(1-\beta)}}} \eta(x) g_\beta(x)dx = \eta(0)(1+o_\beta(1))\left(\int_{\mathbb{B}_R} e^{\alpha_n\frac{n}{n-1}\psi_0(x)}dx + o_\beta(R)\right).$$

On the other hand, we obtain

$$\int_{\{u_\beta > \iota c_\beta\}\setminus \mathbb{B}_{Rr_\beta^{1/(1-\beta)}}} \eta(x) g_\beta(x)dx \leqslant \frac{1}{\iota}\left(\sup_{\mathbb{R}^n}|\eta(x)|\right)\left(1-\int_{\mathbb{B}_R} e^{\alpha_n\frac{n}{n-1}\psi_0(x)}dx + o_\beta(R)\right).$$

It then follows that

$$\lim_{\beta\to 0}\int_{u_\beta > \iota c_\beta} \eta(x) g_\beta(x)dx = \eta(0). \tag{47}$$

Inserting (46) and (47) into (45), letting $\beta \to 0$ then $R \to +\infty$, we see (44) holds.

Multiplying both sides of the equation (8) by c_β, one has

$$-\Delta_n(c_\beta^{\frac{1}{n-1}}u_\beta) + \tau c_\beta u_\beta^{n-1} = \frac{c_\beta u_\beta^{\frac{1}{n-1}}}{\lambda_\beta}\frac{\Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}}. \tag{48}$$

According to (44), $g_\beta(x) = \lambda_\beta^{-1}|x|^{-n\beta} c_\beta u_\beta^{\frac{1}{n-1}} \Phi(n-1, \alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})$ is bounded in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Proceeding as in the proof of ([18], Proposition 3.7), one sees that $c_\beta^{\frac{1}{n-1}} u_\beta$ is bounded in $W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$ for any $1 < q < n$. Hence, we have $c_\beta^{\frac{1}{n-1}} u_\beta \rightharpoonup G_0$ weakly in $W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$. Applying standard elliptic estimates to (48), we get that $c_\beta^{\frac{1}{n-1}} u_\beta \to G_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$. Now (44) implies that $g_\beta \rightharpoonup \delta_0$ in the sense of measure, where δ_0 denotes the Dirac measure centered at 0. In view of (48), G_0 is a distributional solution to (43). This ends the proof of the lemma. □

For the Green function G_0, we have

$$G_0(x) = -\frac{1}{\alpha_n} \log |x|^n + A_0 + \rho(x); \tag{49}$$

here, A_0 is a constant, $\rho(x) \in C^1(\mathbb{R}^n \backslash \{0\})$ satisfying $\rho(x) = O(|x|^n \log^n |x|)$ as $|x| \to 0$. The proof is similar to ([18], Lemma 3.8). We omit the details but refer the reader to [18].

To proceed further, we need the following version of Carleson-Chang's upper bound estimate. The difference is that a singular term $|x|^{-n\beta}$ is involved in our case.

LEMMA 10. *Let $w_\beta \in W_0^{1,n}(\mathbb{B}_r)$ with $\int_{\mathbb{B}_r} |\nabla w_\beta|^n dx \leqslant 1$, $w_\beta \rightharpoonup 0$ weakly in $W_0^{1,n}(\mathbb{B}_r)$ as $\beta \to 0$, w_β is nonnegative and radially symmetric. Then we have*

$$\limsup_{\beta \to 0} \int_{\mathbb{B}_r} |x|^{-n\beta} (e^{\alpha_n(1-\beta) w_\beta^{\frac{n}{n-1}}} - 1) dx \leqslant \frac{\omega_{n-1}}{n} r^n e^{\sum_{k=1}^{n-1} \frac{1}{k}}. \tag{50}$$

Proof Since w_β is radially symmetric, we write $w_\beta(|x|) = w_\beta(x)$. Let

$$v_\beta(x) = (1-\beta)^{\frac{n-1}{n}} w_\beta(|x|^{\frac{1}{1-\beta}}).$$

Then $\int_{\mathbb{B}_1} |\nabla v_\beta|^n dx = \int_{\mathbb{B}_1} |\nabla w_\beta|^n dx$. Furthermore, one can check that $\|v_\beta\|_{W_0^{1,n}(\mathbb{B}_1)}\| = \|w_\beta\|_{W_0^{1,n}(\mathbb{B}_1)} \leqslant 1$. Without loss of generality, we can assume that up to a subsequence, $v_\beta \rightharpoonup v_*$ weakly in $W_0^{1,n}(\mathbb{B}_1)$, $v_\beta \to v_*$ strongly in $L^p(\mathbb{B}_1)$ for any $p > 1$, and $v_\beta \to v_*$ almost everywhere in $\mathbb{B}_1$. It is clear to see $w_\beta \to 0$ almost everywhere in $\mathbb{B}_1$ and hence it follows $v_* = 0$ almost everywhere in $\mathbb{B}_1$. By a change of variable $t = s^{1/(1-\beta)}$, we obtain

$$\begin{aligned}
\int_{\mathbb{B}_1} |x|^{-n\beta} (e^{\alpha_n(1-\beta) w_\beta^{\frac{n}{n-1}}(x)} - 1) dx &= \omega_{n-1} \int_0^1 t^{n-1-n\beta} (e^{\alpha_n(1-\beta) w_\beta^{\frac{n}{n-1}}(t)} - 1) dt \\
&= \frac{\omega_{n-1}}{1-\beta} \int_0^1 s^{n-1} (e^{\alpha_n(1-\beta) w_\beta^{\frac{n}{n-1}}(s^{1/(1-\beta)})} - 1) ds \\
&= \frac{1}{1-\beta} \int_{\mathbb{B}_1} (e^{\alpha_n v_\beta^{\frac{n}{n-1}}(x)} - 1) dx.
\end{aligned}$$

Applying the result of Carleson and Chang [6], we have

$$\begin{aligned}&\limsup_{\beta\to 0}\int_{\mathbb{B}_1}|x|^{-n\beta}(e^{\alpha_n(1-\beta)w_\beta^{\frac{n}{n-1}}(x)}-1)dx\\ \leqslant &\lim_{\beta\to 0}\frac{1}{1-\beta}\frac{\omega_{n-1}}{n}e^{\sum_{k=1}^{n-1}\frac{1}{k}}=\frac{\omega_{n-1}}{n}e^{\sum_{k=1}^{n-1}\frac{1}{k}}. \end{aligned}\tag{51}$$

This proves (50) for $r=1$.

Set $\overline{w}_\beta(x)=w_\beta(rx)$ for any $x\in\mathbb{B}_1$. We can computer

$$\int_{\mathbb{B}_r}|\nabla w_\beta(x)|^n dx=\int_{\mathbb{B}_1}|\nabla\overline{w}_\beta(x)|^n dx$$

and

$$\int_{\mathbb{B}_r}|x|^{-n\beta}(e^{\alpha_n(1-\beta)w_\beta^{\frac{n}{n-1}}}-1)dx=r^{n-n\beta}\int_{\mathbb{B}_1}|x|^{-n\beta}(e^{\alpha_n(1-\beta)\overline{w}_\beta^{\frac{n}{n-1}}}-1)dx.$$

Let $\beta\to 0$ and by virtue of (51), the estimate (50) follows at once. □

Now we will apply Lemma 10 to derive an upper bound of the integral

$$\int_{\mathbb{R}^n}\Phi(n,\alpha_n|u|^{\frac{n}{n-1}})dx.$$

LEMMA 11. *There holds*

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\|u\|_{W^{1,n}(\mathbb{R}^n)}\leqslant 1}\int_{\mathbb{R}^n}\Phi(n,\alpha_n|u|^{n/(n-1)})dx\leqslant\frac{\omega_{n-1}}{n}e^{\alpha_n A_0+\sum_{k=1}^{n-1}\frac{1}{k}},\tag{52}$$

where A_0 is given as in (49).

Proof. Let ν be the unit outward normal to $\partial\mathbb{B}_\delta$. By equation (48), we see that

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus\mathbb{B}_\delta}(|\nabla G_0|^n+\tau|G_0|^n)dx\\ =&-\int_{\partial\mathbb{B}_\delta}G_0|\nabla G_0|^{n-2}\frac{\partial G_0}{\partial\nu}ds+\frac{c_\beta^{\frac{n}{n-1}}}{\lambda_\beta}\int_{\mathbb{R}^n\setminus\mathbb{B}_\delta}u_\beta^{\frac{n}{n-1}}\frac{\Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}}dx.\end{aligned}$$

We estimate the right two terms in the equation. The first term can be calculated by

$$\begin{aligned}-\int_{\partial\mathbb{B}_\delta}G_0|\nabla G_0|^{n-2}\frac{\partial G_0}{\partial\nu}ds&=-G_0(\delta)\int_{\mathbb{B}_\delta}\Delta_n G_0dx\\ &=G_0(\delta)-\tau G_0(\delta)\int_{\mathbb{B}_\delta}G_0^{n-1}dx\end{aligned}$$

since G_0 is a distributional solution of equation (43). A straightforward calculation on the second term reads

$$\frac{c_\beta^{\frac{n}{n-1}}}{\lambda_\beta}\int_{\mathbb{R}^n\setminus\mathbb{B}_\delta} u_\beta^{\frac{n}{n-1}}\frac{\Phi(n-1,\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}})}{|x|^{n\beta}}dx=o_\beta(1).$$

With the aid of (49), we then have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\mathbb{B}_\delta}(|\nabla u_\beta|^n+\tau|u_\beta|^n)dx &= \frac{1}{c_\beta^{\frac{n}{n-1}}}\left(\int_{\mathbb{R}^n\setminus\mathbb{B}_\delta}(|\nabla G_0|^n+\tau|G_0|^n)dx+o_\beta(1)\right)\\ &= \frac{1}{c_\beta^{\frac{n}{n-1}}}\left(G_0(\delta)-\tau G_0(\delta)\int_{\mathbb{B}_\delta}G_0^{n-1}dx+o_\beta(1)\right)\\ &= \frac{1}{c_\beta^{\frac{n}{n-1}}}\left(\frac{n}{\alpha_n}\log\frac{1}{\delta}+A_0+o_\beta(1)+o_\delta(1)\right),\end{aligned}$$

where $o_\delta(1)\to 0$ as $\delta\to 0$. Set

$$\begin{aligned}\kappa_\beta^{n-1} &:= \int_{\mathbb{B}_\delta}|\nabla u_\beta|^n dx\\ &= 1-\int_{\mathbb{R}^n\setminus\mathbb{B}_\delta}(|\nabla u_\beta|^n+\tau u_\beta^n)dx-\tau\int_{\mathbb{B}_\delta}u_\beta^n dx\\ &= 1-\frac{1}{c_\beta^{\frac{n}{n-1}}}\left(\frac{n}{\alpha_n}\log\frac{1}{\delta}+A_0+o_\beta(1)+o_\delta(1)\right)+o_\beta(1).\end{aligned}$$

Denote $s_\beta=\sup_{\partial\mathbb{B}_\delta}u_\beta=u_\beta(\delta)$ and $\widetilde{u}_\beta=(u_\beta-s_\beta)^+$. Obviously, $\widetilde{u}_\beta\in W_0^{1,n}(\mathbb{B}_\delta)$. Moreover, we have $\int_{\mathbb{B}_\delta}|\nabla\widetilde{u}_\beta|^n dx\leqslant\kappa_\beta^{n-1}$. Since $s_\beta=c_\beta^{-\frac{1}{n-1}}(\sup_{\partial\mathbb{B}_\delta}G_0+o_\beta(1))$, we get

$$\begin{aligned}\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}} &\leqslant \alpha_n(1-\beta)(\widetilde{u}_\beta+u_\beta(\delta))^{\frac{n}{n-1}}\\ &\leqslant \alpha_n(1-\beta)\widetilde{u}_\beta^{\frac{n}{n-1}}+\frac{n}{n-1}\alpha_n(1-\beta)\widetilde{u}_\beta^{\frac{1}{n-1}}u_\beta(\delta)+o_\beta(1)\\ &\leqslant \alpha_n(1-\beta)\widetilde{u}_\beta^{\frac{n}{n-1}}+\frac{n}{n-1}\alpha_n(1-\beta)G_0(\delta)+o(1)\\ &\leqslant \alpha_n(1-\beta)\widetilde{u}_\beta^{\frac{n}{n-1}}/\kappa_\beta+n(1-\beta)\log\frac{1}{\delta}+\alpha_n(1-\beta)A_0+o(1).\end{aligned}$$

where $o(1) \to 0$ as $\beta \to 0$ first and next $\delta \to 0$. For any fixed $R > 0$, we have $\mathbb{B}_{Rr_\beta^{1/(1-\beta)}} \subset \mathbb{B}_\delta$ for β small. As a consequence, we have

$$\begin{aligned}
&\int_{\mathbb{B}_{Rr_\beta^{1/(1-\beta)}}} \frac{e^{\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}}}-1}{|x|^{n\beta}}dx \\
&\leqslant \delta^{-n(1-\beta)}e^{\alpha_n(1-\beta)A_0+o(1)}\int_{\mathbb{B}_{Rr_\beta^{1/(1-\beta)}}} \frac{e^{\alpha_n(1-\beta)\widetilde{u}_\beta^{\frac{n}{n-1}}/\kappa_\beta}}{|x|^{n\beta}}dx + o_\beta(1) \\
&= \delta^{-n(1-\beta)}e^{\alpha_n(1-\beta)A_0+o(1)}\int_{\mathbb{B}_{Rr_\beta^{1/(1-\beta)}}} \frac{e^{\alpha_n(1-\beta)\widetilde{u}_\beta^{\frac{n}{n-1}}/\kappa_\beta}-1}{|x|^{n\beta}}dx + o_\beta(1) \\
&\leqslant \delta^{-n(1-\beta)}e^{\alpha_n(1-\beta)A_0+o(1)}\int_{\mathbb{B}_\delta} \frac{e^{\alpha_n(1-\beta)\widetilde{u}_\beta^{\frac{n}{n-1}}/\kappa_\beta}-1}{|x|^{n\beta}}dx + o_\beta(\delta).
\end{aligned}$$

where $o_\beta(\delta) \to 0$ as $\beta \to 0$ for any fixed $\delta > 0$. It follows by Lemma 10 that

$$\lim_{R\to+\infty}\limsup_{\beta\to 0}\int_{\mathbb{B}_{Rr_\beta^{1/(1-\beta)}}} \frac{e^{\alpha_n(1-\beta)u_\beta^{\frac{n}{n-1}}}-1}{|x|^{n\beta}}dx \leqslant \frac{\omega_{n-1}}{n}e^{\alpha_n A_0+\sum_{k=1}^{n-1}\frac{1}{k}}. \tag{53}$$

In view of (9), (32) and (42), we obtain (52). The proof is complete. □

2.3. Exclusion of blow-up phenomenon

Completion of the proof of Theorem 1. According to [18], there exists a function sequence $\phi(x)$ such that

$$\int_{\mathbb{R}^n}\Phi(n,\alpha_n|\phi(x)|^{\frac{n}{n-1}})dx > \frac{\omega_{n-1}}{n}e^{\alpha_n A_0+\sum_{k=1}^{n-1}\frac{1}{k}}.$$

Hence, we have

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\,\|u\|_{W^{1,n}(\mathbb{R}^n)}\leqslant 1}\int_{\mathbb{R}^n}\Phi(n,\alpha_n|u(x)|^{\frac{n}{n-1}})dx > \frac{\omega_{n-1}}{n}e^{\alpha_n A_0+\sum_{k=1}^{n-1}\frac{1}{k}}.$$

which contradicts with (53) and implies that u_β must be uniformly bounded. Then applying elliptic to the equation (8), we conclude that u_β converges in $C^1(\mathbb{R}^n)$ to a maximizer of the supremum (7). Thus this completes the proof of Theorem 1. □

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