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# SOME GENERATOR FUNCTIONS FOR $s$–CONVEX FUNCTIONS IN THE FOURTH SENSE

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*Abstract.* In this paper, some generator functions for $s$-convex functions in the fourth sense and some of their properties are studied. As application examples, some special mean relations and the inequalities involving beta and digamma functions are obtained throught some of the properties.

## 1. Introduction

Convex functions have a very special place in terms of their relation to the optimization theory, algebra and analysis. Many extensions and generalizations have been defined so far and still novel ones are still being exposed. For example, $B$-convexity, $B^{-1}$-convexity, $p$-convexity, quasi-$p$-convexity, log-$p$-convexity and $s$-convexity (in third and fourth senses) are novel ones [6, 18, 8, 3, 13, 2]. As far as we reviewed related literature, some of the novel types have been defined so as to state some inequalities such as predominantly, the Hermite-Hadamard inequalities for these type functions and there are few studies on the characterizations and examples of such type functions ([12, 16, 17, 6, 14, 1, 7, 9] and the references therein). In this paper, we address to this issue for $s$-convex functions in the fourth sense. The generator functions for the $s$-convex functions of the fourth sense are presented by means of the integral and double integral and some properties are shown. Then we give some application examples as inequalities involving digamma and beta functions and some mean relations.

## 2. Preliminaries

The most eminent characterization of convex function on real numbers is given by inequality as follows:

Let $f : A \to \mathbb{R}$ be real valued function. The function $f$ is said to be convex if

$$f(\lambda x + \mu y) \leqslant \lambda f(x) + \mu f(y) \tag{1}$$

for all $x, y \in A$ and $\lambda, \mu \in [0,1]$ with $\lambda + \mu = 1$.

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For the sake of brevity and clarity, let us introduce some notions of $s$-convexity. Let $s$ be fixed number in $(0,1]$ and $a,b \in R$. An $s$-convex combination of $a$ and $b$ is $\lambda a + \mu b$ such that $\lambda^s + \mu^s = 1$ for $\lambda, \mu \in [0,1]$.

$s$-convexity is such a generalization of classical convexity that while the convexity sets the relation (1) between the image of convex combinations of points and the convex combinations of the image of points in a convex sets, $s$-convexity sets the similar relation between mixture of convex and $s$-convex combinations of points and images. The four types of $s$-convex functions are introduced with respect to usage of convex or $s$-convex like combinations in their definitions. The first and third senses of $s$-convex functions are given in [15] and [13], respectively, as follows, which are defined in a $s$-convex subset of real numbers, i.e., a set containing all $s$-convex combinations of its elements. In the case of $s = 1$, s-convex set is a convex set.

Let $A$ be $s$-convex subset of real numbers and $f : A \to \mathbb{R}$ be real valued function. The function $f$ is said to be $s$-convex in the first sense if

$$f(\lambda x + \mu y) \leqslant \lambda^s f(x) + \mu^s f(y)$$

and the function $f$ is said to be $s$-convex in the third sense if

$$f(\lambda x + \mu y) \leqslant \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y)$$

for all $x,y \in A$ and $\lambda, \mu \in [0,1]$ with $\lambda^s + \mu^s = 1$.

The second and fourth senses $s$-convex functions are defined on convex sets and they are given in [5] and [8], respectively.

Let $A$ be a convex subset of real numbers and $f : A \to \mathbb{R}$. The function $f$ is said to be $s$-convex in the second sense if

$$f(\lambda x + \mu y) \leqslant \lambda^s f(x) + \mu^s f(y)$$

and the function $f$ is said to be $s$-convex in the fourth sense if

$$f(\lambda x + \mu y) \leqslant \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y)$$

for all $x,y \in A$ and $\lambda, \mu \in [0,1]$ with $\lambda + \mu = 1$.

The class of $s$-convex functions in the fourth sense is denoted by $K_s^4$. Although they have similar definitions, they have certain distinctions. For example, while $s$-convex functions in the second sense are positive valued, $s$-convex functions in the fourth sense are negative valued functions. Some of their characterizations can be found in [8] and [11].

In [10], it is shown that for $a,b,c \in \mathbb{R}_-$ (negative real numbers) with $c \leqslant 0$,

$$\psi(x) = \begin{cases} a, & \text{if } x = 0 \\ bx^{\frac{1}{s}} + c, & \text{if } x > 0 \end{cases} \tag{2}$$

is an $s$-convex function in the fourth sense on $(0,\infty)$. Under the extra condition $a = c$, $\psi$ is $s$-convex function in the fourth sense on $[0,\infty)$.

## 3. Main results

In this section, we present three types of generator functions for $s$-convex functions in the fourth sense and their properties, which are expressed via single integral or double integral representations. Since some of the properties are based on the Hermite-Hadamard inequality for the $s$-convex functions of the fourth sense, which is stated in [14], let us give it as a lemma.

LEMMA 1. *Let $\psi : [a,b] \to \mathbb{R}_-$ be an $s$-convex function in the fourth sense, integrable on $[a,b]$. Then the following inequality holds,*

$$2^{\frac{1}{s}-1}\psi\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b \psi(x)dx \leqslant \frac{s}{s+1}\left[\psi(a)+\psi(b)\right]. \tag{3}$$

In the following theorem, a generator function is defined for the class $K_s^4$ and some properties of this generator function are given.

THEOREM 1. *Let $\psi$ be an integrable function on $[a,b]$ and let $G$ be defined as follows:*

$$G(t) := \frac{1}{b-a}\int_a^b \psi\left(tx+(1-t)\frac{a+b}{2}\right)dx \tag{4}$$

*for $t \in [0,1]$.*

*$i)$ If $\psi$ is an $s$-convex function in the fourth sense on interval $[a,b]$, then $G$ is an $s$-convex function in the fourth sense on $[0,1]$,*

*$ii)$ If $\psi$ is an $s$-convex function in the fourth sense on interval $[a,b]$, then the following inequality holds,*

$$G(t) \geqslant 2^{\frac{1}{s}-1}\psi\left(\frac{a+b}{2}\right). \tag{5}$$

*Proof.* $i)$ Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geqslant 0$ with $\alpha+\beta = 1$. By using $\psi$ is an $s$-convex function in the fourth sense on interval $[a,b]$, we get

$$\begin{aligned}
&G(\alpha t_1+\beta t_2)\\
&= \frac{1}{b-a}\int_a^b \psi\left((\alpha t_1+\beta t_2)x+(1-\alpha t_1-\beta t_2)\frac{a+b}{2}\right)dx\\
&= \frac{1}{b-a}\int_a^b \psi\left(\alpha\left(t_1x+(1-t_1)\frac{a+b}{2}\right)+\beta\left(t_2x+(1-t_2)\frac{a+b}{2}\right)\right)dx\\
&\leqslant \frac{1}{b-a}\int_a^b \left[\alpha^{\frac{1}{s}}\psi\left(t_1x+(1-t_1)\frac{a+b}{2}\right)+\beta^{\frac{1}{s}}\psi\left(t_2x+(1-t_2)\frac{a+b}{2}\right)\right]dx
\end{aligned}$$

$$= \alpha^{\frac{1}{s}} \frac{1}{b-a} \int_a^b \psi\left(t_1 x + (1-t_1)\frac{a+b}{2}\right) dx + \beta^{\frac{1}{s}} \frac{1}{b-a} \int_a^b \psi\left(t_2 x + (1-t_2)\frac{a+b}{2}\right) dx$$
$$= \alpha^{\frac{1}{s}} G(t_1) + \beta^{\frac{1}{s}} G(t_2).$$

It shows that $G$ is an $s$-convex function in the fourth sense on $[a,b]$.

*ii*) In the case of a $t \in (0,1]$, the following equality is obtained by taking $u = tx + (1-t)\frac{a+b}{2}$

$$G(t) = \frac{1}{p-q} \int_q^p \psi(u) du$$

where $p = tb + (1-t)\frac{a+b}{2}$ and $q = ta + (1-t)\frac{a+b}{2}$.

Applying inequality (3), we have

$$\begin{aligned} G(t) &= \frac{1}{p-q} \int_q^p \psi(u) du \\ &\geqslant 2^{\frac{1}{s}-1} \psi\left(\frac{p+q}{2}\right) \\ &= 2^{\frac{1}{s}-1} \psi\left(\frac{a+b}{2}\right) \end{aligned}$$

and the inequality (5) is obtained.

In the case of $t = 0$, since $\psi\left(\frac{a+b}{2}\right) < 0$ and $2^{\frac{1}{s}-1} \geqslant 1$, the inequality (5) is also provided, i.e.;

$$G(0) = \psi\left(\frac{a+b}{2}\right) \geqslant 2^{\frac{1}{s}-1} \psi\left(\frac{a+b}{2}\right). \quad \square$$

Using (2), hence letting $\psi(x) = -x^{\frac{1}{s}}$ on $[a,b] = [0,1]$ in Theorem 1, we have the following corollary.

COROLLARY 1. *The function*

$$g(t) = \begin{cases} \frac{s}{2^{1+\frac{1}{s}}(1+s)} \frac{(1-t)^{\frac{1}{s}+1} - (1+t)^{\frac{1}{s}+1}}{t}, & t \in (0,1] \\ -2^{-\frac{1}{s}}, & t = 0 \end{cases}$$

*is $s$-convex in the fourth sense.*

THEOREM 2. *Let the functions $G_1$ and $G_2$ be defined on $[0,1]$ as follows:*

$$G_1(t) := \frac{1}{b-a} t^{\frac{1}{s}} \int_a^b \psi(x) dx + (1-t)^{\frac{1}{s}} \psi\left(\frac{a+b}{2}\right)$$

*and*

$$G_2(t) := \frac{s}{s+1}\left(\psi\left(ta+(1-t)\frac{a+b}{2}\right)+\psi\left(tb+(1-t)\frac{a+b}{2}\right)\right).$$

*If $\psi$ is an $s$-convex function in the fourth sense on $[a,b]$, then*

$$G(t) \leqslant \min\left(G_1(t), G_2(t)\right) \tag{6}$$

*for $t \in [0,1]$, where $G(t)$ is defined as in Theorem 1.*

*Proof.* In the case of $t \in (0,1]$, we have the following inequality,

$$\begin{aligned} G(t) &= \frac{1}{p-q}\int_q^p \psi(u)du \\ &\leqslant \frac{s}{1+s}\left(\psi(p)+\psi(q)\right) \\ &= \frac{s}{1+s}\left(\psi\left(tb+(1-t)\frac{a+b}{2}\right)+\psi\left(ta+(1-t)\frac{a+b}{2}\right)\right). \end{aligned}$$

So, $G(t) \leqslant G_2(t)$.

For $G_1(t)$, since $\psi$ is $s$-convex function in the fourth sense, we have

$$\psi\left(tx+(1-t)\frac{a+b}{2}\right) \leqslant t^{\frac{1}{s}}\psi(x)+(1-t)^{\frac{1}{s}}\psi\left(\frac{a+b}{2}\right)$$

and integrating this inequality on $[a,b]$ we get

$$G(t) \leqslant \frac{1}{b-a}t^{\frac{1}{s}}\int_a^b \psi(x)dx+(1-t)^{\frac{1}{s}}\psi\left(\frac{a+b}{2}\right).$$

So, $G(t) \leqslant G_1(t)$.

In the case of $t=0$,

$$G(0) = \frac{1}{b-a}\int_a^b \psi\left(\frac{a+b}{2}\right)dx = \psi\left(\frac{a+b}{2}\right)$$

so, $G(0) = G_1(0)$ and $G_2(0) = \frac{2s}{s+1}\psi\left(\frac{a+b}{2}\right)$. Since $\psi\left(\frac{a+b}{2}\right) < 0$ and $0 < \frac{2s}{s+1} < 1$, we get

$$G(0) = \psi\left(\frac{a+b}{2}\right) \leqslant \frac{2s}{s+1}\psi\left(\frac{a+b}{2}\right).$$

Thus, $G(t) \leqslant G_2(t)$. □

THEOREM 3. *Let $\psi$ be an $s$-convex function in the fourth sense on $[a,b]$ and let $G_1, G_2$ be defined as in Theorem 2. If $\widetilde{G}(t) := \max\left(G_1(t), G_2(t)\right)$ for $t \in [0,1]$, then*

$$\widetilde{G}(t) \leqslant \frac{s}{s+1}\left(t^{\frac{1}{s}}\left(\psi(a)+\psi(b)\right)+(1-t)^{\frac{1}{s}}2\psi\left(\frac{a+b}{2}\right)\right).$$

*Proof.* By inequality (3) and since $0<\frac{2s}{s+1}<1,$ we can write the following inequalities,

$$\frac{1}{b-a}t^{\frac{1}{s}}\int_{a}^{b}\psi(x)dx\leqslant\frac{s}{s+1}t^{\frac{1}{s}}\left(\psi(a)+\psi(b)\right),$$

and

$$(1-t)^{\frac{1}{s}}\psi\left(\frac{a+b}{2}\right)\leqslant(1-t)^{\frac{1}{s}}\frac{2s}{s+1}\psi\left(\frac{a+b}{2}\right)$$

for $t\in(0,1)$. So that

$$\begin{aligned}G_1(t)&=\frac{1}{b-a}t^{\frac{1}{s}}\int_{a}^{b}\psi(x)dx+(1-t)^{\frac{1}{s}}\psi\left(\frac{a+b}{2}\right)\\&\leqslant t^{\frac{1}{s}}\frac{s}{s+1}\left(\psi(a)+\psi(b)\right)+(1-t)^{\frac{1}{s}}\frac{2s}{s+1}\psi\left(\frac{a+b}{2}\right)\\&=\frac{s}{s+1}\left[t^{\frac{1}{s}}\left(\psi(a)+\psi(b)\right)+(1-t)^{\frac{1}{s}}2\psi\left(\frac{a+b}{2}\right)\right].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}G_2(t)&=\frac{s}{s+1}\left[\psi\left(ta+(1-t)\frac{a+b}{2}\right)+\psi\left(tb+(1-t)\frac{a+b}{2}\right)\right]\\&\leqslant\frac{s}{s+1}\left[t^{\frac{1}{s}}\psi(a)+(1-t)^{\frac{1}{s}}\psi\left(\frac{a+b}{2}\right)+t^{\frac{1}{s}}\psi(b)+(1-t)^{\frac{1}{s}}\psi\left(\frac{a+b}{2}\right)\right]\\&=\frac{s}{s+1}\left[t^{\frac{1}{s}}\left(\psi(a)+\psi(b)\right)+(1-t)^{\frac{1}{s}}2\psi\left(\frac{a+b}{2}\right)\right].\end{aligned}$$

Thus, we get $\widetilde{G}\leqslant\frac{s}{s+1}\left(t^{\frac{1}{s}}\left(\psi(a)+\psi(b)\right)+(1-t)^{\frac{1}{s}}2\psi\left(\frac{a+b}{2}\right)\right).$ □

Another generator function can be given via integral representation as follows:

THEOREM 4. *Let us define the following function on* $[0,1]$

$$J(t)=\frac{1}{2(b-a)}\int_a^b\left[\psi\left(\frac{1+t}{2}a+\frac{1-t}{2}x\right)+\psi\left(\frac{1+t}{2}b+\frac{1-t}{2}x\right)\right]dx.$$

$J(t)$ *is* $s$*-convex in the fourth sense.*

*Proof.* Let $\alpha,\beta\geqslant0$ with $\alpha+\beta=1$ and $t_1,t_2\in[0,1]$. Then

$$\begin{aligned}&J\left(\alpha t_1+\beta t_2\right)\\&=\frac{1}{2(b-a)}\int_a^b\left[\psi\left(\frac{1+(\alpha t_1+\beta t_2)}{2}a+\frac{1-(\alpha t_1+\beta t_2)}{2}x\right)\right.\\&\quad\left.+\psi\left(\frac{1+(\alpha t_1+\beta t_2)}{2}b+\frac{1-(\alpha t_1+\beta t_2)}{2}x\right)\right]dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2(b-a)} \int_a^b \left[ \psi\left( \alpha \frac{(1+t_1)a+(1-t_1)x}{2} + \beta \frac{(1+t_2)a+(1-t_2)x}{2} \right) \right. \\
&\quad \left. + \psi\left( \alpha \frac{(1+t_1)b+(1-t_1)x}{2} + \beta \frac{(1+t_2)b+(1-t_2)x}{2} \right) \right] dx \\
&\leqslant \frac{\alpha^{\frac{1}{s}}}{2(b-a)} \int_a^b \left[ \psi\left( \frac{(1+t_1)a+(1-t_1)x}{2} \right) + \psi\left( \frac{(1+t_1)b+(1-t_1)x}{2} \right) \right] dx \\
&\quad + \frac{\beta^{\frac{1}{s}}}{2(b-a)} \int_a^b \left[ \psi\left( \frac{(1+t_2)a+(1-t_2)x}{2} \right) + \psi\left( \frac{(1+t_2)b+(1-t_2)x}{2} \right) \right] dx \\
&= \alpha^{\frac{1}{s}} J(t_1) + \beta^{\frac{1}{s}} J(t_2). \quad \square
\end{aligned}
$$

Letting $[a,b]=[0,1]$ and $\psi(x) = -x^{\frac{1}{s}}$ in Theorem 4, we have the following corollary.

COROLLARY 2. *The function*

$$
j(t) = \begin{cases} \frac{1}{2^{\frac{1}{s}+1}} \frac{s}{(s+1)} \frac{(1-t)^{\frac{1}{s}+1}+2^{\frac{1}{s}+1}-(t+1)^{\frac{1}{s}+1}}{t-1}, & t \in [0,1) \\ -\frac{1}{2}, & t = 1 \end{cases}
$$

*is $s$-convex in the fourth sense.*

In the following theorem, a new generator function for the class $K_s^4$ is given by a double integral function and its properties are mentioned.

THEOREM 5. *Let $\psi$ be integrable function on $[a,b]$ and*

$$
F(t) := \frac{1}{(b-a)^2} \int_a^b \int_a^b \psi(tx + (1-t)y)\, dx\, dy
$$

*for $t \in [0,1]$. If $\psi$ is $s$-convex in the fourth sense on $[a,b]$, then $F$ is also $s$-convex in the fourth sense.*

*Proof.* Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$. Then

$$
\begin{aligned}
F(\alpha t_1 + \beta t_2) &= \frac{1}{(b-a)^2} \int_a^b \int_a^b \psi((\alpha t_1 + \beta t_2)x + (1 - \alpha t_1 - \beta t_2)y)\, dx\, dy \\
&= \frac{1}{(b-a)^2} \int_a^b \int_a^b \psi(\alpha(t_1 x + (1-t_1)y) + \beta(t_2 x + (1-t_2)y)\, dx\, dy
\end{aligned}
$$

$$\leqslant \alpha^{\frac{1}{s}} \frac{1}{(b-a)^2} \int_a^b \int_a^b \psi(t_1 x + (1-t_1)y)\, dxdy$$

$$+\beta^{\frac{1}{s}} \frac{1}{(b-a)^2} \int_a^b \int_a^b \psi(t_2 x + (1-t_2)y) dxdy$$

$$= \alpha^{\frac{1}{s}} F(t_1) + \beta^{\frac{1}{s}} F(t_2). \quad \square$$

Letting $[a,b]=[0,1]$ and $\psi(x) = -x^{\frac{1}{s}}$ in Theorem 5, we have the following corollary.

COROLLARY 3. *The function*

$$f(t) = \begin{cases} \frac{s^2}{(2s+1)(s+1)} \frac{(1-t)^{\frac{1}{s}+2} + t^{\frac{1}{s}+2} - 1}{t(1-t)}, & t \in (0,1) \\ -\frac{s}{s+1}, & t = 0 \ \ or \ \ t = 1 \end{cases}$$

*is $s$-convex in the fourth sense.*

THEOREM 6. *Let $\psi$ be an $s$-convex function in the fourth sense on $[a,b]$. Then for $t \in [0,1]$, we have*

*i)* $2^{1-\frac{1}{s}} F(t) \geqslant \frac{1}{(b-a)^2} \int_a^b \int_a^b \psi(\frac{x+y}{2}) dxdy,$

*ii)* $F(t) \geqslant 2^{\frac{1}{s}-1} \max(G(t), G(1-t)),$

*iii)* $F(t) \leqslant \left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right) \frac{1}{b-a} \int_a^b \psi(x) dx,$

*iv)* $F(t) \leqslant \frac{s^2}{(s+1)^2} [\psi(a) + \psi(ta + (1-t)b) + \psi(tb + (1-t)a) + \psi(b)].$

*Proof.* *i)* From $s$-convexity in the fourth sense of $\psi$, we get

$$\psi(\frac{x+y}{2}) \leqslant \frac{\psi(tx + (1-t)y) + \psi(ty + (1-t)x)}{2^{\frac{1}{s}}}$$

for all $t \in [0,1]$ and $x, y \in [a,b]$. Integrating this inequality on $[a,b]^2$, we get

$$\int_a^b \int_a^b \psi(\frac{x+y}{2}) dxdy \leqslant \int_a^b \int_a^b \frac{\psi(tx + (1-t)y) + \psi(ty + (1-t)x)}{2^{\frac{1}{s}}} dxdy$$

here

$$\int_a^b \int_a^b \psi(tx + (1-t)y) dxdy = \int_a^b \int_a^b \psi(ty + (1-t)x) dxdy$$

which yields $(i)$.

$ii)$ For $y \in [a,b]$, let us define a function $G_y(t)$ as follows,

$$G_y(t) := \frac{1}{b-a} \int_a^b \psi(tx + (1-t)y)dx.$$

In the case of $t \in (0,1]$ we take $u = tx + (1-t)y$, and we get

$$G_y(t) = \frac{1}{p-q} \int_q^p \psi(u)du$$

where $p = tb + (1-t)y$ and $q = ta + (1-t)y$. Applying inequality (3), we get

$$\frac{1}{p-q} \int_q^p \psi(v)dv \geqslant 2^{\frac{1}{s}-1} \psi\left(\frac{p+q}{2}\right) = 2^{\frac{1}{s}-1} \psi\left(t\frac{a+b}{2} + (1-t)y\right)$$

and integrating this inequality on $[a,b]$ with respect to $y$ we obtain,

$$F(t) \geqslant 2^{\frac{1}{s}-1} G(1-t).$$

Since $F(t) = F(1-t)$ we get $(ii)$.

In the case of $t = 0$ we have

$$F(0) = \frac{1}{b-a} \int_a^b \psi(y)dy$$

$$G(0) = \psi\left(\frac{a+b}{2}\right)$$

and

$$\psi\left(\frac{a+b}{2}\right) 2^{s-1} \geqslant \psi\left(\frac{a+b}{2}\right) 2^{\frac{1}{s}-1}.$$

Applying inequality (3), we get

$$2^{\frac{1}{s}-1} \psi\left(\frac{a+b}{2}\right) = 2^{\frac{1}{s}-1} G(0) \leqslant \frac{1}{b-a} \int_a^b \psi(x) = F(0).$$

That is,

$$F(0) \geqslant 2^{\frac{1}{s}-1} G(0). \tag{7}$$

$$2^{\frac{1}{s}-1} G(1) = 2^{\frac{1}{s}-1} \frac{1}{b-a} \int_a^b \psi(x)dx \leqslant \frac{1}{b-a} \int_a^b \psi(x)dx = G(1) = F(0).$$

So,

$$F(0) \geqslant 2^{\frac{1}{s}-1} G(1). \tag{8}$$

From inequalities (7) and (8), we get $F(0) \geqslant 2^{\frac{1}{s}-1} \max(G(0), G(1))$.

*iii*) If we integrate the following inequality over $[a,b]^2$,

$$\psi(tx+(1-t)y) \leqslant t^{\frac{1}{s}} f(x) + (1-t)^{\frac{1}{s}} f(y),$$

we have

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b \psi(tx+(1-t)y)dxdy \leqslant \left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right) \frac{1}{b-a} \int_a^b \psi(x)dx.$$

That is

$$F(t) \leqslant \left(t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right) \frac{1}{b-a} \int_a^b \psi(x)dx.$$

*iv*) Now observe that, in the notation above, we have

$$G_y(t) = \frac{1}{p-q} \int_q^p \psi(u)du \leqslant \frac{s}{s+1} \left(\psi(ta+(1-t)y) + \psi(tb+(1-t)y)\right)$$

so that integrating this inequality on $[a,b]$, we get

$$\begin{aligned} &\frac{1}{b-a} \int_a^b \left(\frac{1}{p-q} \int_q^p f(u)du\right) dy \\ &\leqslant \int_a^b \frac{s}{s+1} \frac{(\psi(ta+(1-t)y) + \psi(tb+(1-t)y))}{b-a} dy \\ &\leqslant \frac{s}{(s+1)(b-a)} \left[\int_a^b \psi(ta+(1-t)y)dy + \int_a^b \psi(tb+(1-t)y)dy\right]. \end{aligned}$$

As above we have

$$G_y(t) \leqslant \frac{s}{s+1} \left(\psi(ta+(1-t)y) + \psi(tb+(1-t)y)\right)$$

for $y \in [a,b]$. From this inequality, we have the following inequalities:

$$G_a(1-t) \leqslant \frac{s}{s+1} (\psi(a) + \psi(ta+(1-t)b)),$$

$$G_b(1-t) \leqslant \frac{s}{s+1} (\psi(b) + \psi(tb+(1-t)a))$$

and adding them together we get $(iv)$. $\square$

## 4. Applications

We consider the applications of Theorem 1 and Theorem 6 to get some inequalities related to the special means.

Let $a,b,p$ be positive numbers with $a \neq b$ and $p \neq 1$,

$$A(a,b) = \frac{a+b}{2},$$
$$L_p(a,b) = \begin{cases} a, & \text{if } a=b \\ \left(\frac{a^p-b^p}{p(a-b)}\right)^{1/(p-1)}, & \text{if } a \neq b \end{cases},$$

are called arithmetic mean, Stolarsky mean (generalized logarithmic mean), respectively.

PROPOSITION 1. Let $a,b \in R_+$ with $a<b$ and $s \in (0,1]$. For $t \in [0,1]$, the following inequality holds

$$\left[L_{\frac{s+1}{s}}\left(ta+(1-t)\tfrac{a+b}{2}, tb+(1-t)\tfrac{a+b}{2}\right)\right]^{\frac{1}{s}} \leqslant 2^{\frac{1}{s}-1}A^{\frac{1}{s}}(a,b). \tag{9}$$

*Proof.* Let $\psi(x) = -x^{\frac{1}{s}}$ with $s \in (0,1]$ on $[a,b]$. Applying $i)$ in Theorem 1, we have

$$G(t) = -\tfrac{s}{s+1}\tfrac{1}{b-a}\tfrac{1}{t}\left[\left(tb+(1-t)\tfrac{a+b}{2}\right)^{\frac{1}{s}+1} - \left(ta+(1-t)\tfrac{a+b}{2}\right)^{\frac{1}{s}+1}\right]$$

for $t \in (0,1]$. Since $\lim_{t\to 0} G(t) = -\left(\frac{a+b}{2}\right)^{\frac{1}{s}}$, we can consider $G(0) = -\left(\frac{a+b}{2}\right)^{\frac{1}{s}}$. Expressing $G(t)$ as generalized logarithmic mean and applying $(ii)$ in Theorem 1, we have the inequality (9) for $t \in (0,1]$. For $t=0$, from the definition of generalized logarithmic mean,

$$\left[L_{\frac{s}{s+1}}\left(\tfrac{a+b}{2}, \tfrac{a+b}{2}\right)\right]^{\frac{1}{s}} = \left(\tfrac{a+b}{2}\right)^{\frac{1}{s}} \leqslant 2^{\frac{1}{s}-1}A^{\frac{1}{s}}(a,b)$$

is shown. Thus, inequality (9) holds for all $t \in [0,1]$. □

For $t=1$ in Proposition 1, the following inequality is obtained:

COROLLARY 4. *Let $a,b \in R_+$ with $a<b$ and $s \in (0,1]$. Then*

$$\left[L_{\frac{s+1}{s}}(a,b)\right]^{\frac{1}{s}} \leqslant 2^{\frac{1}{s}-1}A^{\frac{1}{s}}(a,b).$$

PROPOSITION 2. Let $a,b,x$ be positive real numbers with $a<b$. For $x \geqslant 3$, the following inequality holds

$$A^{x-2}(a,b) + A(a^{x-2}, b^{x-2}) \leqslant \frac{4}{x(b-a)^2}\left(A(a^x,b^x) - A^x(a,b)\right).$$

*Proof.* Let $\psi(x) = -x^{\frac{1}{s}}$ on $[a,b]$ in Theorem 6 (iv). Then multiplying both sides by $\frac{s+1}{s}$, we have

$$\frac{1}{(b-a)^2}\frac{s}{(2s+1)}\frac{(a-at+bt)^{\frac{1}{s}+2}-a^{\frac{1}{s}+2}+(b+at-bt)^{\frac{1}{s}+2}-b^{\frac{1}{s}+2}}{t(1-t)}$$
$$\leqslant -\left(a^{\frac{1}{s}}+(b+at-bt)^{\frac{1}{s}}+(a-at+bt)^{\frac{1}{s}}+b^{\frac{1}{s}}\right)$$

for $t \in [0,1]$. Rewriting the inequality for $t = \frac{1}{2}$, we have

$$2\left(\frac{1}{2}a+\frac{1}{2}b\right)^{\frac{1}{s}}+a^{\frac{1}{s}}+b^{\frac{1}{s}} \leqslant -\frac{s}{(2s+1)(a-b)^2}\left(4a^{\frac{1}{s}+2}-8\left(\frac{1}{2}a+\frac{1}{2}b\right)^{\frac{1}{s}+2}+4b^{\frac{1}{s}+2}\right).$$

From the definition of arithmetic mean,

$$A^{\frac{1}{s}}(a,b)+A(a^{\frac{1}{s}},b^{\frac{1}{s}}) \leqslant -\frac{4s}{(2s+1)(a-b)^2}\left(A(a^{\frac{1}{s}+2}b^{\frac{1}{s}+2})-A^{\frac{1}{s}+2}(a,b)\right)$$

is obtained. Making the substitution $\frac{1}{s}+2 = x$ and considering $s \in (0,1]$ yields desired result for $x \geqslant 3$. $\square$

Also, the obtained results can be used to get an lower and upper bounds for special functions.

PROPOSITION 3. For $x \geqslant 3$,

$$\Psi(x) \leqslant 2^{x-2} - 2^{-1} - \gamma.$$

where $\Psi(x)$ is digamma function, i.e.

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \text{ for } x > 0$$

and $\gamma$ is Euler-Mascheroni constant, i.e. $\gamma \approx 0.5772156649\ldots$.

*Proof.* Applying $t \in [0,1]$, $a = 0$ and $b = 1$, $\psi(x) = -x^{\frac{1}{s}}$ in Theorem 1 (ii) and using Corollary 1, we have

$$\frac{s}{2^{\frac{1}{s}+1}(s+1)}\frac{(t+1)^{\frac{1}{s}+1}-(1-t)^{\frac{1}{s}+1}}{t} \leqslant \frac{1}{2}.$$

The substitution $r = \frac{1-t}{1+t}$ gives

$$\frac{1-r^{\frac{1}{s}+1}}{1-r} \leqslant \frac{s+1}{2s}(1+r)^{\frac{1}{s}}.$$

$t\in[0,1]$ implies $r\in[0,1]$. Integrating the expression with respect to $r$ on $[0,1]$ gives

$$\int_0^1 \frac{1-r^{\frac{1}{s}+1}}{1-r}dr \leqslant 2^{\frac{1}{s}}-2^{-1}.$$

Using the following integral representation of digamma function

$$\Psi(r)=\int_0^1 \frac{1-t^{r-1}}{1-t}dt-\gamma$$

for $r>0$, we have

$$\Psi(\frac{1}{s}+2)+\gamma \leqslant 2^{\frac{1}{s}}-2^{-1}.$$

The substitution $x-2=\frac{1}{s}$ yields to the desired result. □

PROPOSITION 4. For $x\geqslant 2$,

$$B(3,x)\leqslant \frac{3x^3+8x^2-6x-4}{x(3x-2)(x+2)(x+1)}.$$

*Proof.* Let us use $\psi(x)=-x^{\frac{1}{s}}$ on $[0,1]$ in Theorem 6 (iii). Using Corollary 3, we have

$$\frac{s}{t(2s+1)(1-t)}\left((1-t)^{\frac{1}{s}+2}+t^{\frac{1}{s}+2}-1\right)\leqslant\left((1-t)^{\frac{1}{s}}+t^{\frac{1}{s}}\right).$$

Multiplying both sides with $t(1-t)$ then integrating on $t$ yields

$$\frac{s}{(2s+1)}\left(2\frac{s}{s+3}-1\right)\leqslant\frac{s^2}{(2s+1)(s+1)}-B\left(3,1+\frac{1}{s}\right)+\frac{s}{2s+1}-\frac{s}{3s+1}.$$

Then substitution $1+\frac{1}{s}=x$ gives desired inequality for $x\geqslant 2$. □

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