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A DERIVATIVE–HILBERT OPERATOR ACTING FROM BESOV SPACES INTO BLOCH SPACE

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Abstract. If μ is a positive Borel measure on the interval $[0,1)$, we let $\mathscr{H}_\mu$ be the Hankel matrix $\mathscr{H}_\mu = (\mu_{n,k})_{n,k\geqslant 0}$ with entries $\mu_{n,k} = \mu_{n+k}$ and $\mu_n = \int_{[0,1)} t^n d\mu(t)$. Using $\mathscr{H}_\mu$, Ye and Zhou first defined the Derivative-Hilbert operator as

$$\mathscr{DH}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)(n+1)z^n, \quad z \in \mathbb{D},$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function in $\mathbb{D}$. In this paper, we characterize the measure μ for which $\mathscr{DH}_\mu$ is a bounded (resp., compact) operator from Besov space B_p into Bloch space $\mathscr{B}$ with $1 < p < \infty$.

1. Introduction

If μ is a positive Borel measure on the interval $[0,1)$, we let $\mathscr{H}_\mu$ be the Hankel matrix $\mathscr{H}_\mu = (\mu_{n,k})_{n,k\geqslant 0}$ with entries $\mu_{n,k} = \mu_{n+k}$ and $\mu_n = \int_{[0,1)} t^n d\mu(t)$. For any analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, generalized Hilbert operator is defined as

$$\mathscr{H}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)z^n, \quad z \in \mathbb{D},$$

on the space of analytic functions in $\mathbb{D}$. In recent decades, in complex setting, generalized Hilbert operator $\mathscr{H}_\mu$ has been studied extensively. For example, Galanopoulos and Peláez [7] characterized the Borel measure μ for which the Hankel operator is a bounded (resp., compact) operator on Hardy and Bergman space. Girela and Merchán [6] extended the study of Hilbert operator to all the conformally invariant spaces. Li and Zhou [10] studied the essential norm of generalized Hilbert matrix from Bloch type spaces into BMOA and Bloch space.

In 2020, Ye and Zhou [13] defined the Derivative-Hilbert operator for the first time using Hankel matrix. They defined it as

$$\mathscr{DH}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)(n+1)z^n, \quad z \in \mathbb{D},$$

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on the space of analytic functions in $\mathbb{D}$. Since

$$\mathscr{D}\mathscr{H}_\mu(f)(z) = (z\mathscr{H}_\mu(f)(z))',$$

$\mathscr{D}\mathscr{H}_\mu$ is called the Derivative-Hilbert operator. Another generalized integral operator related to $\mathscr{D}\mathscr{H}_\mu$ (denoted by $\mathscr{I}_{\mu_\alpha}$, $\alpha \in \mathbb{N}^+$) is defined by

$$\mathscr{I}_{\mu_\alpha}(f)(z) = \int_{[0,1)} \frac{f(t)}{(1-tz)^\alpha} d\mu(t).$$

Ye and Zhou [13] characterized the measure μ for which $\mathscr{I}_{\mu_2}$ and $\mathscr{D}\mathscr{H}_\mu$ is bounded (resp., compact) on Bloch space. They did the similar research on Bergman spaces in [14].

In this paper, we consider the operators

$$\mathscr{D}\mathscr{H}_\mu, \mathscr{I}_{\mu_2} : B_p \to \mathscr{B}, \ 1 < p < \infty.$$

The aim is to study the boundedness (resp.,compactness) of $\mathscr{I}_{\mu_2}$ and $\mathscr{D}\mathscr{H}_\mu$.

The rest of this paper is organized as follows. In section 2, we state some notation and preliminaries which will be used in the sequel. Section 3 gives the sufficient condition such that $\mathscr{D}\mathscr{H}_\mu$ and $\mathscr{I}_{\mu_\alpha}(\alpha \in \mathbb{N}^+)$ are well defined in B_p. Section 4 devotes to study the boundedness (resp., compactness) of $\mathscr{I}_{\mu_2}$ and $\mathscr{D}\mathscr{H}_\mu$.

NOTATION. Throughout this paper, C denotes a positive constant which may be different from one occurrence to the next.

2. Notation and preliminaries

Let $\mathbb{D}$ and $\partial\mathbb{D} = \{z : |z| = 1\}$ denote respectively the open unit disc and the unit circle in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ and $dA(z) = \frac{1}{\pi}dxdy$ the normalized Lebesgue area measure on $\mathbb{D}$.

For $1 < p < \infty$, the analytic Besov space B_p consists of functions $f \in H(\mathbb{D})$ with

$$\|f\|_{B_p} = \left(|f(0)|^p + \rho_p(f)^p\right)^{\frac{1}{p}} < \infty,$$

where

$$\rho_p(f) = \left(\int_{\mathbb{D}} (1-|z|^2)^{p-2} |f'(z)|^p dA(z)\right)^{\frac{1}{p}}.$$

We refer to [1, 5, 8, 16, 18] for the theory of Besov spaces.

If $0 < p < \infty$, the Bergman space A^p is the set of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty.$$

We refer to [18] for the notation and results regarding Bergman spaces.

The Bloch space $\mathscr{B}$ is the set of functions $f \in H(\mathbb{D})$ with

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

It is known that $\mathscr{B}$ is a Banach space with the norm $\|f\|_{\mathscr{B}}$, a classical reference for the Bloch space is [17].

For an arc $I \subseteq \partial\mathbb{D}$, let $|I| = \frac{1}{2\pi}\int_I |d\xi|$ be the normalized length of I and $S(I)$ is the Carleson box based on I with

$$S(I) = \{z = re^{it} : e^{it} \in I; 1-|I| \leqslant r < 1\}.$$

Clearly, if $I = \partial\mathbb{D}$, then $S(I) = \mathbb{D}$.

For $0 < s < \infty$, we say that a positive Borel measure on $\mathbb{D}$ is a s-Carleson measure (see [4]) if

$$\sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty.$$

If $s = 1$, 1-Carleson measure is the classical Carleson measure. When the positive Borel measure μ on $\mathbb{D}$ satisfies the following equation

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^s} = 0,$$

μ is a vanishing s-Carleson measure. If $s = 1$, the vanishing 1-Carleson measure is the vanishing Carleson measure.

For $0 \leqslant \alpha < \infty$ and $0 < s < \infty$, we say that a positive Borel measure on $\mathbb{D}$ is a α-logarithmic s-Carleson measure (see [15]) if

$$\sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))(\log\frac{2\pi}{|I|})^\alpha}{|I|^s} < \infty.$$

If a positive Borel measure μ on $\mathbb{D}$ satisfies the following equation

$$\lim_{|I|\to 0} \frac{\mu(S(I))(\log\frac{2\pi}{|I|})^\alpha}{|I|^s} = 0,$$

μ is a vanishing α-logarithmic s-Carleson measure (see [11]).

Suppose μ is a s-Carleson measure on $\mathbb{D}$, the Carleson norm of μ is

$$\mathscr{N}_1(\mu) = \sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s},$$

we use $\mathscr{N}_2(\mu)$ denote the norm of identity mapping i from A^1 into $L^1(\mathbb{D},\mu)$. More important, $\mathscr{N}_1(\mu)$ is equivalent to $\mathscr{N}_2(\mu)$. Set $d\mu_r(z) = \mathscr{X}_{r<|z|<1}(t)d\mu(t)$. Then μ is a vanishing s-Carleson measure if and only if

$$\lim_{r\to 1^-} \mathscr{N}_1(\mu_r) = 0 \quad \text{or} \quad \lim_{r\to 1^-} \mathscr{N}_2(\mu_r) = 0. \tag{2.1}$$

A positive Borel measure on $[0,1)$ can be seen as a Borel measure on $\mathbb{D}$ by identifying it with the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(E) = \mu(E \cap [0,1)),$$

for any Borel subset E of $\mathbb{D}$. Then a positive Borel measure μ on $[0,1)$ can be seen as an s-Carleson measure on $\mathbb{D}$, if

$$\sup_{t\in[0,1)} \frac{\mu([t,1))}{(1-t)^s} < \infty.$$

On the interval $[0,1)$, we have similar statements for vanishing s-Carleson measure, α-logarithmic β-Carleson measure and vanishing α-logarithmic β-Carleson measure.

3. Conditions such that $\mathscr{DH}_\mu$ and $\mathscr{I}_{\mu_\alpha}$ are well defined in B_p

In this section, we find the sufficient condition such that $\mathscr{I}_{\mu_\alpha}$ and $\mathscr{DH}_\mu$ are well defined in B_p $(1<p<\infty)$ and obtain that $\mathscr{DH}_\mu(f)=\mathscr{I}_{\mu_2}(f)$, for all $f\in B_p$, with the certain condition.

THEOREM 3.1. *Suppose* $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ *and let* μ *be a positive Borel measure on* $[0,1)$. *If* μ *satisfies* $\int_{[0,1)}\left(\log\frac{2}{1-t}\right)^{\frac{1}{q}}d\mu(t)<\infty$, *then for any* $f\in B_p$, $\alpha\in\mathbb{N}^+$, $\mathscr{I}_{\mu_\alpha}(f)(z)$ *uniformly converges on any compact subset of* $\mathbb{D}$.

Proof. Let $M=\int_{[0,1)}\left(\log\frac{2}{1-t}\right)^{\frac{1}{q}}d\mu(t)$, it follows from Holland [8] that there exists a positive constant C, such that

$$|f(z)|\leqslant C\|f\|_{B_p}\left(\log\frac{2}{1-|z|}\right)^{\frac{1}{q}}, \quad f\in B_p. \tag{3.1}$$

For any $f\in B_p$, $0<r<1$, $|z|\leqslant r$, using (3.1) we have

$$\begin{aligned}\int_{[0,1)}\frac{|f(t)|}{|1-tz|^\alpha}d\mu(t) &\leqslant \frac{1}{(1-r)^\alpha}\int_{[0,1)}|f(t)|d\mu(t)\\ &\leqslant \frac{C\|f\|_{B_p}}{(1-r)^\alpha}\int_{[0,1)}\left(\log\frac{2}{1-t}\right)^{\frac{1}{q}}d\mu(t)\\ &=\frac{CM\|f\|_{B_p}}{(1-r)^\alpha}.\end{aligned}$$

Hence $\mathscr{I}_{\mu_\alpha}(f)(z)$ uniformly converges on any compact subset of $\mathbb{D}$. □

THEOREM 3.2. *Suppose* $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\in\mathbb{N}^+$ *and let* μ *be a positive Borel measure on* $[0,1)$. *If the operator* $\mathscr{I}_{\mu_\alpha}$ *is well defined in* B_p, *then for any* $\gamma<\frac{1}{q}$, *we have* $\int_{[0,1)}\left(\log\frac{2}{1-t}\right)^\gamma d\mu(t)<\infty$.

Proof. Let $\gamma<\frac{1}{q}$, note that the function $F(z)=\left(\log\frac{2}{1-z}\right)^\gamma$ belongs to B_p (see [8], Theorem 1). From the suppose, $\mathscr{I}_{\mu_\alpha}(F)(z)$ is well defined for every $z\in\mathbb{D}$. Take

$z = 0$, we have

$$\mathscr{I}_{\mu_\alpha}(F)(0) = \int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{\gamma} d\mu(t),$$

it is a complex number. Since μ is a positive Borel measure on $[0,1)$ and $\left(\log \frac{2}{1-t}\right)^{\gamma} > 0$ for all $t \in [0,1)$, we obtain that

$$\int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{\gamma} d\mu(t) < \infty. \quad \square$$

The following two lemmas will be used in finding the condition such that $\mathscr{DH}_\mu$ is well defined in B_p.

LEMMA 3.3. [6] *Suppose that $0 \leqslant \alpha \leqslant \beta$, $s \geqslant 1$ and let μ be a positive Borel measure on $[0,1)$ which is a β-logarithmic s-Carleson measure. Then*

$$\int_{[0,1)} t^k \left(\log \frac{2}{1-t}\right)^{\alpha} d\mu(t) = \mathscr{O}\left(\frac{(\log k)^{\alpha-\beta}}{k^s}\right), \text{ as } k \to \infty.$$

LEMMA 3.4. (I) [8] *Suppose that $1 < p \leqslant 2$. Then there exists a positive constant C_p such that if $f \in B_p$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k (z \in \mathbb{D})$, then $\sum_{k=1}^{\infty} k^{p-1}|a_k|^p \leqslant C_p(\rho_p(f))^p$.*

(II) [12] *Suppose that $2 < p < \infty$. Then there exists a positive constant C_p such that if $f \in B_p$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k (z \in \mathbb{D})$, then $\sum_{k=1}^{\infty} k|a_k|^p \leqslant C_p(\rho_p(f))^p$.*

THEOREM 3.5. *Suppose $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and let μ be a finite positive Borel measure on $[0,1)$.*

(I) *If $1 < p \leqslant 2$ and $\sum_{k=1}^{\infty} \frac{\mu_k{}^q}{k} < \infty$, then the operator $\mathscr{DH}_\mu$ is well defined in B_p.*

(II) *If $2 < p < \infty$ and $\sum_{k=1}^{\infty} \frac{\mu_k{}^q}{k^{\frac{q}{p}}} < \infty$, then the operator $\mathscr{DH}_\mu$ is well defined in B_p.*

Proof. For any $f \in B_p$, let $f(z) = \sum_{k=0}^{\infty} a_k z^k (z \in \mathbb{D})$. Since

$$\mu_{n+1} - \mu_n = \int_{[0,1)} t^n (t-1) d\mu(t) < 0,$$

the non-negative sequence $\{\mu_n\}_{n=0}^{\infty}$ is decreasing, we have

$$\sum_{k=1}^{\infty} |\mu_{n+k}||a_k| \leqslant \sum_{k=1}^{\infty} |\mu_k||a_k|, \quad n \geqslant 0. \tag{3.2}$$

First, we prove (I):

Let $1 < p \leqslant 2$ and $\sum_{k=1}^{\infty} \frac{\mu_k{}^q}{k} < \infty$. Using (3.2), Hölder inequality and Lemma 3.4 (I), we obtain that

$$\begin{aligned}
\sum_{k=1}^{\infty} |\mu_{n+k}||a_k| &\leqslant \sum_{k=1}^{\infty} |\mu_k||a_k| \\
&= \sum_{k=1}^{\infty} k^{1-\frac{1}{p}} |a_k| \frac{|\mu_k|}{k^{\frac{1}{q}}} \\
&\leqslant \left(\sum_{k=1}^{\infty} k^{p-1} |a_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} \frac{|\mu_k|^q}{k} \right)^{\frac{1}{q}} \\
&\leqslant C\rho_p(f) \left(\sum_{k=1}^{\infty} \frac{|\mu_k|^q}{k} \right)^{\frac{1}{q}} \\
&< \infty.
\end{aligned}$$

So in this condition, $\mathscr{DH}_\mu$ is well defined in B_p.

Second, we prove (II):

Let $2 < p < \infty$ and $\sum_{k=1}^{\infty} \frac{|\mu_k|^q}{k^{\frac{q}{p}}} < \infty$. Applying (3.2), Hölder inequality and Lemma 3.4 (II), we have

$$\begin{aligned}
\sum_{k=1}^{\infty} |\mu_{n+k}||a_k| &\leqslant \sum_{k=1}^{\infty} |\mu_k||a_k| \\
&= \sum_{k=1}^{\infty} k^{\frac{1}{p}} |a_k| \frac{|\mu_k|}{k^{\frac{1}{p}}} \\
&\leqslant \left(\sum_{k=1}^{\infty} k |a_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} \frac{|\mu_k|^q}{k^{\frac{q}{p}}} \right)^{\frac{1}{q}} \\
&\leqslant C\rho_p(f) \left(\sum_{k=1}^{\infty} \frac{|\mu_k|^q}{k^{\frac{q}{p}}} \right)^{\frac{1}{q}} \\
&< \infty.
\end{aligned}$$

In this case, we see also $\mathscr{DH}_\mu$ is well defined in B_p. □

THEOREM 3.6. *Suppose that $1 < p < \infty$ and let μ be a positive Borel measure on $[0,1)$. If μ is a 1-Carleson measure, then the operator $\mathscr{DH}_\mu$ is well defined in B_p.*

Proof. Since μ is a 1-Carleson measure, using Lemma 3.3 with $\alpha = 0$ and $\beta = 0$, we have

$$\mu_k = \int_{[0,1)} t^k d\mu(t) = \mathscr{O}\left(\frac{1}{k}\right), \quad k \to \infty.$$

Hence there exists a positive constant C and a positive integer N such that

$$\mu_k \leqslant \frac{C}{k}, \quad k > N. \tag{3.3}$$

When $1 < p \leqslant 2$ and $q = \frac{p}{p-1}$, using (3.3), we have

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{{\mu_k}^q}{k} &= \sum_{k=1}^{N} \frac{{\mu_k}^q}{k} + \sum_{k=N+1}^{\infty} \frac{{\mu_k}^q}{k} \\ &\leqslant \sum_{k=1}^{N} \frac{{\mu_k}^q}{k} + C \sum_{k=N+1}^{\infty} \frac{1}{k^{q+1}} \\ &< \infty,\end{aligned}$$

then by Theorem 3.5(I), $\mathscr{DH}_\mu$ is well defined in B_p.

When $2 < p < \infty$ and $q = \frac{p}{p-1}$, using (3.3), we have

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{{\mu_k}^q}{k^{\frac{q}{p}}} &= \sum_{k=1}^{N} \frac{{\mu_k}^q}{k^{\frac{q}{p}}} + \sum_{k=N+1}^{\infty} \frac{{\mu_k}^q}{k^{\frac{q}{p}}} \\ &\leqslant \sum_{k=1}^{N} \frac{{\mu_k}^q}{k^{\frac{q}{p}}} + C \sum_{k=N+1}^{\infty} \frac{1}{k^{q+\frac{q}{p}}} \\ &= \sum_{k=1}^{N} \frac{{\mu_k}^q}{k^{\frac{q}{p}}} + C \sum_{k=N+1}^{\infty} \frac{1}{k^{\frac{2q}{p}+1}} \\ &< \infty,\end{aligned}$$

then Theorem 3.5(II) yields that $\mathscr{DH}_\mu$ is well defined in B_p. $\square$

THEOREM 3.7. *Suppose* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $s \geqslant 1$, $\alpha \in \mathbb{N}^+$ *and let* μ *be the positive Borel measure on* $[0,1)$. *If* μ *is a* $\frac{1}{q}$*-logarithmic* s*-Carleson measure, then the operator* $\mathscr{DH}_\mu$ *and* $\mathscr{I}_{\mu_\alpha}$ *are well defined in* B_p *and* $\mathscr{DH}_\mu(f) = \mathscr{I}_{\mu_2}(f)$, *for all* $f \in B_p$.

Proof. Let $d\nu(t) = \left(\log \frac{2}{1-t}\right)^{\frac{1}{q}} d\mu(t)$. Proposition 2.5 of [6] gives that ν is a s-Carleson measure. From the definition of s-Carleson measure, we know that there exists a positive constant C, such that

$$\int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{\frac{1}{q}} d\mu(t) = \nu([0,1)) \leqslant C(1-0)^s \leqslant C,$$

then

$$\int_{[0,1)} |t^n f(t)| d\mu(t) \leqslant \int_{[0,1)} |f(t)| d\mu(t) \leqslant C\|f\|_{B_p} \int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{\frac{1}{q}} d\mu(t) \leqslant C.$$

Thus for all $n \in \mathbb{N}$, the integral $\int_{[0,1)} t^n f(t) d\mu(t)$ converges absolutely. From the proof of Theorem 3.1, we know the integral $\int_{[0,1)} \frac{f(t)}{(1-tz)^2} d\mu(t)$ converges absolutely. We have

$$\int_{[0,1)} \frac{f(t)}{(1-tz)^2} d\mu(t) = \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n f(t) d\mu(t) \right) (n+1) z^n.$$

Hence, from the given condition we can imply that $\mathscr{I}_{\mu_2}$ is well defined in B_p and

$$\mathscr{I}_{\mu_2}(f)(z) = \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n f(t) d\mu(t) \right) (n+1) z^n.$$

Since μ is a $\frac{1}{q}$-logarithmic s-Carleson measure on $[0,1)$, Theorem 3.6 gives that $\mathscr{DH}_\mu$ is well defined in B_p. From the proof of Theorem 3.5 and Theorem 3.6, we get that $\sum_{k=0}^{\infty} |\mu_{n,k} a_k| \leqslant C$, hence we can interchange the order of summation in the expression defining $\mathscr{DH}_\mu(f)(z)$.

Therefore, for any $f \in B_p$, let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, using Fubini's theorem, we obtain that

$$\begin{aligned}
\mathscr{DH}_\mu(f)(z) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) (n+1) z^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_k \int_{[0,1)} t^{n+k} d\mu(t) \right) (n+1) z^n \\
&= \sum_{n=0}^{\infty} \left(\int_{[0,1)} \left(\sum_{k=0}^{\infty} a_k t^k \right) t^n d\mu(t) \right) (n+1) z^n \\
&= \sum_{n=0}^{\infty} \left(\int_{[0,1)} f(t) t^n d\mu(t) \right) (n+1) z^n \\
&= \mathscr{I}_{\mu_2}(f)(z).
\end{aligned}$$

The proof is complete. □

4. Boundedness and compactness of $\mathscr{DH}_\mu$ and $\mathscr{I}_{\mu_2}$

The following two lemmas will be very useful in the proof of the boundedness and compactness of $\mathscr{DH}_\mu$ and $\mathscr{I}_{\mu_2}$.

LEMMA 4.1. *Suppose that $\mathscr{I}_{\mu_2}$ is a bounded operator from B_p $(1 < p < \infty)$ into $\mathscr{B}$. Then $\mathscr{I}_{\mu_2}$ is compact if and only if for any bounded sequence $\{f_n\}_{n=0}^{\infty} \subseteq B_p$ which converges to 0 uniformly on every compact subset of $\mathbb{D}$, we have $\mathscr{I}_{\mu_2}(f_n) \to 0$ in $\mathscr{B}$.*

The proof of Lemma 4.1 is referred to the Proposition 3.11 of [3]. From [9] and the closed graph theorem we can easy obtain the following result.

LEMMA 4.2. *Let $1 \leqslant m \leqslant n < \infty$, then $A^m \subseteq L^n(\mathbb{D}, d\mu)$ if and only if μ is a $\frac{2n}{m}$-Carleson measure.*

THEOREM 4.3. *Suppose* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ *and let* μ *be a positive Borel measure on* $[0,1)$.

(I) *If* μ *is a* $\frac{1}{q}$*-logarithmic* 2*-Carleson measure, then* $\mathscr{I}_{\mu_2}$ *is a bounded operator from* B_p *into* $\mathscr{B}$.

(II) *If* μ *is a vanishing* $\frac{1}{q}$*-logarithmic* 2*-Carleson measure, then* $\mathscr{I}_{\mu_2}$ *is a compact operator from* B_p *into* $\mathscr{B}$.

Proof. Let $M = \int_{[0,1)} \left(\log \frac{2}{1-t}\right)^{\frac{1}{q}} d\mu(t)$, since μ is a $\frac{1}{q}$-logarithmic 2-Carleson measure, by the proof of Theorem 3.7, we know that $M < \infty$ and $\mathscr{I}_{\mu_2}$ is well defined in B_p. Using (3.1), we obtain that

$$\int_{[0,1)} |f(t)| d\mu(t) \leqslant CM \|f\|_{B_p}.$$

If $0 \leqslant r < 1$, for any $f \in B_p$, $g \in A^1$, let $g_r(z) = g(rz)$, $z \in \mathbb{D}$, by Theorem 11.6 of [18], we get that $\|g_r\|_{A^1} \leqslant \|g\|_{A^1}$, then we have

$$\begin{aligned} \int_{\mathbb{D}} \int_{[0,1)} \left| \frac{f(t)g(rz)}{(1-rtz)^2} \right| d\mu(t) dA(z) &\leqslant \frac{CM\|f\|_{B_p}}{(1-r)^2} \int_{\mathbb{D}} |g(rz)| dA(z) \\ &= \frac{CM\|f\|_{B_p}}{(1-r)^2} \|g_r\|_{A^1} \\ &\leqslant \frac{CM\|f\|_{B_p}}{(1-r)^2} \|g\|_{A^1}. \end{aligned} \tag{4.1}$$

For $0 < k < 1$, the Bergman kernel function of $k\mathbb{D} = \{kz : z \in \mathbb{D}\}$ is $K(z,\xi) = \frac{1}{k^2(1-\frac{\bar{\xi}z}{k^2})^2}$.

The reproducing property is

$$f(z) = \int_{k\mathbb{D}} f(\xi) K(z,\xi) dA(\xi), \ f \in A^1. \tag{4.2}$$

When $0 \leqslant r < 1$, $f \in \mathscr{B}$, $g \in A^1$, using (4.1), Fubini's theorem and (4.2), we imply that

$$\int_{\mathbb{D}} \overline{\mathscr{I}_{\mu_2}(f)(rz)} g(rz) dA(z) = \int_{[0,1)} \overline{f(t)} g(r^2 t) d\mu(t), \tag{4.3}$$

which is referred to the proof of Theorem 2.3 in [13].

Now we begin the proof of (I):

Let $d\nu(t) = (\log \frac{2}{1-t})^{\frac{1}{q}} d\mu(t)$, by Proposition 2.5 of [6], ν is a 2-Carleson measure. From Lemma 4.2, we obtain that $A^1 \subseteq L^1(\mathbb{D}, d\nu)$. Hence there exists $C > 0$, such that for any $g \in A^1$,

$$\int_{[0,1)} |g(t)| d\nu(t) \leqslant \int_{\mathbb{D}} |g(z)| d\nu(z) \leqslant C \int_{\mathbb{D}} |g(z)| dA(z) = C\|g\|_{A^1}. \tag{4.4}$$

For any $f \in B_p$, $g \in A^1$, combining (4.3) and (3.1) with (4.4), we get

$$\begin{aligned} \left| \int_{\mathbb{D}} \overline{\mathscr{I}_{\mu_2}(f)(rz)} g(rz) dA(z) \right| &= \left| \int_{[0,1)} \overline{f(t)} g(r^2 t) d\mu(t) \right| \\ &\leqslant C \|f\|_{B_p} \int_{[0,1)} \left| g(r^2 t) \right| \left(\log \frac{2}{1-t} \right)^{\frac{1}{q}} d\mu(t) \\ &= C \|f\|_{B_p} \int_{[0,1)} \left| g(r^2 t) \right| d\nu(t) \\ &\leqslant C \|f\|_{B_p} \|g\|_{A^1}, \end{aligned} \tag{4.5}$$

we know that $(A^1)^* \cong \mathscr{B}$ (see [17]), under the pairing

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_{\mathbb{D}} \overline{f(rz)} g(rz) dA(z), \quad f \in \mathscr{B},\ g \in A^1. \tag{4.6}$$

Combining (4.5) with (4.6), we have that

$$\begin{aligned} \langle \mathscr{I}_{\mu_2}(f), g \rangle &= \lim_{r \to 1^-} \left| \int_{\mathbb{D}} \overline{\mathscr{I}_{\mu_2}(f)(rz)} g(rz) dA(z) \right| \\ &\leqslant C \|f\|_{B_p} \|g\|_{A^1}. \end{aligned}$$

Hence, $\mathscr{I}_{\mu_2}$ is a bounded operator from B_p into $\mathscr{B}$.

Next we start the proof of (II):

Let $d\nu(t) = (\log \frac{2}{1-t})^{\frac{1}{q}} d\mu(t)$, since μ is a vanishing $\frac{1}{q}$-logarithmic 2-Carleson measure, then by Proposition 2.5 of [6], ν is a vanishing 2-Carleson measure. For $0 < r < 1$, let $d\nu_r(z) = \mathscr{X}_{r<|z|<1}(t) d\nu(t)$ and $\mathscr{N}$ be the norm of identity mapping i from A^1 into $L^1(\mathbb{D}, d\nu)$, by (2.1), $\mathscr{N}(\nu_r) \to 0$ $(r \to 1^-)$. Suppose $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in B_p which converges to 0 uniformly on every compact subset of $\mathbb{D}$. For any $g \in A^1$, $r \in [0,1)$, we have

$$\begin{aligned} &\int_{[0,1)} |f_n(t)||g(t)| d\mu(t) \\ &= \int_{[0,r)} |f_n(t)||g(t)| d\mu(t) + \int_{[r,1)} |f_n(t)||g(t)| d\mu(t) \\ &\leqslant \int_{[0,r)} |f_n(t)||g(t)| d\mu(t) + C \|f_n\|_{B_p} \int_{[r,1)} |g(t)| \left(\log \frac{2}{1-t} \right)^{\frac{1}{q}} d\mu(t) \\ &= \int_{[0,r)} |f_n(t)||g(t)| d\mu(t) + C \|f_n\|_{B_p} \int_{[0,1)} |g(t)| d\nu_r(t) \\ &\leqslant \int_{[0,r)} |f_n(t)||g(t)| d\mu(t) + C \|f_n\|_{B_p} \mathscr{N}(\nu_r) \|g\|_{A^1}. \end{aligned}$$

Then $\mathscr{N}(\nu_r) \to 0$ $(r \to 1^-)$ and the condition $\{f_n\} \to 0 (n \to 0)$ uniformly on every compact subset of $\mathbb{D}$ imply that

$$\lim_{n \to \infty} \int_{[0,1)} |f_n(t)||g(t)| d\mu(t) = 0, \quad g \in A^1, \tag{4.7}$$

combining (4.7) with (4.3), we get that

$$\begin{aligned}\lim_{n\to\infty}\langle \mathscr{I}_{\mu_2}(f_n),g\rangle &= \lim_{n\to\infty}\left(\lim_{r\to 1^-}\left|\int_{\mathbb{D}}\overline{\mathscr{I}_{\mu_2}(f_n)(rz)}g(rz)dA(z)\right|\right)\\ &= \lim_{n\to\infty}\left(\lim_{r\to 1^-}\left|\int_{[0,1)}\overline{f_n(t)}g(r^2t)d\mu(t)\right|\right)\\ &\leqslant \lim_{n\to\infty}\int_{[0,1)}|f_n(t)||g(t)|d\mu(t)\\ &= 0.\end{aligned}$$

Hence when $n\to 0$, $\mathscr{I}_{\mu_2}(f_n)\to 0$ in $\mathscr{B}$. Then, Lemma 4.1 implies that $\mathscr{I}_{\mu_2}$ is a compact operator from B_p into $\mathscr{B}$. □

Using Theorem 3.7 and Theorem 4.3, we obtain the following corollary.

COROLLARY 4.4. *(I) If μ is a $\frac{1}{q}$-logarithmic 2-Carleson measure, then $\mathscr{DH}_\mu$ is a bounded operator from B_p into $\mathscr{B}$.*

(II) If μ is a vanishing $\frac{1}{q}$-logarithmic 2-Carleson measure, then $\mathscr{DH}_\mu$ is a compact operator from B_p into $\mathscr{B}$.

THEOREM 4.5. *Suppose $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ and let μ be a positive Borel measure on $[0,1)$ which satisfies $\int_{[0,1)}(\log\frac{2}{1-t})^{\frac{1}{q}}d\mu(t)<\infty$.*

(I) If $\mathscr{DH}_\mu$ is a bounded operator from B_p into $\mathscr{B}$, then μ is a γ-logarithmic 2-Carleson measure. $(\gamma<\frac{1}{q})$

(II) If $\mathscr{DH}_\mu$ is a compact operator from B_p into $\mathscr{B}$, then μ is a vanishing γ-logarithmic 2-Carleson measure. $(\gamma<\frac{1}{q})$

Proof. Since μ satisfies $\int_{[0,1)}(\log\frac{2}{1-t})^{\frac{1}{q}}d\mu(t)<\infty$, it follows from Theorem 3.1 that $\mathscr{I}_{\mu_2}$ is well defined in B_p. Let $f(z)=\sum_{n=0}^{\infty}a_kz^k\in\mathscr{B}$, the proof of Theorem 2.3 in [6] gives that $\sum_{k=0}^{\infty}|\mu_{n,k}a_k|\leqslant\|f\|_{\mathscr{B}}$. $B_p\subset\mathscr{B}$ implies that for every $f=\sum_{k=0}^{\infty}a_kz^k\in B_p$, $\sum_{k=0}^{\infty}|\mu_{n,k}a_k|\leqslant C$. Then by the similar proof of Theorem 3.7, we get that

$$\mathscr{DH}_\mu(f)=\mathscr{I}_{\mu_2}(f)\quad\text{for all}\quad f\in B_p.$$

(I) From the given condition and the above proof, we know that $\mathscr{I}_{\mu_2}$ is a bounded operator from B_p into $\mathscr{B}$. For $b\in(0,1)$, we take two test functions $F_b(z)=(\log\frac{2}{1-bz})^\gamma$ and $g_b(z)=(\frac{1-b^2}{(1-bz)^2})^2$, $z\in\mathbb{D}$. We already mention that the function $F(z)=(\log\frac{1}{1-z})^\gamma$ belongs to B_p.

Let $\varphi(z)=bz$, $z\in\mathbb{D}$, then φ is a analytic function from $\mathbb{D}$ into $\mathbb{D}$ and $F_b(z)=F(\varphi(z))$. Using the Theorem 11.6 in [18], we have

$$\begin{aligned}\|F_b\|_{B_p}^p &= |F_b(0)|+\int_{\mathbb{D}}|F_b'(z)|^p(1-|z|^2)^{p-2}dA(z)\\ &= |F(0)|+b^p\int_{\mathbb{D}}|F'(\varphi(z))|^p(1-|z|^2)^{p-2}dA(z)\end{aligned}$$

$$
\begin{aligned}
&< |F(0)| + \int_{\mathbb{D}} |F'(\varphi(z))|^p (1-|z|^2)^{p-2} dA(z) \\
&< |F(0)| + \int_{\mathbb{D}} |F'(z)|^p (1-|z|^2)^{p-2} dA(z) \\
&= \|F\|_{B_p}^p .
\end{aligned}
$$

Hence $\|F_b\|_{B_p} < \|F\|_{B_p}$ and $\|g_b\|_{A^1} = 1$, this implies that $F_b(z) \in B_p$, $g_b(z) \in A^1$ and

$$
\sup_{0<b<1} \|F_b\|_{B_p} \leqslant \|F\|_{B_p}, \ \sup_{0<b<1} \|g_b\|_{A^1} = 1.
$$

Combing (4.3) with (4.6), due to the boundedness of $\mathscr{I}_{\mu_2}$, there exists a positive constant C such that

$$
\left| \int_{[0,1)} \overline{f(t)} g(r^2 t) d\mu(t) \right| \leqslant C \|f\|_{B_p} \|g\|_{A^1}, \quad 0 < r < 1, \ f \in B_p, \ g \in A^1.
$$

Hence, we have

$$
\begin{aligned}
\infty &> C \sup_{0<b<1} \|F\|_{B_p} \sup_{0<b<1} \|g_b\|_{A^1} \\
&\geqslant \left| \int_{[0,1)} \overline{F_b(t)} g_b(r^2 t) d\mu(t) \right| \\
&\geqslant \int_{[b,1)} \left(\frac{1-b^2}{(1-br^2t)^2} \right)^2 \left(\log \frac{2}{1-bt} \right)^{\gamma} d\mu(t) \\
&\geqslant C \frac{\left(\log \frac{2}{1-b^2} \right)^{\gamma}}{(1-b^2)^2} \mu([b,1)),
\end{aligned}
$$

thus, μ is a γ-logarithmic 2-Carleson measure.

(II) Since $\mathscr{DH}_\mu(f) = \mathscr{I}_{\mu_2}(f)$, $\mathscr{I}_{\mu_2}$ is a compact operator from B_p into $\mathscr{B}$. Take any sequence $\{b_n\} \subset (0,1)$ and $\lim_{n\to\infty} b_n = 1$. Set

$$
g_{b_n}(z) = \left(\frac{1-b_n^2}{(1-b_n z)^2} \right)^2 ,
$$

thus $\|g_{b_n}\|_{A^1} = 1$ and $g_{b_n} \in A^1$, for all $n \in \mathbb{N}$. Let $f_{b_n}(z) = \frac{1}{\log \frac{2}{1-b_n^2}} (\log \frac{2}{1-b_n z})^{\gamma+1}$ and $F_{b_n}(z) = (\log \frac{2}{1-b_n z})^{\gamma}$, from the proof of (I), we know $F_{b_n} \in B_p$. Let

$$
M = |f'_{b_n}(z)|^p (1-|z|^2)^{p-2} = \left(\frac{b_n (\gamma+1) \left(\log \frac{2}{|1-b_n z|} \right)^{\gamma}}{|1-b_n z| \log \frac{2}{1-b_n^2}} \right)^p \left(1-|z|^2\right)^{p-2}.
$$

When $|z| \leqslant b_n$,

$$
M \leqslant \left(\frac{b_n (\gamma+1) \left(\log \frac{2}{|1-b_n z|} \right)^{\gamma-1}}{|1-b_n z|} \right)^p (1-|z|^2)^{p-2} = C |F'_{b_n}(z)|^p \left(1-|z|^2\right)^{p-2}.
$$

Since

$$\begin{aligned}\|f_{b_n}(z)\|_{B_p}^p &= |f_{b_n}(0)| + \int_{|z|\leqslant b_n} M dA(z) + \int_{b_n<|z|\leqslant 1} M dA(z)\\ &\leqslant |f_{b_n}(0)| + C\int_{|z|\leqslant b_n} |F'_{b_n}(z)|^p \left(1-|z|^2\right)^{p-2} dA(z) + \int_{b_n<|z|\leqslant 1} M dA(z)\\ &\leqslant C|F_{b_n}(0)| + C\int_{\mathbb{D}} |F'_{b_n}(z)|^p \left(1-|z|^2\right)^{p-2} dA(z) + \int_{b_n<|z|\leqslant 1} M dA(z)\\ &\leqslant C\|F_{b_n}\|_{B_p}^p + \int_{b_n<|z|\leqslant 1} M dA(z),\end{aligned}$$

we obtain

$$\lim_{n\to\infty} \|f_{b_n}\|_{B_p}^p \leqslant C \lim_{n\to\infty} \|F_{b_n}\|_{B_p}^p = C\|F\|_{B^p}^p.$$

The above calculations show that $f_{b_n} \in B_p$ and $\sup_{n\geqslant 1} \|f_{b_n}\|_{B_p} < \infty$. Then $\{f_{b_n}\}$ is a bounded sequence in B_p and $\{f_{b_n}\}$ converges to 0 uniformly on any compact subset of $\mathbb{D}$. Lemma 4.1 implies that $\mathscr{I}_{\mu_2}(f_{b_n})$ converges to 0 in $\mathscr{B}$. Using (4.3), we have

$$\begin{aligned}&\lim_{n\to\infty} \int_{[0,1)} \overline{f_{b_n}(t)} g_{b_n}(r^2 t) d\mu(t)\\ &= \lim_{n\to\infty} \int_{\mathbb{D}} \overline{\mathscr{I}_{\mu_2}(f_{b_n})(rz)} g_{b_n}(rz) dA(z)\\ &= 0.\end{aligned}$$

Now we imply that

$$\begin{aligned}&\int_{[0,1)} \overline{f_{b_n}(t)} g_{b_n}(r^2 t) d\mu(t)\\ &\geqslant \int_{[b_n,1)} \left(\frac{1-b_n^2}{(1-b_n r^2 t)^2}\right)^2 \frac{1}{\log\frac{2}{1-b_n^2}} \left(\log\frac{2}{1-b_n t}\right)^{\gamma+1} d\mu(t)\\ &\geqslant C\frac{(\log\frac{2}{1-b_n^2})^{\gamma}}{(1-b_n^2)^2} \mu([b_n,1)).\end{aligned}$$

Since $\{b_n\} \subset (0,1)$ and $\lim_{n\to\infty} b_n = 1$,

$$\lim_{b\to 1^-} \frac{(\log\frac{2}{1-b^2})^{\gamma}}{(1-b^2)^2} \mu([b,1)) = 0.$$

It is clear that μ is a vanishing γ-logarithmic 2-Carleson measure. □

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