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ON SOME HILBERT–PACHPATTE INEQUALITIES WITH ALTERNATING SIGNS

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Dedicated to Professor Vandanjav Adiyasuren on the occasion of his 60th birthday

(*Communicated by T. Burić*)

Abstract. Motivated by the results of Zhao and Cheung, we deduce a Hilbert-Pachpatte inequality with alternating signs involving non-homogeneous kernels. We also obtain a generalization of a related result known from the literature.

1. Introduction

The Hilbert-Pachpatte inequality is one of interesting inequalities in mathematical analysis and its applications (see [1], [4], [5], [7]–[9]). Although classical, it is still of interest to numerous authors.

Recently, C. J. Zhao and W. S. Cheung [7],[8] obtained some Hilbert-Pachpatte inequalities with alternating signs. Their result is contained in the following theorem:

THEOREM 1. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, *and let* $0 \leqslant b_{2n+1} \leqslant b_{2n} \leqslant \cdots \leqslant b_2 \leqslant b_1$, $0 \leqslant a_{2m+1} \leqslant a_{2m} \leqslant \cdots \leqslant a_2 \leqslant a_1$ *for* $n=0,1,\ldots,r+1$ *and* $m=0,1,\ldots,s+1$, *respectivily. Further, let* $\overline{A}_m=\sum_{\ell=1}^{2m+1}(-1)^{\ell+1}a_\ell$ *and* $\overline{B}_n=\sum_{k=1}^{2n+1}(-1)^{k+1}b_k$. *If* f *and* g *are convex, nondecreasing and nonnegative functions on* $[0,b_1]$ *and* $[0,a_1]$, *respectivily, then the following inequality holds*

$$\sum_{m=0}^{s+1}\sum_{n=0}^{r+1}\frac{pqf(\overline{B}_n)g(\overline{A}_m)}{q(m+1)+p(n+1)} \leqslant D_{p,q,r,s}\left[\sum_{m=1}^{s+2}(g(a_{2m-1})-g(a_{2m}))^q(s-m+3)\right]^{\frac{1}{q}} \times\left[\sum_{n=1}^{r+2}(f(b_{2n-1})-f(b_{2n}))^p(r-n+3)\right]^{\frac{1}{p}}, \tag{1}$$

where

$$D_{p,q,r,s}=\frac{q(s+2)+p(r+2)}{pq}. \tag{2}$$

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In 2019, Ts. Batbold, L. E. Azar and M. Krnić [1] considered generalizations of Hilbert-Pachpatte inequality with non-homogeneous kernel. For some similar works on the extensions and generalizations of Hilbert-Pachpatte inequality, we refer the reader to [2], [7]–[9].

In this paper, following the way of [1], [7], we give a generalization of Hilbert-Pachpatte inequality (1). More precisely, in the sequel, we deduce more general form of inequality (1) containing non-homogeneous kernels. Furhermore, a generalization of a related inequality is also obtained.

2. Main results

We have already mentioned, we will extend this inequality for a class of non-homogeneous kernels defined by $K(m,n)=(\lambda_m+\rho_n)^{-\mu}$, $m,n\in\mathbb{N}\cup\{0\}$, $\mu>0$. Here, and throughout this paper $(\lambda_m)_{m\in\mathbb{N}\cup\{0\}}$ and $(\rho_n)_{n\in\mathbb{N}\cup\{0\}}$ are positive sequences of real numbers.

LEMMA 1. (Szegö's inequality, [6]) *If* $0\leqslant b_{2n+1}\leqslant b_{2n}\leqslant\cdots\leqslant b_2\leqslant b_1$ *and* f *is convex on* $[0,b_1]$*, then*

$$f\left(\sum_{k=1}^{2n+1}(-1)^{k+1}b_k\right)\leqslant\sum_{k=1}^{2n+1}(-1)^{k+1}f(b_k). \tag{3}$$

THEOREM 2. *Let* $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, *and let* $0\leqslant b_{2n+1}\leqslant b_{2n}\leqslant\cdots\leqslant b_2\leqslant b_1$, $0\leqslant a_{2m+1}\leqslant a_{2m}\leqslant\cdots\leqslant a_2\leqslant a_1$ *for* $n=0,1,\ldots,r+1$ *and* $m=0,1,\ldots,s+1$, *respectivily. Further, let* $\overline{A}_m=\sum_{\ell=1}^{2m+1}(-1)^{\ell+1}a_\ell$ *and* $\overline{B}_n=\sum_{k=1}^{2n+1}(-1)^{k+1}b_k$. *If* f *and* g *are convex, nondecreasing and nonnegative functions on* $[0,b_1]$ *and* $[0,a_1]$, *respectivily, then the following inequality holds*

$$\sum_{m=0}^{s+1}\sum_{n=0}^{r+1}\frac{f(\overline{B}_n)g(\overline{A}_m)}{(\lambda_m+\rho_n)^\mu}\leqslant C_{\lambda,\rho}\left[\sum_{m=1}^{s+2}(g(a_{2m-1})-g(a_{2m}))^\alpha(s-m+3)\right]^{\frac{1}{\alpha}}\times\left[\sum_{n=1}^{r+2}(f(b_{2n-1})-f(b_{2n}))^\beta(r-n+3)\right]^{\frac{1}{\beta}}, \tag{4}$$

where

$$C_{\lambda,\rho}=\alpha^{-\frac{\mu}{\alpha}}\beta^{-\frac{\mu}{\beta}}\left(\sum_{m=0}^{s+1}\frac{m+1}{\lambda_m^\mu}\right)^{\frac{1}{\beta}}\left(\sum_{n=0}^{r+1}\frac{n+1}{\rho_n^\mu}\right)^{\frac{1}{\alpha}}. \tag{5}$$

Proof. Using the following relation

$$\frac{1}{(u+v)^\mu}=\frac{1}{\Gamma(\mu)}\int_0^\infty x^{\mu-1}e^{-(u+v)x}dx, \tag{6}$$

which follows from the definition of the Gamma function, the left-hand side of inequality (4) can be rewritten in the following form:

$$\begin{aligned}\sum_{m=0}^{s+1}\sum_{n=0}^{r+1}\frac{f(\overline{B}_n)g(\overline{A}_m)}{(\lambda_m+\rho_n)^{\mu}} &= \frac{1}{\Gamma(\mu)}\sum_{m=0}^{s+1}\sum_{n=0}^{r+1} f(\overline{B}_n)g(\overline{A}_m)\int_0^{\infty} x^{\mu-1}e^{-(\lambda_m+\rho_n)x}dx \\ &= \frac{1}{\Gamma(\mu)}\int_0^{\infty}\left(x^{\frac{\mu-1}{\beta}}\sum_{m=0}^{s+1}e^{-\lambda_m x}g(\overline{A}_m)\right)\left(x^{\frac{\mu-1}{\alpha}}\sum_{n=0}^{r+1}e^{-\rho_n x}f(\overline{B}_n)\right)dx. \end{aligned} \tag{7}$$

From the Szegö's inequality, we have

$$\begin{aligned} f(\overline{B}_n) &= f\left(\sum_{k=1}^{2n+1}(-1)^{k+1}b_k\right) \\ &\leqslant \sum_{k=1}^{2n+1}(-1)^{k+1}f(b_k) \\ &= \sum_{k=1}^{n+1}(f(b_{2k-1})-f(b_{2k})), \end{aligned} \tag{8}$$

and

$$g(\overline{A}_m) \leqslant \sum_{\ell=1}^{m+1}(g(a_{2\ell-1})-g(a_{2\ell})), \tag{9}$$

where, for the sake of simplicity, we write $f(b_{2n+2}) = g(a_{2m+2}) = 0$.

By the above inequalities (7), (8), (9), and Hölder inequality, we have

$$\begin{aligned}\sum_{m=0}^{s+1}\sum_{n=0}^{r+1}\frac{f(\overline{B}_n)g(\overline{A}_m)}{(\lambda_m+\rho_n)^{\mu}} &\leqslant \frac{1}{\Gamma(\mu)}\int_0^{\infty}\left(x^{\frac{\mu-1}{\beta}}\sum_{m=0}^{s+1}e^{-\lambda_m x}\sum_{\ell=1}^{m+1}(g(a_{2\ell-1})-g(a_{2\ell}))\right) \\ &\quad\times\left(x^{\frac{\mu-1}{\alpha}}\sum_{n=0}^{r+1}e^{-\rho_n x}\sum_{k=1}^{n+1}(f(b_{2k-1})-f(b_{2k}))\right)dx \\ &\leqslant \frac{1}{\Gamma(\mu)}\left[\int_0^{\infty}x^{\mu-1}\left(\sum_{m=0}^{s+1}e^{-\lambda_m x}\sum_{\ell=1}^{m+1}(g(a_{2\ell-1})-g(a_{2\ell}))\right)^{\beta}dx\right]^{\frac{1}{\beta}} \\ &\quad\times\left[\int_0^{\infty}x^{\mu-1}\left(\sum_{n=0}^{r+1}e^{-\rho_n x}\sum_{k=1}^{n+1}(f(b_{2k-1})-f(b_{2k}))\right)^{\alpha}dx\right]^{\frac{1}{\alpha}}. \end{aligned} \tag{10}$$

Using again Hölder inequality and relation (6), we obtain

$$\sum_{m=0}^{s+1}\sum_{n=0}^{r+1}\frac{f(\overline{B}_n)g(\overline{A}_m)}{(\lambda_m+\rho_n)^{\mu}} \leqslant \frac{1}{\Gamma(\mu)}\left[\int_0^{\infty} x^{\mu-1}\left[\sum_{m=0}^{s+1}\left((m+1)^{\frac{1}{\beta}}e^{-\lambda_m x}\right)\right.\right.$$
$$\left.\left.\times\left((m+1)^{-\frac{1}{\beta}}\sum_{\ell=1}^{m+1}(g(a_{2\ell-1})-g(a_{2\ell}))\right)\right]^{\beta}dx\right]^{\frac{1}{\beta}}$$
$$\times\left[\int_0^{\infty} x^{\mu-1}\left[\sum_{n=0}^{r+1}\left((n+1)^{\frac{1}{\alpha}}e^{-\rho_n x}\right)\right.\right.$$
$$\left.\left.\times\left((n+1)^{-\frac{1}{\alpha}}\sum_{k=1}^{n+1}(f(b_{2k-1})-f(b_{2k}))\right)\right]^{\alpha}dx\right]^{\frac{1}{\alpha}}$$
$$\leqslant \frac{1}{\Gamma(\mu)}\left[\int_0^{\infty} x^{\mu-1}\left[\sum_{m=0}^{s+1}(m+1)e^{-\beta\lambda_m x}\right]\right.$$
$$\left.\times\left[\sum_{m=0}^{s+1}(m+1)^{-\frac{\alpha}{\beta}}\left(\sum_{\ell=1}^{m+1}(g(a_{2\ell-1})-g(a_{2\ell}))\right)^{\alpha}\right]^{\frac{\beta}{\alpha}}dx\right]^{\frac{1}{\beta}}$$
$$\times\left[\int_0^{\infty} x^{\mu-1}\left[\sum_{n=0}^{r+1}(n+1)e^{-\alpha\rho_n x}\right]\right.$$
$$\left.\times\left[\sum_{n=0}^{r+1}(n+1)^{-\frac{\beta}{\alpha}}\left(\sum_{k=1}^{n+1}(f(b_{2k-1})-f(b_{2k}))\right)^{\beta}\right]^{\frac{\alpha}{\beta}}dx\right]^{\frac{1}{\alpha}}$$
$$= C_{\lambda,\rho}\left[\sum_{m=0}^{s+1}(m+1)^{-\frac{\alpha}{\beta}}\left(\sum_{\ell=1}^{m+1}(g(a_{2\ell-1})-g(a_{2\ell}))\right)^{\alpha}\right]^{\frac{1}{\alpha}}$$
$$\times\left[\sum_{n=0}^{r+1}(n+1)^{-\frac{\beta}{\alpha}}\left(\sum_{k=1}^{n+1}(f(b_{2k-1})-f(b_{2k}))\right)^{\beta}\right]^{\frac{1}{\beta}}.$$

Finally, applying the Hölder inequality and interchanging the order of summation, we obtain

$$\sum_{m=0}^{s+1}\sum_{n=0}^{r+1}\frac{f(\overline{B}_n)g(\overline{A}_m)}{(\lambda_m+\rho_n)^{\mu}}$$
$$\leqslant C_{\lambda,\rho}\left[\sum_{m=0}^{s+1}(m+1)^{-\frac{\alpha}{\beta}}\left[\left(\sum_{\ell=1}^{m+1}1\right)^{\frac{1}{\beta}}\left(\sum_{\ell=1}^{m+1}(g(a_{2\ell-1})-g(a_{2\ell}))^{\alpha}\right)^{\frac{1}{\alpha}}\right]^{\alpha}\right]^{\frac{1}{\alpha}}$$

$$\times\left[\sum_{n=0}^{r+1}(n+1)^{-\frac{\beta}{\alpha}}\left[\left(\sum_{k=1}^{n+1}1\right)^{\frac{1}{\alpha}}\left(\sum_{k=1}^{n+1}(f(b_{2k-1})-f(b_{2k}))^{\beta}\right)^{\frac{1}{\beta}}\right]^{\beta}\right]^{\frac{1}{\beta}}$$

$$=C_{\lambda,\rho}\left[\sum_{m=0}^{s+1}\sum_{\ell=1}^{m+1}(g(a_{2\ell-1})-g(a_{2\ell}))^{\alpha}\right]^{\frac{1}{\alpha}}\left[\sum_{n=0}^{r+1}\sum_{k=1}^{n+1}(f(b_{2k-1})-f(b_{2k}))^{\beta}\right]^{\frac{1}{\beta}}$$

$$=C_{\lambda,\rho}\left[\sum_{\ell=1}^{s+2}(g(a_{2\ell-1})-g(a_{2\ell}))^{\alpha}(s-\ell+3)\right]^{\frac{1}{\alpha}}$$

$$\times\left[\sum_{k=1}^{r+2}(f(b_{2k-1})-f(b_{2k}))^{\beta}(r-k+3)\right]^{\frac{1}{\beta}}$$

$$=C_{\lambda,\rho}\left[\sum_{m=1}^{s+2}(g(a_{2m-1})-g(a_{2m}))^{\alpha}(s-m+3)\right]^{\frac{1}{\alpha}}$$

$$\times\left[\sum_{n=1}^{r+2}(f(b_{2n-1})-f(b_{2n}))^{\beta}(r-n+3)\right]^{\frac{1}{\beta}},$$

which concludes the proof. □

REMARK 1. It should be noticed here that if $\alpha=q$, $\beta=p$, $\lambda_m=\frac{m+1}{p}$, $\rho_n=\frac{n+1}{q}$ and $\mu=1$, the constant reduces to $C_{\lambda,\rho}=(s+2)^{\frac{1}{p}}(r+2)^{\frac{1}{q}}$. By using Young's inequality we get the Theorem 1.

REMARK 2. Taking $\lambda_m=(m+1)^{\frac{1}{\mu}}$, $\rho_n=(n+1)^{\frac{1}{\mu}}$ in (4), we have

$$\sum_{m=0}^{s+1}\sum_{n=0}^{r+1}\frac{f(\overline{B}_n)g(\overline{A}_m)}{\left((m+1)^{\frac{1}{\mu}}+(n+1)^{\frac{1}{\mu}}\right)^{\mu}}\leqslant C_{\lambda,\rho}\left[\sum_{m=1}^{s+2}(g(a_{2m-1})-g(a_{2m}))^{\alpha}(s-m+3)\right]^{\frac{1}{\alpha}}$$
$$\times\left[\sum_{n=1}^{r+2}(f(b_{2n-1})-f(b_{2n}))^{\beta}(r-n+3)\right]^{\frac{1}{\beta}}, \tag{11}$$

where

$$C_{\lambda,\rho}=\alpha^{-\frac{\mu}{\alpha}}\beta^{-\frac{\mu}{\beta}}(s+2)^{\frac{1}{\beta}}(r+2)^{\frac{1}{\alpha}}. \tag{12}$$

THEOREM 3. *Let* $p,q\geqslant 1$, $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, *and let* (a_m), (b_n) *be positive non-increasing sequences of real numbers. Further, let* $\overline{A}_m=\sum_{\ell=1}^{2m+1}(-1)^{\ell+1}a_\ell$ *and*

$\overline{B}_n = \sum_{k=1}^{2n+1}(-1)^{k+1}b_k$. *Then holds the inequality*

$$\sum_{m=0}^{s}\sum_{n=0}^{r}\frac{\overline{A}_m^p\overline{B}_n^q}{(\lambda_m+\rho_n)^\mu} \leqslant \tilde{C}_{\lambda,\rho}\left[\sum_{m=1}^{s+1}(s-m+2)((a_{2m-1}-a_{2m})\tilde{A}_m^{p-1})^\beta\right]^{\frac{1}{\beta}} \times\left[\sum_{n=1}^{r+1}(r-n+2)((b_{2n-1}-b_{2n})\tilde{B}_n^{q-1})^\alpha\right]^{\frac{1}{\alpha}}, \quad (13)$$

where

$$\tilde{C}_{\lambda,\rho} = \alpha^{-\frac{\mu}{\alpha}}\beta^{-\frac{\mu}{\beta}}\left(\sum_{m=0}^{s}\frac{m+1}{\lambda_m^\mu}\right)^{\frac{1}{\alpha}}\left(\sum_{n=0}^{r}\frac{n+1}{\rho_n^\mu}\right)^{\frac{1}{\beta}}, \quad (14)$$

and $\tilde{A}_m = \sum_{\ell=1}^{m+1}(a_{2\ell-1}-a_{2\ell}), \quad \tilde{B}_n = \sum_{k=1}^{n+1}(b_{2k-1}-b_{2k})$.

Proof. By virtue of (6), it follows that

$$\sum_{m=0}^{s}\sum_{n=0}^{r}\frac{\overline{A}_m^p\overline{B}_n^q}{(\lambda_m+\rho_n)^\mu} = \frac{1}{\Gamma(\mu)}\int_0^\infty\left(x^{\frac{\mu-1}{\alpha}}\sum_{m=1}^{s}e^{-\lambda_m x}\overline{A}_m^p\right)\left(x^{\frac{\mu-1}{\beta}}\sum_{n=1}^{r}e^{-\rho_n x}\overline{B}_n^q\right)dx. \quad (15)$$

Using the following inequality (see [3])

$$\left(\sum_{m=1}^{n}z_m\right)^\gamma \leqslant \gamma\sum_{m=1}^{n}z_m\left(\sum_{k=1}^{m}z_k\right)^{\gamma-1},$$

where $\gamma \geqslant 0$ is a constant and $z_m \geqslant 0$, it easily follows that

$$\overline{A}_m^p = \tilde{A}_m^p \leqslant p\sum_{\ell=1}^{m+1}(a_{2\ell-1}-a_{2\ell})\tilde{A}_\ell^{p-1}. \quad (16)$$

and

$$\overline{B}_n^q = \tilde{B}_n^q \leqslant q\sum_{k=1}^{n+1}(b_{2k-1}-b_{2k})\tilde{B}_k^{q-1}, \quad (17)$$

and we assume that $a_{2n+2} = b_{2m+2} = 0$.

From (15)–(17) and using Hölder inequality, we have

$$\sum_{m=0}^{s}\sum_{n=0}^{r}\frac{\overline{A}_m^p\overline{B}_n^q}{(\lambda_m+\rho_n)^\mu} = \frac{pq}{\Gamma(\mu)}\int_0^\infty\left(x^{\frac{\mu-1}{\alpha}}\sum_{m=0}^{s}e^{-\lambda_m x}\sum_{\ell=1}^{m+1}(a_{2\ell-1}-a_{2\ell})\tilde{A}_\ell^{p-1}\right) \times\left(x^{\frac{\mu-1}{\beta}}\sum_{n=0}^{r}e^{-\rho_n x}\sum_{k=1}^{n+1}(b_{2k-1}-b_{2k})\tilde{B}_k^{q-1}\right)dx$$

$$\leqslant \frac{1}{\Gamma(\mu)}\left[\int_0^\infty x^{\mu-1}\left(\sum_{m=0}^{s}e^{-\lambda_m x}\sum_{\ell=1}^{m+1}(a_{2\ell-1}-a_{2\ell})\tilde{A}_\ell^{p-1}\right)^\alpha dx\right]^{\frac{1}{\alpha}} \times\left[\int_0^\infty x^{\mu-1}\left(\sum_{n=0}^{r}e^{-\rho_n x}\sum_{k=1}^{n+1}(b_{2k-1}-b_{2k})\tilde{B}_k^{q-1}\right)^\beta dx\right]^{\frac{1}{\beta}}. \quad (18)$$

Using again Hölder inequality and relation (6), we obtain

$$\sum_{m=0}^{s}\sum_{n=0}^{r}\frac{\overline{A}_m^p\overline{B}_n^q}{(\lambda_m+\rho_n)^\mu} \leqslant \frac{pq}{\Gamma(\mu)}\left[\int_0^\infty x^{\mu-1}\left[\sum_{m=0}^{s}\left((m+1)^{\frac{1}{\alpha}}e^{-\lambda_m x}\right)\right.\right.$$
$$\left.\left.\times\left((m+1)^{-\frac{1}{\alpha}}\sum_{\ell=1}^{m+1}(a_{2\ell-1}-a_{2\ell})\tilde{A}_\ell^{p-1}\right)\right]^\alpha dx\right]^{\frac{1}{\alpha}}$$
$$\times\left[\int_0^\infty x^{\mu-1}\left[\sum_{n=0}^{r}\left((n+1)^{\frac{1}{\beta}}e^{-\rho_n x}\right)\right.\right.$$
$$\left.\left.\times\left((n+1)^{-\frac{1}{\beta}}\sum_{k=1}^{n+1}(b_{2k-1}-b_{2k})\tilde{B}_k^{q-1}\right)\right]^\beta dx\right]^{\frac{1}{\beta}}.$$
$$\leqslant\frac{pq}{\Gamma(\mu)}\left[\int_0^\infty x^{\mu-1}\left[\sum_{m=0}^{s}(m+1)e^{-\alpha\lambda_m x}\right]\right.$$
$$\left.\times\left[\sum_{m=1}^{s}(m+1)^{-\frac{\beta}{\alpha}}\left(\sum_{\ell=1}^{m+1}(a_{2\ell-1}-a_{2\ell})\tilde{A}_\ell^{p-1}\right)^\beta\right]^{\frac{\alpha}{\beta}}dx\right]^{\frac{1}{\alpha}}$$
$$\times\left[\int_0^\infty x^{\mu-1}\left[\sum_{n=0}^{r}(n+1)e^{-\beta\rho_n x}\right]\right.$$
$$\left.\times\left[\sum_{n=0}^{r}(n+1)^{-\frac{\alpha}{\beta}}\left(\sum_{k=1}^{n+1}(b_{2k-1}-b_{2k})\tilde{B}_k^{q-1}\right)^\alpha\right]^{\frac{\beta}{\alpha}}dx\right]^{\frac{1}{\beta}}$$
$$=\tilde{C}_{\lambda,\rho}\left[\sum_{m=0}^{s}(m+1)^{-\frac{\beta}{\alpha}}\left(\sum_{\ell=1}^{m+1}(a_{2\ell-1}-a_{2\ell})\tilde{A}_\ell^{p-1}\right)^\beta\right]^{\frac{1}{\beta}}$$
$$\times\left[\sum_{n=0}^{r}(n+1)^{-\frac{\alpha}{\beta}}\left(\sum_{k=1}^{n+1}(b_{2k-1}-b_{2k})\tilde{B}_k^{q-1}\right)^\alpha\right]^{\frac{1}{\alpha}}.$$

Now, using the Hölder inequality and interchanging the order of summation, we have

$$\sum_{m=0}^{s}\sum_{n=0}^{r}\frac{\overline{A}_m^p\overline{B}_n^q}{(\lambda_m+\rho_n)^\mu}$$
$$\leqslant\tilde{C}_{\lambda,\rho}\left[\sum_{m=0}^{s}\sum_{\ell=1}^{m+1}((a_{2\ell-1}-a_{2\ell})\tilde{A}_\ell^{p-1})^\beta\right]^{\frac{1}{\beta}}\left[\sum_{n=0}^{r}\sum_{k=1}^{n+1}((b_{2k-1}-b_{2k})\tilde{B}_k^{q-1})^\alpha\right]^{\frac{1}{\alpha}}$$

$$= \tilde{C}_{\lambda,\rho}\left[\sum_{\ell=1}^{s+1}((a_{2\ell-1}-a_{2\ell})\tilde{A}_{\ell}^{p-1})^{\beta}(s-\ell+2)\right]^{\frac{1}{\beta}}$$

$$\times\left[\sum_{k=1}^{r+1}((b_{2k-1}-b_{2k})\tilde{B}_{k}^{q-1})^{\alpha}(r-k+2)\right]^{\frac{1}{\alpha}}$$

$$= \tilde{C}_{\lambda,\rho}\left[\sum_{m=1}^{s+1}(s-m+2)((a_{2m-1}-a_{2m})\tilde{A}_{m}^{p-1})^{\beta}\right]^{\frac{1}{\beta}}$$

$$\times\left[\sum_{n=1}^{r+1}(r-n+2)((b_{2n-1}-b_{2n})\tilde{B}_{n}^{q-1})^{\alpha}\right]^{\frac{1}{\alpha}}. \quad \square$$

REMARK 3. If $\mu = 1$, $\lambda_m = (m+1)/\beta$ and $\rho_n = (n+1)/\alpha$, so our inequality (13) reduces to the corresponding result established in [8].

REMARK 4. Taking $\lambda_m = (m+1)^{\frac{1}{\mu}}$, $\rho_n = (n+1)^{\frac{1}{\mu}}$ in (13), we have

$$\sum_{m=0}^{s}\sum_{n=0}^{r}\frac{\overline{A}_m^p\overline{B}_n^q}{\left((m+1)^{\frac{1}{\mu}}+(n+1)^{\frac{1}{\mu}}\right)^{\mu}} \leqslant \tilde{C}_{\lambda,\rho}\left[\sum_{m=1}^{s+1}(s-m+2)((a_{2m-1}-a_{2m})\tilde{A}_{m}^{p-1})^{\beta}\right]^{\frac{1}{\beta}} \times\left[\sum_{n=1}^{r+1}(r-n+2)((b_{2n-1}-b_{2n})\tilde{B}_{n}^{q-1})^{\alpha}\right]^{\frac{1}{\alpha}}, \tag{19}$$

where

$$\tilde{C}_{\lambda,\rho} = \alpha^{-\frac{\mu}{\alpha}}\beta^{-\frac{\mu}{\beta}}(s+1)^{\frac{1}{\alpha}}(r+1)^{\frac{1}{\beta}}. \tag{20}$$

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