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APPROXIMATION PROPERTIES OF THE RIEMANN–LIOUVILLE FRACTIONAL INTEGRAL TYPE SZÁSZ–MIRAKYAN–KANTOROVICH OPERATORS

NAZIM MAHHMUDOV AND MUSTAFA KARA

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Abstract. In the present paper, we introduce the Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators. We investigate the order of convergence by using Lipschitz-type maximal functions, second order modulus of smoothness and Peetre's K-functional. Weighted approximation properties of these operators in terms of modulus of continuity have been discussed. Then, Voronskaja-type type theorem are obtained. Moreover, bivariate the Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators are constructed. The last section is devoted to graphical representation and numerical results for these operators.

1. Introduction

Approximation theory plays an important role in mathematical analysis problems and in many fields of science. In particular, linear positive operators have an important role in approximation theory. One of the best known of these operators is the Szász–Mirakjan operator. The classical Szász–Mirakjan operator (see [8] and [9]) is defined for bounded functions $\varphi(\tau)$ on $[0,\infty)$ by the formula

$$S_n(\varphi;\tau) = \sum_{k=0}^{n} s_{n,k}(\tau)\, \varphi\left(\frac{k}{n}\right)$$

where $n \in \mathbb{N},\ \tau \in [0,\infty)$ and $s_{n,k}(\tau) = e^{-n\tau}\frac{(n\tau)^k}{k!}$. In 1954, Butzer [10] generalized into integral modification of the Szász-Mirakjan operators as follows:

$$K_n(\varphi;\tau) = n\sum_{k=0}^{\infty} s_{n,k}(\tau) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \varphi(t)\, dt. \tag{1}$$

The above operators can be rewritten as follows:

$$K_n(\varphi;\tau) = \sum_{k=0}^{\infty} s_{n,k}(\tau) \int_{0}^{1} \varphi(\frac{k+t}{n})\, dt. \tag{2}$$

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In the literature, there are a lot of studies that involve Szász operators, Szász-Kantorovich operators and their generalizations. For instance, see [1]–[7] and [20]–[31].

Recently, Fractional calculus and its applications have been paid more and more attention. Fractional calculus deals with the study of fractional degree derivative and integral operators on complex or real fields and their applications. There are several known forms of fractional integrals. The most well-known is:

$$(I_{a+}^{\alpha}\varphi)(\tau) = \frac{1}{\Gamma(\alpha)}\int_{a}^{\tau}\frac{\varphi(t)}{(\tau-t)^{1-\alpha}}dt \qquad (\tau > a;\ \mathrm{Re}(\alpha) > 0)$$

where $\Gamma(z) = \int_{0}^{\infty} t^{z-1}e^{-t}dt$.

The aim of the present paper is to construct the Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators and discuss their approximation properties.

The paper is organized as follows. In Section 2, we introduce the Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators and give some moments and central moments. In section 3, Korovkin-type approximation theorem is given, and the rate of convergence of these types of operators is obtained for Lipschitz-type maximal functions, second order modulus of smoothness and Peetre's K-functional. In Section 4, weighted approximation properties of the Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators in terms of modulus of continuity has been discussed. In Section 5, we provide the quantitative Vorononskaja-type asymptotic formula. In section 6, we introduce bivariate the Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators and their approximation behaviors. Finally, some graphs and error estimation tables are given for the convergence of Szász-Mirakyan-Kantorovich operators of the Riemann-Liouville fractional integral type.

2. The Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators

Now, for the Szász-Mirakjan operators, the following Lemma is known (see [11] and [12]).

LEMMA 1. ([11] and [12]) *Let* $e_i(\tau) = \tau^i,\ i = 1,2,3,4$. *Then, for each* $\tau \geqslant 0$, *we have*

$$\begin{aligned} S_n(1;\tau) &= 1 \\ S_n(t;\tau) &= \tau \\ S_n(t^2;\tau) &= \tau^2 + \frac{\tau}{n} \\ S_n(t^3;\tau) &= \tau^3 + \frac{3\tau^2}{n} + \frac{\tau}{n^2} \end{aligned}$$

$$S_n(t^4;\tau) = \tau^4 + \frac{6\tau^3}{n} + \frac{7\tau^2}{n^2} + \frac{\tau}{n^3}$$

DEFINITION 2. Let $\alpha > 0$ and $n \in \mathbb{N}$. For $\varphi \in C[0,\infty)$, Riemann-Liouville integral type Szász-Mirakyan-Kantorovich operators can be defined by

$$K_n^{(\alpha)}(\varphi;\tau) = \sum_{k=0}^{\infty} \alpha s_{n,k}(\tau) \int_0^1 \frac{\varphi\left(\frac{k+t}{n}\right)}{(1-t)^{1-\alpha}} dt. \tag{3}$$

where $s_{n,k}(\tau) = e^{-n\tau}\frac{(n\tau)^k}{k!}$.

If $\alpha = 1$, then the operator reduces to classical Szász-Mirakyan-Kantorovich operators.

The moments of the $K_n^{(\alpha)}$ operators plays significant role in our main results. We derive the following formula to obtain them.

LEMMA 3. *Let $\alpha > 0, m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. Then for $\tau \geqslant 0$, we have*

$$K_n^{(\alpha)}(t^m;\tau) = \sum_{j=0}^{m} \binom{m}{j} \frac{\alpha n^j B(m-j+1,\alpha)}{n^m} S_n(t^j;\tau) \tag{4}$$

where

$$S_n(\varphi;\tau) = \sum_{k=0}^{n} s_{n,k}(\tau)\ \varphi\left(\frac{k}{n}\right)$$

and

$$B(a,b) = \int_0^1 \tau^{a-1}(1-\tau)^{b-1}, \quad a,b > 0.$$

Proof. From (3), we can write

$$\begin{aligned} K_n^{(\alpha)}(t^m;\tau) &= \sum_{k=0}^{\infty} \alpha s_{n,k}(\tau) \int_0^1 \left(\frac{k+t}{n}\right)^m (1-t)^{\alpha-1} dt \\ &= \sum_{k=0}^{\infty} \alpha s_{n,k}(\tau) \sum_{j=0}^{m} \binom{m}{j} \frac{k^j}{n^m} \int_0^1 t^{m-j}(1-t)^{\alpha-1} dt \\ &= \sum_{j=0}^{m} \binom{m}{j} \frac{\alpha n^j B(m-j+1,\alpha)}{n^m} \sum_{k=0}^{\infty} s_{n,k}(\tau) \frac{k^j}{n^j} \\ &= \sum_{j=0}^{m} \binom{m}{j} \frac{\alpha n^j B(m-j+1,\alpha)}{n^m} S_n(t^j;\tau). \quad \square \end{aligned}$$

The following Lemma for $K_n^{\alpha}(t^j;\tau)$ $(j = 0,1,2,3,4)$ can be written immediately.

LEMMA 4. *Let* $\alpha > 0$ *and* $n \in \mathbb{N}$. *Then for* $\tau \geqslant 0$, *we have*

$$\text{(i)} \qquad K_n^{(\alpha)}(1;\tau) = 1,$$

$$\text{(ii)} \qquad K_n^{(\alpha)}(t;\tau) = \tau + \frac{1}{n(\alpha+1)},$$

$$\text{(iii)} \qquad K_n^{(\alpha)}(t^2;\tau) = \frac{2}{(\alpha+1)(\alpha+2)n^2} + \frac{(\alpha+3)}{(\alpha+1)n}\tau + \tau^2,$$

$$\text{(iv)} \qquad K_n^{(\alpha)}(t^3;\tau) = \frac{6}{(\alpha+1)(\alpha+2)(\alpha+3)n^3} + \frac{6+(\alpha+2)(4\alpha+7)}{(\alpha+1)(\alpha+2)n^2}\tau + \frac{3(2+\alpha)}{(\alpha+1)n}\tau^2 + \tau^3$$

$$\text{(v)} \qquad K_n^{(\alpha)}(t^4;\tau) = \frac{24}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)n^4} + \frac{24+(\alpha+3)[12+(\alpha+2)(\alpha+5)]}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)n^3}\tau + \frac{12+(\alpha+2)(7\alpha+19)}{(\alpha+1)(\alpha+2)n^2}\tau^2 + \frac{(6\alpha+10)}{(\alpha+1)n}\tau + \tau^4$$

Proof. Since they have the same proof technique, we only give for $K_n^{(\alpha)}(t^2;\tau)$. Using recurrence formula (4) and Lemma 1, we get

$$\begin{aligned} K_{n,k}^{(\alpha)}(t^2;\tau) &= \frac{\alpha B(3,\alpha)}{n^2} S_{n,k}(1;\tau) + \frac{2n\alpha B(2,\alpha)}{n^2} S_{n,k}(t;\tau) + \frac{n^2\alpha B(1,\alpha)}{n^2} S_{n,k}(t^2;\tau) \\ &= \frac{2}{(\alpha+1)(\alpha+2)n^2} + \frac{2}{(\alpha+1)n}\tau + \left(\tau^2 + \frac{\tau}{n}\right) \\ &= \frac{2}{(\alpha+1)(\alpha+2)n^2} + \frac{(\alpha+3)}{(\alpha+1)n}\tau + \tau^2. \quad \square \end{aligned}$$

We are now ready to present the central moments of the operators $K_n^{(\alpha)}$.

LEMMA 5. *Let* $\alpha > 0$. *For every* $\tau \geqslant 0$, *there holds*

$$\begin{aligned} K_n^{(\alpha)}(t-\tau;\tau) &= \frac{1}{(\alpha+1)n}, \\ K_n^{(\alpha)}((t-\tau)^2;\tau) &= \frac{2}{(\alpha+1)(\alpha+2)n^2} + \frac{\tau}{n}, \\ K_n^{(\alpha)}((t-\tau)^4;\tau) &= \frac{24}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)n^4} + \frac{12+(\alpha+2)(\alpha+5)}{(\alpha+1)(\alpha+2)n^3}\tau + \frac{3}{n^2}\tau^2. \end{aligned}$$

Proof. Since they have the same proof technique, we only give for $K_n^{(\alpha)}((t-\tau)^2;\tau)$. From the linearity of $K_n^{(\alpha)}(t;\tau)$ and lemma 4, we get

$$\begin{aligned}K_n^{(\alpha)}((t-\tau)^2;\tau) &= K_n^{(\alpha)}(t^2;\tau)-2\tau K_n^{(\alpha)}(t;\tau)+\tau^2K_n^{(\alpha)}(1;\tau)\\ &= \frac{2}{(\alpha+1)(\alpha+2)n^2}+\frac{(\alpha+3)}{(\alpha+1)n}\tau+\tau^2\\ &\quad -2\tau\left(\tau+\frac{1}{n(\alpha+1)}\right)+\tau^2. \quad \square\end{aligned}$$

LEMMA 6. *Let $\alpha>0$ and $n\in\mathbb{N}$. For every $\tau\geqslant 0$, there holds*

$$\lim_{n\to\infty} nK_{n,k}^{(\alpha)}((t-\tau);\tau)=\frac{1}{(\alpha+1)}, \tag{5}$$

$$\lim_{n\to\infty} nK_{n,k}^{(\alpha)}((t-\tau)^2;\tau)=\tau, \tag{6}$$

and

$$\lim_{n\to\infty} nK_{n,k}^{(\alpha)}((t-\tau)^4;\tau)=0. \tag{7}$$

Proof. From Lemma 5, for every $\tau\geqslant 0$, we have

$$\lim_{n\to\infty} nK_{n,k}^{(\alpha)}((t-\tau);\tau)=\frac{1}{(\alpha+1)},$$

$$\lim_{n\to\infty} nK_{n,k}^{(\alpha)}((t-\tau)^2;\tau)=\lim_{n\to\infty} n\left(\frac{2}{(\alpha+1)(\alpha+2)n^2}+\frac{\tau}{n}\right)=\tau$$

and

$$\lim_{n\to\infty} nK_{n,k}^{(\alpha)}((t-\tau)^4;\tau)=0. \quad \square$$

In [17], Becker proved the following explicit formula for the moments of S_n.

LEMMA 7. [17] *For $m\in\mathbb{N}$, there holds*

$$S_n(t^m;\tau)=\sum_{j=1}^{m} a_{m,j}\frac{\tau^j}{n^{m-j}} \tag{8}$$

with positive coefficients $a_{m,j}$. In particular $S_n(t^m;\tau)$ is a polynomial of degree m without a constant term.

For the next main results we need to define the following classes of functions:

1. $B_m[0,\infty)=\left\{\varphi:[0,\infty)\to\mathbb{R};|\varphi(\tau)|\leqslant M_\varphi(1+\tau^m)\right\}$, where M_φ is constant depending on the function $\varphi, m>0$.
2. $C_m[0,\infty)=B_m[0,\infty)\cap C[0,\infty)$, $m>0$.

3. $C_m^{\star}[0,\infty)=\left\{\varphi : C_m[0,\infty) : \lim_{|\tau|\to\infty}\frac{|\varphi(\tau)|}{1+\tau^m}<\infty\right\}$, $m>0$.

The norm on the space $C_m^{\star}[0,\infty)$ is showed as $\|\varphi(\tau)\|_m=\sup_{\tau\in[0,\infty)}\frac{|\varphi(\tau)|}{1+\tau^m}$, where $m>0$.

LEMMA 8. *Let $m\in\mathbb{N}\cup\{0\}$ and $\alpha>0$ be fixed. We have*

$$\left\|K_n^{(\alpha)}(1+t^m;\tau)\right\|_m\leqslant C_{m,j}(\alpha),\ n\in\mathbb{N}. \tag{9}$$

where $C_m(\alpha)$ is a positive constant. Morever, we have

$$\left\|K_n^{(\alpha)}(\varphi;\tau)\right\|_m\leqslant C_{m,j}(\alpha)\|\varphi\|_m,\ \ n\in\mathbb{N}, \tag{10}$$

where $\varphi\in C_m^{*}[0,\infty)$. Thus, for any $m\in\mathbb{N}\cup\{0\}$, $K_n^{(\alpha)}:C_m^{*}[0,\infty)\to C_m^{*}[0,\infty)$ is a linear positive operator.

Proof. For $m=0$, inequality (9) is obvious.
For $m\geqslant 1$, combining Lemma (4) and inequality (8), we have

$$\begin{aligned}\frac{1}{\tau^m+1}K_n^{(\alpha)}(1+t^m;\tau)&=\frac{1}{\tau^m+1}+\frac{1}{\tau^m+1}K_n^{(\alpha)}(t^m;\tau)\\&=\frac{1}{\tau^m+1}+\frac{1}{\tau^m+1}\sum_{j=0}^{m}\binom{m}{j}\frac{\alpha n^jB(m-j+1,\alpha)}{n^m}\sum_{j_0=1}^{j}a_{j,j_0}\frac{\tau^{j_0}}{n^{j-j_0}}\\&\leqslant 1+k_{m,j}(\alpha)=C_{m,j}(\alpha).\end{aligned}$$

$C_{m,j}(\alpha)$ is a positive constant with depend on m,j and α. Moreover,

$$\left\|K_n^{(\alpha)}(\varphi;\tau)\right\|_m\leqslant\|\varphi\|_m\left\|K_n^{(\alpha)}(1+t^m;\tau)\right\|_m \tag{11}$$

for every $\varphi\in C_m^{*}[0,\infty)$. Therefore, from (9), we get

$$\left\|K_n^{(\alpha)}(\varphi;\tau)\right\|_m\leqslant C_{m,j}(\alpha)\|\varphi\|_m.\quad\square$$

3. Direct results

Let $C_B[0,\infty)$ denote the space of all real-valued continuous and bounded functions φ on $[0,\infty)$. The norm on the space $C_B[0,\infty)$ is showed as

$$\|\varphi\|_{C_B[0,\infty)}=\sup_{\tau\in[0,\infty)}|\varphi(\tau)|.$$

Then, the modulus of continuity of $\varphi\in C_B[0,\infty)$ is given by

$$w(\varphi,\delta)=\sup_{0<h\leqslant\delta}\ \sup_{\tau\in[0,\infty)}|\varphi(\tau+h)-\varphi(\tau)|.$$

Further, Peetre's K-functional is defined by

$$K_2(\varphi;\delta)=\inf_{g\in w^2}\left\{\|\varphi-g\|+\delta\left\|g^{''}\right\|\right\}\qquad \delta>0,$$

where $w^2:=\left\{g\in C_B[0,\infty):g^{'},g^{''}\in C_B[0,\infty)\right\}$. By Theorem 2.4 in [13], there exists an absolute constant $L>0$ such that

$$K_2(\varphi;\delta)\leqslant L\omega_2\left(\varphi;\sqrt{\delta}\right). \tag{12}$$

where $\delta>0$ are absolute constant.

Here, $\omega_2(\varphi;\delta)$ is the second order modulus of smoothness of $\varphi\in C_B[0,\infty)$ and defined as

$$\omega_2(\varphi;\delta)=\sup_{0<h\leqslant\delta}\ \sup_{\tau\in[0,\infty)}|\varphi(\tau+2h)-2\varphi(\tau+h)+\varphi(\tau)|$$

LEMMA 9. *Let $\varphi\in C_B[0,\infty)$ and $\alpha>0$, Consider the operators*

$$^*K_n^{(\alpha)}(\varphi;\tau)=K_n^{(\alpha)}(\varphi;\tau)+\varphi(\tau)-\varphi\left(\tau+\frac{1}{(\alpha+1)n}\right) \tag{13}$$

Then, for all $g\in w^2$, we have

$$\left|^*K_n^{(\alpha)}(g;\tau)-g(\tau)\right|\leqslant\left(\frac{2}{(\alpha+1)(\alpha+2)n^2}+\frac{\tau}{n}+\left(\frac{1}{(\alpha+1)n}\right)^2\right)\left\|g^{''}\right\|. \tag{14}$$

Proof. From (13) we have

$$\begin{aligned}^*K_n^{(\alpha)}((t-\tau);\tau)&=K_n^{(\alpha)}((t-\tau);\tau)-\left(\tau+\frac{1}{(\alpha+1)n}-\tau\right)\\&=K_n^{(\alpha)}(t;\tau)-\tau K_n^{(\alpha)}(1;\tau)-\left(\tau+\frac{1}{(\alpha+1)n}\right)+\tau=0.\end{aligned} \tag{15}$$

Let $\tau\in[0,\infty)$ and $g\in w^2$. Using the Taylor's formula,

$$g(t)-g(\tau)=(t-\tau)g^{'}(\tau)+\int_\tau^t(t-u)g^{''}(u)du, \tag{16}$$

Applying $^*K_n^{(\alpha)}$ and using (15), we can get

$$\begin{aligned}&^*K_n^{(\alpha)}(g;\tau)-g(\tau)\\&={}^*K_n^{(\alpha)}\left((t-\tau)g^{'}(\tau);\tau\right)+{}^*K_n^{(\alpha)}\left(\int_\tau^t(t-u)g^{''}(u)du;\tau\right)\end{aligned}$$

$$= g'(\tau)^* K_n^{(\alpha)}\left((t-\tau);\tau\right) + K_n^{(\alpha)}\left(\int_\tau^t (t-u)g''(u)du;\tau\right)$$

$$-\int_\tau^{\tau+\frac{1}{(\alpha+1)n}} \left(\tau+\frac{1}{(\alpha+1)n}-u\right)g''(u)du$$

$$= K_n^{(\alpha)}\left(\int_\tau^t (t-u)g''(u)du;\tau\right) - \int_\tau^{\tau+\frac{1}{(\alpha+1)n}} \left(\tau+\frac{1}{(\alpha+1)n}-u\right)g''(u)du.$$

On the other hand, since

$$\int_\tau^t |t-u|\left|g''(u)\right|du \leqslant \left\|g''\right\| \int_\tau^t |t-u|du \leqslant (t-\tau)^2\left\|g''\right\|$$

and

$$\left|\int_\tau^{\tau+\frac{1}{(\alpha+1)n}} \left(\tau+\frac{1}{(\alpha+1)n}-u\right)g''(u)du\right| \leqslant \left(\frac{1}{(\alpha+1)n}\right)^2\left\|g''\right\|,$$

we conclude that

$$\begin{aligned}
&\left|{}^*K_n^{(\alpha)}(g;\tau)-g(\tau)\right| \\
&= \left|K_n^{(\alpha)}\left(\int_\tau^t (t-u)g''(u)du;\zeta\right) - \int_\tau^{\tau+\frac{1}{(\alpha+1)n}} \left(\tau+\frac{1}{(\alpha+1)n}-u\right)g''(u)du\right| \\
&\leqslant \left\|g''\right\| K_n^{(\alpha)}\left((t-\tau)^2;\tau\right) + \left(\frac{1}{(\alpha+1)n}\right)^2\left\|g''\right\|.
\end{aligned}$$

From Lemma 5, we can write

$$\left|{}^*K_n^{(\alpha)}(g;\tau)-g(\tau)\right| \leqslant \frac{2}{(\alpha+1)(\alpha+2)n^2}+\frac{\tau}{n}+\left(\frac{1}{(\alpha+1)n}\right)^2\left\|g''\right\|. \quad \square$$

THEOREM 10. *Let* $\varphi \in C_B[0,\infty)$ *and* $\alpha > 0$. *Then, for every* $\tau \in [0,\infty)$, *there exists a constant* $M > 0$ *such that*

$$\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right| \leqslant M\omega_2\left(\varphi;\sqrt{\delta_n(\tau)}\right)+\omega\left(\varphi;\beta_n^{(\alpha)}(\tau)\right)$$

where

$$\delta_n^{(\alpha)}(\tau) = \frac{2}{(\alpha+1)(\alpha+2)n^2}+\frac{\tau}{n}+\left(\frac{1}{(\alpha+1)n}\right)^2\left\|g''\right\|$$

and

$$\beta_{n,k}^{(\alpha)}(\tau) = \left|\frac{1}{(\alpha+1)n}\right|$$

Proof. It follows from Lemma 9, that

$$\begin{aligned}\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right| &\leqslant \left|{}^*K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right| + \left|\varphi(\tau)-\varphi\left(\tau+\frac{1}{(\alpha+1)n}\right)\right| \\ &\leqslant \left|{}^*K_n^{(\alpha)}(\varphi-g;\tau)-(\varphi-g)(\tau)\right| \\ &\quad + \left|\varphi(\tau)-\varphi\left(\tau+\frac{1}{(\alpha+1)n}\right)\right| + \left|{}^*K_n^{(\alpha)}(g;\tau)-g(\tau)\right| \\ &\leqslant \left|{}^*K_n^{(\alpha)}(\varphi-g;\tau)\right| + |(\varphi-g)(\tau)| \\ &\quad + \left|\varphi(\tau)-\varphi\left(\tau+\frac{1}{(\alpha+1)n}\right)\right| + \left|{}^*K_n^{(\alpha)}(g;\tau)-g(\tau)\right|.\end{aligned}$$

Now, considering the boundedness of the ${}^*K_n^{(\alpha)}$ and inequality (14), we get

$$\begin{aligned}\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right| &\leqslant 4\|\varphi-g\| + \left|\varphi(\tau)-\varphi\left(\tau+\frac{1}{(\alpha+1)n}\right)\right| \\ &\quad + \frac{2}{(\alpha+1)(\alpha+2)n^2} + \frac{\tau}{n} + \left(\frac{1}{(\alpha+1)n}\right)^2 \left\|g''\right\| \\ &\leqslant 4\|\varphi-g\| + \omega\left(\varphi;\left|\frac{1}{(\alpha+1)n}\right|\right) + \delta_n(\tau)\left\|g''\right\|.\end{aligned}$$

Now, taking infimum on the right hand side over all $g \in w^2$ and using the property of Peetre's K-functional (12), we can get

$$\begin{aligned}\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right| &\leqslant 4K_2\left(\varphi;\delta_n^{(\alpha)}(\tau)\right) + \omega\left(\varphi;\beta_n^{(\alpha)}(\tau)\right) \\ &\leqslant M\omega_2\left(\varphi;\sqrt{\delta_n^{(\alpha)}(\tau)}\right) + \omega\left(\varphi;\beta_n^{(\alpha)}(\tau)\right). \quad \square\end{aligned}$$

COROLLARY 11. *Let $\alpha > 0$. For any $A > 0$ and $\varphi \in C_B[0,\infty)$, then $K_n^{(\alpha)}(\varphi;\tau) \rightrightarrows \varphi$ on $[0,A]$, where the symbol $\rightrightarrows$ denotes the uniform convergence.*

THEOREM 12. *Let $K_n^{(\alpha)}$ be the operators defined by (3), $\alpha > 0$, $\gamma \in (0,1]$ and D be any subset of the interval $[0,\infty)$. if $\varphi \in C_B[0,\infty)$ is locally Lip(γ) on D, i.e., if φ satisfies the following inequality:*

$$|\varphi(t)-\varphi(\tau)| \leqslant C_{\varphi,\rho}|t-\tau|^\rho, \qquad t \in D \text{ and } \tau \in [0,\infty) \tag{17}$$

then for each $\tau \in [0,\infty)$, we have

$$\left|K_{n,k}^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right| \leqslant C_{\varphi,\gamma}\left\{\left(K_n^{(\alpha)}\left((t-\tau)^2;\tau\right)\right)^{\frac{\rho}{2}} + 2d^\rho(\tau,D)\right\}$$

where $C_{\varphi,\rho}$ is constant depending on φ and ρ and $d(\tau,D)$ is the distance between τ and D defined by

$$d(\tau,D)=\inf\{|t-\tau|:t\in D\}.$$

Proof. Let $\overline{D}$ be the closure of D. Using the properties of infimum, there is at least a point $t_0\in\overline{D}$ such that $d(\tau,D)=|\tau-t_0|$. By the triangle inequality

$$|\varphi(t)-\varphi(\tau)|\leqslant|\varphi(t)-\varphi(t_0)|+|\varphi(\tau)-\varphi(t_0)|.$$

Applying $K_n^{(\alpha)}$ to the above inequality and using ((17)), we can get

$$\begin{aligned}\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right| &\leqslant K_n^{(\alpha)}\left(|\varphi(t)-\varphi(t_0)|;\tau\right)+K_n^{(\alpha)}\left(|\varphi(\tau)-\varphi(t_0)|;\tau\right)\\ &\leqslant C_{\varphi,\gamma}\left\{K_n^{(\alpha)}\left(|t-t_0|^{\rho};\tau\right)+|\tau-t_0|^{\rho}\right\}\\ &\leqslant C_{\varphi,\gamma}\left\{K_n^{(\alpha)}\left(|t-\tau|^{\rho}+|\tau-t_0|^{\rho};\tau\right)+|\tau-t_0|^{\rho}\right\}\\ &= C_{\varphi,\gamma}\left\{K_n^{(\alpha)}\left(|t-\tau|^{\rho};\tau\right)+2|\tau-t_0|^{\rho}\right\}.\end{aligned}$$

Choosing $a_1=\frac{2}{\rho}$ and $a_2=\frac{2}{2-\rho}$ and applying Hölder inequality, we have:

$$\begin{aligned}\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right| &\leqslant C_{\varphi,\gamma}\left\{\left(K_n^{(\alpha)}\left(|t-\tau|^{a_1};\tau\right)\right)^{\frac{1}{a_1}}\left(K_n^{(\alpha)}\left(1^{a_2};\tau\right)\right)^{\frac{1}{a_2}}+2d^{\rho}(\tau,D)\right\}\\ &\leqslant C_{\varphi,\gamma}\left\{\left(K_n^{(\alpha)}\left((t-\tau)^2;\tau\right)\right)^{\frac{\rho}{2}}+2d^{\rho}(\tau,D)\right\}. \qquad\square\end{aligned}$$

In [14], Lipcshitz type maximal function of the order γ defined as

$$\phi_{\rho}(\varphi;\tau)=\sup_{\tau,t\in[0,\infty),\tau\neq t}\frac{|\varphi(t)-\varphi(\tau)|}{|t-\tau|^{\rho}} \tag{18}$$

where $\tau\in[0,\infty)$ and $\gamma\in(0,1]$. In the next theorem we obtain local direct estimate of the operators $K_n^{(\alpha)}$ by using (18).

THEOREM 13. *Let $\varphi\in C_B[0,\infty)$, $\alpha>0$ and $\rho\in(0,1]$. Then, for all $\tau\in[0,\infty)$, we have*

$$\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right|\leqslant\phi_{\rho}(\varphi;\tau)\left(K_n^{(\alpha)}\left((t-\tau)^2;\tau\right)\right)^{\frac{\rho}{2}}.$$

Proof. From the equation (18), we have

$$\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right|\leqslant\phi_{\rho}(\varphi;\tau)K_n^{(\alpha)}\left(|t-\tau|^{\rho};\tau\right)$$

Applying the Hölder inequality with $a_1=\frac{2}{\rho}$ and $a_2=\frac{2}{2-\rho}$, we get

$$\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right|\leqslant\phi_{\rho}(\varphi;\tau)\left(K_n^{(\alpha)}\left((t-\tau)^2;\tau\right)\right)^{\frac{\rho}{2}}. \qquad\square$$

THEOREM 14. *For* $\alpha>0$, $\varphi\in C_2[0,\infty)$, $w_{a+1}(\varphi;\delta)$ *is the modulus of continuity of* φ *on the interval* $[0,a+1]\subset[0,\infty)$, $a>0$. *Then, we have*

$$\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right|\leqslant 4N_\varphi\left(1+a^2\right)\delta_n(\tau)+2w_{a+1}\left(\psi;\sqrt{\delta_n(\tau)}\right).$$

where $\delta_n(\tau)=K_n^{(\alpha)}\left((t-\tau)^2;\tau\right)$ *given by Lemma 5.*

Proof. For $0\leqslant\tau\leqslant a$ and $a+1<t$, since $1<t-\tau$, we have

$$\begin{aligned}|\varphi(t)-\varphi(\tau)|&\leqslant M_\varphi\left(\tau^2+t^2+2\right)\\&\leqslant M_\varphi\left(2(t-\tau)^2+2+3\tau^2\right)\\&\leqslant M_\varphi(t-\tau)^2\left(4+3\tau^2\right)\\&\leqslant 4M_\varphi(t-\tau)^2\left(1+a^2\right)\end{aligned}\tag{19}$$

Also, for $0\leqslant\tau\leqslant a$ and $a+1\geqslant t$, we have

$$\begin{aligned}|\varphi(t)-\varphi(\tau)|&\leqslant w_{a+1}(\varphi;|t-\tau|)\\&\leqslant\left(1+\frac{|t-\tau|}{\delta}\right)w_{a+1}(\varphi;\delta)\end{aligned}\tag{20}$$

with $\delta>0$.

For $0\leqslant\tau\leqslant a$ and $t\geqslant 0$, combining (19) and (20) gives

$$|\varphi(t)-\varphi(\tau)|\leqslant 4M_\varphi(t-\tau)^2\left(1+a^2\right)+\left(1+\frac{|t-\tau|}{\delta}\right)w_{a+1}(\varphi;\delta),\tag{21}$$

From the above inequality (21) and Cauchy-Schwarz's inequality, we get

$$\begin{aligned}&\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right|\\&\leqslant K_n^{(\alpha)}(|\varphi(t)-\varphi(\tau)|;\tau)\\&\leqslant 4M_\varphi\left(1+a^2\right)K_n^{(\alpha)}\left((t-\tau)^2;\tau\right)+\left(1+\frac{\sqrt{K_n^{(\alpha)}((t-\tau)^2;\tau)}}{\delta}\right)w_{a+1}(\varphi;\delta)\\&\leqslant 4M_\varphi\left(1+a^2\right)K_n^{(\alpha)}\left((t-\tau)^2;\tau\right)+2w_{a+1}(\varphi;\delta_n(\tau))\end{aligned}$$

on choosing $\delta:=\delta_n(\tau)=\sqrt{K_n^{(\alpha)}((t-\tau)^2;\tau)}$. □

4. Weighted approximation

THEOREM 15. *Let* $\alpha>0$. *Then for each* $\varphi\in C_2^*[0,\infty)$, *we have:*

$$\lim_{n\to\infty}\left\|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right\|_2=0.$$

Proof. By the Korovkin type theorem on the weighted approximation([15]), we need to verify the following

$$\lim_{n\to\infty}\left\|K_n^{(\alpha)}(t^m;\tau)-\tau^m\right\|_2=0,\quad m=0,1,2. \tag{22}$$

- Since $K_n^{(\alpha)}(1;\tau)=1$, then inequlity (22) holds for $m=0$.
- From Lemma 4, we can write:

$$\begin{aligned}\lim_{n\to\infty}\left\|K_n^{(\alpha)}(t;\tau)-\tau\right\|_2 &= \sup_{\tau\geqslant 0}\frac{\left|K_n^{(\alpha)}(t;\tau)-\tau\right|}{1+\tau^2}\\ &= \sup_{\tau\geqslant 0}\frac{1}{1+\tau^2}\left|\frac{1}{(\alpha+1)n}\right|\\ &\leqslant \frac{1}{(\alpha+1)n}\sup_{\tau\geqslant 0}\frac{1}{1+\tau^2}\\ &\leqslant \frac{1}{(\alpha+1)n}\to 0,\ n\to\infty\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\left\|K_n^{(\alpha)}(t^2;\tau)-\tau^2\right\|_2 &= \sup_{\tau\geqslant 0}\frac{\left|K_n^{(\alpha)}(t^2;\tau)-\tau^2\right|}{1+\tau^2}\\ &= \sup_{\tau\geqslant 0}\frac{1}{1+\tau^2}\left|\frac{(\alpha+3)}{(\alpha+1)n}\tau+\frac{2}{(\alpha+1)(\alpha+2)n^2}\right|\\ &\leqslant \frac{(\alpha+3)}{(\alpha+1)n}\sup_{\tau\geqslant 0}\frac{\tau}{1+\tau^2}+\frac{2}{(\alpha+1)(\alpha+2)n^2}\sup_{\tau\geqslant 0}\frac{1}{1+\tau^2}\\ &\leqslant \frac{(\alpha+3)}{(\alpha+1)n}+\frac{2}{(\alpha+1)(\alpha+2)n^2}\to 0,\ n\to\infty.\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\left\|K_{n,k}^{(\alpha)}(t^m;\tau)-\tau^m\right\|_2=0,\quad m=0,1,2.\quad\square$$

In the next theorem, we present a weighted approximation theorem for $\varphi\in C_2^*[0,\infty)$. In [16], Doğru studied for the classical Szász operators,

THEOREM 16. *Let* $\alpha>0$. *For each* $\varphi\in C_2^*[0,\infty)$ *and* $\beta>0$, *we have*

$$\lim_{n\to\infty}\sup_{\tau\geqslant 0}\frac{\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right|}{(1+\tau^2)^{1+\beta}}=0.$$

Proof. Let $\tau_0 \in [0,\infty)$ be arbitrary but fixed. Then

$$\begin{aligned}
\sup_{\tau\in[0,\infty)} \frac{\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right|}{(1+\tau^2)^{1+\beta}} &= \sup_{\tau\in[0,\tau_0]} \frac{\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right|}{(1+\tau^2)^{1+\beta}} + \sup_{\tau\in(\tau_0,\infty)} \frac{\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right|}{(1+\tau^2)^{1+\beta}} \\
&\leqslant \left\|K_n^{(\alpha)}(\varphi)-\varphi\right\|_{C[0,\tau_0]} + \|\varphi\|_2 \sup_{\tau\in(\tau_0,\infty)} \frac{\left|K_{n,k}^{(\alpha)}\left(1+t^2;\tau\right)\right|}{(1+\tau^2)^{1+\beta}} \\
&\quad + \sup_{\tau\in(\tau_0,\infty)} \frac{|\varphi(\tau)|}{(1+\tau^2)^{1+\beta}} \\
&= H_1 + H_2 + H_3.
\end{aligned}$$

Since $|\varphi(\tau)| \leqslant N_\varphi(1+\tau^2)$, we have

$$H_3 = \sup_{\tau\in(\tau_0,\infty)} \frac{|\varphi(\tau)|}{\left(1+\tau^2\right)^{1+\beta}} \leqslant \sup_{\tau\in(\tau_0,\infty)} \frac{N_\varphi}{\left(1+\tau^2\right)^{\beta}} \leqslant \frac{N_\varphi}{\left(1+\tau_0^2\right)^{\beta}}.$$

By Theorem 15, we can get

$$\begin{aligned}
H_2 &= \|\varphi\|_2 \lim_{n\to\infty} \sup_{\tau\in(\tau_0,\infty)} \frac{\left|K_n^{(\alpha)}\left(1+t^2;\tau\right)\right|}{\left(1+\tau^2\right)^{1+\beta}} \\
&= \sup_{\tau\in(\tau_0,\infty)} \frac{\left(1+\tau^2\right)}{\left(1+\tau^2\right)^{1+\beta}} \|\varphi\|_2 \\
&= \sup_{\tau\in(\tau_0,\infty)} \frac{\|\varphi\|_2}{\left(1+\tau^2\right)^{\beta}} \leqslant \frac{\|\varphi\|_2}{\left(1+\tau_0^2\right)^{\beta}}.
\end{aligned}$$

Using Theorem 14, we can see that

$$H_1 \text{ goes to zero as } n \to \infty.$$

Moreover, if we choose $\tau_0 > 0$ large enough, we can see that

$$H_2 \to 0 \text{ and } H_3 \to 0 \text{ as } n \to \infty,$$

Thus, combining, H_1, H_2 and H_3, we get desired result. □

In the next theorem we obtain direct estimation in terms of weighted modulus of continuity. For every $\varphi \in C_m^*[0,\infty)$ the weighted modulus of continuity defined as follows

$$\Omega_m(\varphi,\delta) = \sup_{\tau\geqslant 0,\, 0<h\leqslant\delta} \frac{|\varphi(\tau+h)-\varphi(\tau)|}{1+(\tau+h)^m}, \tag{23}$$

LEMMA 17. [18] *If* $\varphi \in C_m^*[0,\infty)$, $m \in \mathbb{N}$, *then*
(i) $\Omega_m(\varphi,\delta)$ *is a monotone increasing function of* δ,
(ii) $\lim_{\delta\to 0^+} \Omega_m(\varphi,\delta) = 0$,
(iii) for any $\rho \in [0,\infty)$, $\Omega_m(\varphi,\rho\delta) \leqslant (1+\rho)\Omega_m(\varphi,\delta)$.

In the next theorem, we express the approximation error of $K_n^{(\alpha)}$ by using Ω_m.

THEOREM 18. *Let $m \in \mathbb{N}$. For $\varphi \in C_m^*[0,\infty)$, we have*

$$\left\| K_n^{(\alpha)}(\varphi) - \varphi \right\|_{m+1} \leqslant N\Omega_m(\varphi, (1/\sqrt{n})),$$

where N is a constant independent of φ and n.

Proof. From (23) and Lemma 17, we can write

$$\begin{aligned}|\varphi(t)-\varphi(\tau)| &\leqslant (1+(\tau+|t-\tau|)^m)\left(\frac{|t-\tau|}{\delta}+1\right)\Omega_m(\varphi,\delta)\\ &\leqslant (1+(2\tau+t)^m)\left(\frac{|t-\tau|}{\delta}+1\right)\Omega_m(\varphi,\delta).\end{aligned}$$

Then

$$\begin{aligned}&\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right|\\ &\leqslant K_n^{(\alpha)}\,|(\varphi(t)-\varphi(\tau)|\,;\tau)\\ &\leqslant \Omega_m(\varphi,\delta)\left(K_n^{(\alpha)}((1+(2\tau+t)^m);\tau)+K_n^{(\alpha)}\left((1+(2\tau+t)^m)\frac{|t-\tau|}{\delta};\tau\right)\right).\\ &=\Omega_m(\varphi,\delta)\left(K_n^{(\alpha)}\,(1+(2\tau+t)^m;\tau)+I_1\right).\end{aligned}$$

Applying Cauchy-Schwartz inequality to the I_1, we get

$$I_1 \leqslant (K_n^{(\alpha)}\,((1+(2\tau+t)^m)^2\,;\tau))^{1/2}\left(K_n^{(\alpha)}\left(\frac{|t-\tau|^2}{\delta^2};\tau\right)\right)^{1/2}.$$

Therefore,

$$\begin{aligned}&\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right| \\ &\leqslant \Omega_m(\varphi,\delta)\Bigg(K_n^{(\alpha)}((1+(2\tau+t)^m);\tau)+(K_n^{(\alpha)}\,((1+(2\tau+t)^m)^2\,;\tau))^{1/2}\\ &\quad\times\left(K_n^{(\alpha)}\left(\frac{|t-\tau|^2}{\delta^2};\tau\right)\right)^{1/2}\Bigg).\end{aligned} \tag{24}$$

By Lemma 8 and 7,

$$K_n^{(\alpha)}(1+(2\tau+t)^m;\tau) \leqslant C_m(\alpha)\,(1+\tau^m),$$

$$(K_n^{(\alpha)}\,((1+(2\tau+t)^m)^2\,;\tau))^{1/2} \leqslant C_m^1(\alpha)\,(1+\tau^m). \tag{25}$$

and

$$\left(K_n^{(\alpha)}\left(\frac{|t-\tau|^2}{\delta^2};\tau\right)\right)^{1/2} \leqslant \frac{1}{\delta}\sqrt{\frac{2}{(\alpha+1)(\alpha+2)n^2}+\frac{\tau}{n}} \leqslant \frac{(1+\tau)}{\delta\sqrt{n}}. \tag{26}$$

Combining (25), (25) and (26), we have

$$\begin{aligned}\left|K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right| &\leqslant \Omega_m(\varphi,\delta)\left(C_m(\alpha)(1+\tau^m)+C_m^1(\alpha)\frac{(1+\tau^m)(1+\tau)}{\delta\sqrt{n}}\right)\\ &= \Omega_m(\varphi,\delta)\left(C_m(\alpha)(1+\tau^m)+C_m^1(\alpha)C_1\frac{\left(1+\tau^{m+1}\right)}{\delta\sqrt{n}}\right),\end{aligned}$$

where

$$C_1 = \sup_{\tau\geqslant 0}\frac{\left(1+\tau^m+\tau+\tau^{m+1}\right)}{1+\tau^{m+1}}.$$

if we take $\delta = (1/\sqrt{n})$ in the above inequality, we obtain the desired result. □

Next result is a Voronovskaja type formula for the operators $K_n^{(\alpha)}(\varphi;\tau)$.

5. Voronovskaja type results

THEOREM 19. *Let* $\alpha > 0$. *For any* $\varphi \in C_2^*[0,\infty)$ *such that* φ', $\varphi'' \in C_2^*[0,\infty)$ *the following equality holds*

$$\lim_{n\to\infty} n\left[K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right] = \frac{1}{(\alpha+1)}\varphi'(\tau)+\frac{1}{2}\tau\varphi''(\tau).$$

Proof. By the Taylor's formula, we can write

$$\varphi(t) = \varphi(\tau)+\varphi'(\tau)(t-\tau)+\frac{1}{2}\varphi''(\tau)(t-\tau)^2+r(t,\tau)(t-\tau)^2 \tag{27}$$

where $r(t,\tau)$ is Peano form of remainder, $r(.,\tau) \in C_2^*[0,\infty)$ and $\lim_{t\to\tau} r(t,\tau) = 0$.

Applying $K_n^{(\alpha)}$ to the both sides of (27), we get

$$\begin{aligned}&n\left(K_n^{(\alpha)}(\varphi;\tau)-\varphi(\tau)\right)\\ &= \varphi'(\tau)nK_n^{(\alpha)}((t-\tau);\tau)+\frac{1}{2}\varphi''(\tau)nK_n^{(\alpha)}\left((t-\tau)^2;\tau\right)+nK_n^{(\alpha)}\left(r(t,\tau)(t-\tau)^2;\tau\right).\end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$K_n^{(\alpha)}\left(r(t,\tau)(t-\tau)^2;\tau\right) \leqslant \sqrt{K_n^{(\alpha)}\left(r^2(t,\tau);\tau\right)}\sqrt{K_n^{(\alpha)}\left((t-\tau)^4;\tau\right)}. \tag{28}$$

Observe that $r^2(t,\tau) = 0$ and $r^2(.,\tau) \in C_2^*[0,\infty)$.

Then, it follows from that Corollary 11,

$$\lim_{n\to\infty} nK_n^{(\alpha)}\left(r^2(t,\tau);\tau\right) = r^2(\tau,\tau) = 0. \tag{29}$$

Moreover, from (7), (28) and (29), we can obtain

$$\lim_{n\to\infty} K_n^{(\alpha)}\left(r(t,\tau)(t-\tau)^2;\tau\right) = 0 \tag{30}$$

Hence, combining (5), (6) and (30), we get

$$\lim_{n\to\infty} n\left[K_n^{(\alpha)}(\varphi;\tau) - \varphi(\tau)\right] = \frac{1}{(\alpha+1)}\varphi'(\tau) + \frac{1}{2}\tau\varphi''(\tau). \quad \square$$

6. Bivariate Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators

In this section,we introduce the bivariate Riemann-Liouville fractional integral type of $K_n^{(\alpha)}(\varphi;\tau)$ (3)as follows:

$$\begin{aligned} &K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(\varphi;\tau,\gamma) \\ &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \alpha_1\alpha_2 s_{n_1,k_1}(\tau)s_{n_2,k_2}(\gamma)\int_0^1\int_0^1 \varphi\left(\frac{k_1+t_1}{n_1},\frac{k_2+t_2}{n_2}\right)(1-t_1)^{\alpha_1-1}(1-t_2)^{\alpha_2-1}dt_1dt_2 \end{aligned}$$

where $(\tau,\gamma)\in I^2 = [0,\infty)\times[0,\infty)$ and $\alpha_1,\alpha_2>0$.

Bivariate Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators can be rewritten as

$$K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(\cdot;\tau,\gamma) = K_{n_1}^{\alpha_1}(\cdot;\tau)\times K_{n_2}^{\alpha_1}(\cdot;\gamma)$$

LEMMA 20. *Let* $e_{ij}(\tau,\gamma)=\tau^i\gamma^j$, $0\leqslant i+j\leqslant 2$ *and* $\alpha_1,\alpha_2>0$. *For* $(\tau,\gamma)\in I^2 = [0,\infty)\times[0,\infty)$, *we have*

$$\begin{aligned} K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(e_{00};\tau,\gamma) &= 1, \\ K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(e_{10};\tau,\gamma) &= \tau + \frac{1}{n_1(\alpha_1+1)}, \\ K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(e_{10};\tau,\gamma) &= \gamma + \frac{1}{n_2(\alpha_2+1)}, \\ K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(e_{20};\tau,\gamma) &= \frac{2}{(\alpha_1+1)(\alpha_1+2)n_1^2} + \frac{(\alpha_1+3)}{(\alpha_1+1)n_1}\tau + \tau^2, \\ K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(e_{02};\tau,\gamma) &= \frac{2}{(\alpha_2+1)(\alpha_2+2)n_2^2} + \frac{(\alpha_2+3)}{(\alpha_2+1)n_2}\tau + \tau^2. \end{aligned}$$

LEMMA 21. *According to above Lemma 20, we get*

$$\begin{aligned}
K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(e_{10}-\tau;\tau,\gamma) &= \frac{1}{(\alpha_1+1)(n_1+1)},\\
K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(e_{01}-\gamma;\tau,\gamma) &= \frac{1}{(\alpha_2+1)(n_2+1)},\\
K_{n_1,n_2}^{(\alpha_1,\alpha_2)}((e_{10}-\tau)^2;\tau,\gamma) &= \frac{2}{(\alpha_1+1)(\alpha_1+2)n_1^2}+\frac{\tau}{n_1}=\delta_{n_1}^{(\alpha_1)}(\tau),\\
K_{n_1,n_2}^{(\alpha_1,\alpha_2)}((e_{01}-\gamma)^2;\tau,\gamma) &= \frac{2}{(\alpha_2+1)(\alpha_2+2)n_2^2}+\frac{\gamma}{n_2}=\delta_{n_2}^{(\alpha_2)}(\gamma).
\end{aligned}$$

For bivariate real functions, modulus of continuity is given by

$$w(\varphi;\delta_n,\delta_m)=\sup\left\{|\varphi(t,s)-\varphi(\tau,\gamma)|:(t,s),(\tau,\gamma)\in I^2,|t-\tau|\leqslant\delta_n,|s-\gamma|\leqslant\delta_m\right\}.$$

THEOREM 22. *Let $\varphi\in C(I^2)$ and and $\alpha_1,\alpha_2>0$. Then for all $(\tau,\gamma)\in I^2$, the inequality*

$$\left|K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(\varphi;\tau,\gamma)-\varphi(\tau,\gamma)\right|\leqslant 4w\left(\varphi;\delta_{n_1}^{(\alpha_1)}(\tau),\delta_{n_2}^{(\alpha_2)}(\gamma)\right)$$

holds, where $\delta_{n_1}^{(\alpha_1)}(\tau),\delta_{n_2}^{(\alpha_2)}(\gamma)$ are as in Lemma 21.

Proof. By the positivity and linearity properties of the $K_{n_1,n_2}^{(\alpha_1,\alpha_2)}$, we can write

$$\begin{aligned}
\left|K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(\varphi;\tau,\gamma)-\varphi(\tau,\gamma)\right| &\leqslant K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(|\varphi(t,s)-\varphi(\tau,\gamma)|;\tau,\gamma)\\
&\leqslant w(\varphi;\delta_1,\delta_2)\left(K_{n_1}^{(\alpha_1)}(1;\tau)+\frac{1}{\delta_1}K_{n_2}^{(\alpha_2)}(|t-\tau|;\tau)\right)\\
&\quad\times\left(K_{n_2}^{(\alpha_2)}(1;\gamma)+\frac{1}{\delta_2}K_{n_2}^{(\alpha_2)}(|s-\gamma|;\gamma)\right)
\end{aligned}$$

Applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
K_{n_1}^{(\alpha_1)}(|t-\tau|;\tau) &\leqslant K_{n_1}^{(\alpha_1)}\left((t-\tau)^2;\tau\right)^{\frac{1}{2}}\\
K_{n_2}^{(\alpha_2)}(|s-\gamma|;\gamma) &\leqslant K_{n_2}^{(\alpha_2)}\left((s-\gamma)^2;\gamma\right)^{\frac{1}{2}}
\end{aligned}$$

Choosing $\delta_1=\delta_{n_1}^{(\alpha_1)}(\tau)$ and $\delta_2=\delta_{n_2}^{(\alpha_2)}(\gamma)$, we have desired result. □

7. Graphical simulations

In this section, we present some graphical simulations and numerical results for $K_n^{(\alpha)}$ and $K_{n_1,n_2}^{(\alpha_1,\alpha_2)}$ obtained by using Matlab. This simulations agree with our theoretical results.

EXAMPLE 23. Consider $f(x)=\begin{cases} -6x^3+9x^2-\left(\frac{66}{25}\right)x, & x\in[0,1] \\ \frac{9}{25}, & x>1 \end{cases}$. Here we take the value of $n\in\{50,100,150\}$ and $\alpha=10$ for $K_n^{(\alpha)}(f;x)$ with $x\in[0,1]$. The Figure 1 demonstrate the convergence of operators $K_n^{(\alpha)}$ to $f(x)$ for fixed α and increasing values of n . Then, some numerical values of $E_n^{\alpha}(f;x)=\left|K_n^{(\alpha)}(f;x)-f(x)\right|$ at certain points on the interval $[0,1]$ for $n\in\{50,100,150\}$ and $\alpha=10$ are given in Table 1.

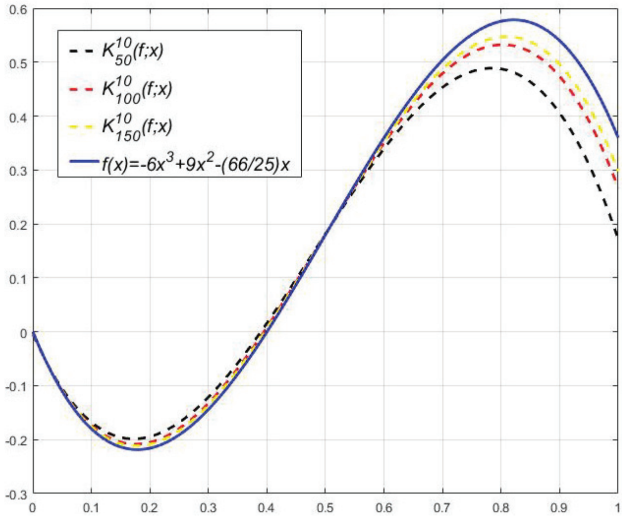

Figure 1: *Approximation to* f *by* $K_n^{(\alpha)}(f;x)$ *for* $\alpha=10$, $f(x)=-6x^3+9x^2-\frac{66}{25}x$ *and* $n=50,100,150$.

x	$\left\lvert K_{50}^{10}(f;x)-f(x)\right\rvert$	$\left\lvert K_{100}^{10}(f;x)-f(x)\right\rvert$	$\left\lvert K_{150}^{10}(f;x)-f(x)\right\rvert$
0	0.004745622	0.002386385	0.001593946
0.1	0.012283469	0.006207252	0.004152721
0.2	0.021458014	0.010873615	0.007281206
0.3	0.022778014	0.011612706	0.007791509
0.4	0.016243469	0.008424524	0.00568363
0.5	0.001854378	0.00130907	0.00095757
0.6	0.020389259	0.009733657	0.006386673
0.7	0.050487441	0.024703657	0.016349097
0.8	0.088440168	0.04360093	0.028929703
0.9	0.134247441	0.066425476	0.044128491
1	0.187909259	0.093177294	0.061945461

Table 1: *Estimation of the absolute error function* E_n^{10} *with* $f(x)=-6x^3+9x^2-\frac{66}{25}x$ *for some values of* x *in* $[0,1]$ *and* $n\in\{50,100,150\}$.

As we increase the value of n and fixed α, the approximation is good, i.e for the largest value of n and fixed α, the error is minumum.

EXAMPLE 24. Let $f(x)=\begin{cases}-6x^3+9x^2-\left(\frac{66}{25}\right)x, & x\in[0,0.5]\\ \frac{117}{100}, & x>0.5\end{cases}$. Here we take the value of $\alpha\in\{0.5,1,5\}$ and $n=50$ with $x\in[0.2,0.5]$. The Figure 2 demonstrate the convergence of operators $K_n^{(\alpha)}$ to $f(x)$ for increasing values of α and fixed n. Secondly, the absolute error function $E_n^{\alpha}(f;x)=\left|K_n^{(\alpha)}(f;x)-f(x)\right|$ is illustrated in figure 3. Finally, some numerical values of $E_n^{\alpha}(f;x)$ at certain points on the interval $[0.2,0.5]$ for $\alpha\in\{0.5,1,5\}$ and $n=50$ are given in Table 2.

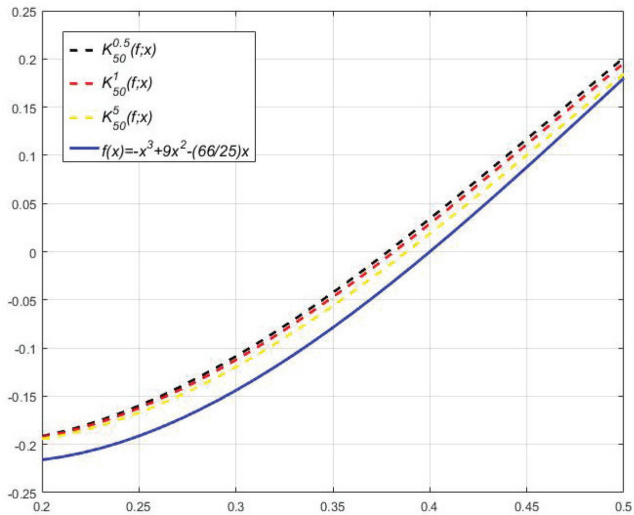

Figure 2: *Approximation to* f *by* $K_n^{(\alpha)}(f;x)$ *for* $n=50$, $f(x)=-6x^3+9x^2-\frac{66}{25}x$ *and* $\alpha=0.5,1,5$.

x	$\left\lvert K_{50}^{0.5}(f;x)-f(x)\right\rvert$	$\left\lvert K_{50}^{1}(f;x)-f(x\right\rvert$	$\left\lvert K_{50}^{5}(f;x)-f(x)\right\rvert$
0.2	0.024490057	0.023508	0.021782
0.3	0.035386057	0.031668	0.024387714
0.4	0.034282057	0.029028	0.018593429
0.5	0.021178057	0.015588	0.004399143

Table 2: *Estimation of the absolute error function* $E_{50}^{(\alpha)}$ *with* $f(x)=-6x^3+9x^2-\frac{66}{25}x$ *for some values of* x *in* $[0.2,0.5]$ *and* $\alpha\in\{0.5,1,5\}$.

Now, we are present some graphs and numerical results for the convergence of bivariate Riemann-Liouville integral type Szász-Mirakyan-Kantorovich operators $K_{n_1,n_2}^{(\alpha_1,\alpha_2)}$ by considering the function $f(x,y)=x+xy+12y^2$.

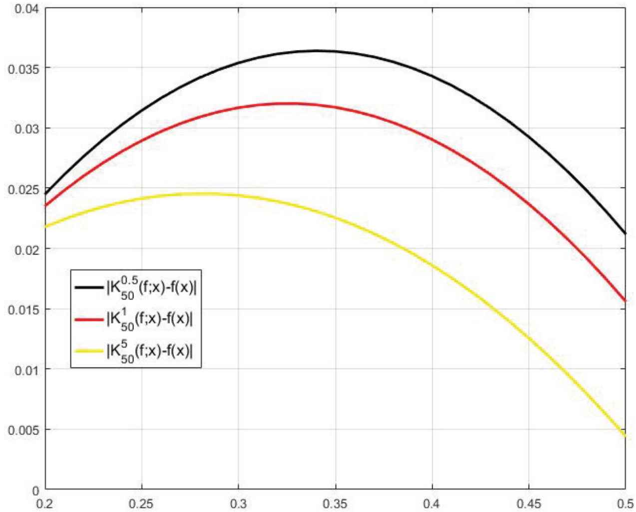

Figure 3: *Absolute error function* $E_n^{(\alpha)}(f;x)$ *for* $\alpha = 10$, $f(x) = -6x^3 + 9x^2 - \frac{66}{25}x$ *and* $n = 50$.

EXAMPLE 25. Consider, $f(x,y) = \begin{cases} x + xy + 12y^2, & (x,y) \in [0,1] \times [0,1], \\ 14, & (x,y) \in (1,\infty) \times (1,\infty) \end{cases}$. Here we take the value of $n_1, n_2 \in \{10, 20\}$ and $\alpha_1 = \alpha_2 = 10$ with $(x,y) \in [0,1] \times [0,1]$. The Figure 4 explains the convergence of the operators $K_{n_1,n_2}^{(\alpha_1,\alpha_2)}$ towards the

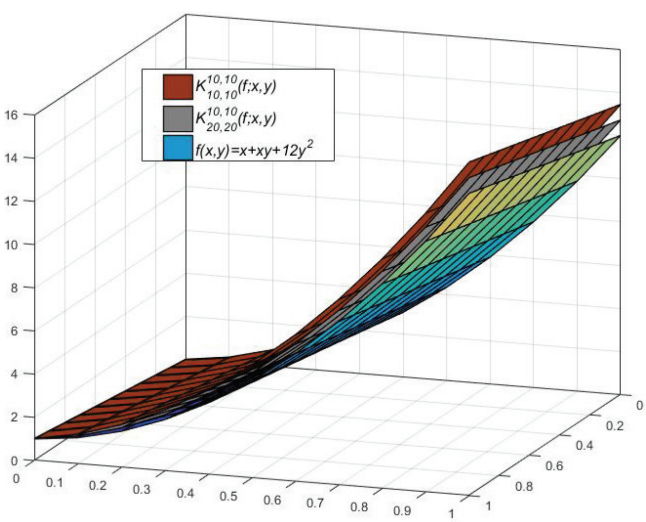

Figure 4: *Convergence of the operators* $K_{n_1,n_2}^{(10,10)}$ *to the function* $f(x,y) = x + xy + 12y^2$ *for the increasing values of* $n_1, n_2 \in \{10, 20\}$ *on the interval* $[0,1] \times [0,1]$.

function $f(x,y)$ for increasing values of n_1, n_2 and fixed α_1 and α_2 . Then, some numerical values of $E_n^{\alpha}(f;x) = \left|K_n^{(\alpha)}(f;x) - f(x)\right|$ at certain points on the interval $[0,1] \times [0,1]$ for $n_1, n_2 \in \{10,20\}$ and $\alpha_1 = \alpha_2 = 10$ are given in Table 3.

τ	y	$\left\|K_{10,10}^{10,10}(f;\tau) - f(\tau)\right\|$	$\left\|K_{20,20}^{10,10}(f;\tau) - f(\tau)\right\|$
1	0.1	0.162809917	0.080929752
1	0.2	0.30553719	0.152293388
1	0.3	0.448264463	0.223657025
1	0.4	0.590991736	0.295020661
1	0.5	0.733719008	0.366384298
1	0.6	0.876446281	0.437747934
1	0.7	1.019173554	0.50911157
1	0.8	1.161900826	0.580475207
1	0.9	1.304628099	0.651838843
1	1	1.447355372	0.723202479

Table 3: *Estimation of the absolute error function* $E_{n_1,n_2}^{(10,10)}$ *with* $f(x,y) = x + xy + 12y^2$ *for some values of* (x,y) *in* $[0,1] \times [0,1]$ *and* $n_1, n_2 \in \{10,20\}$.

As we increase the value of n_1 and n_2 and fixed α_1 and α_2, the approximation is good, i.e for the largest value of n_1 and n_2 and fixed α_1 and α_2, the error is minumum.

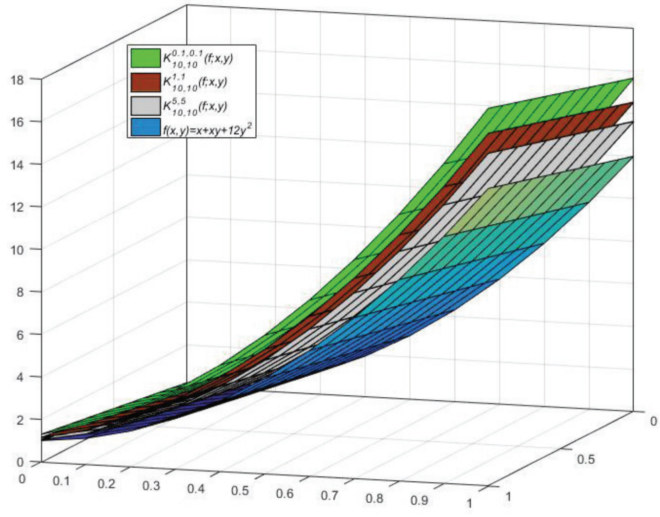

Figure 5: *Convergence of the operators* $K_{10,10}^{(\alpha_1,\alpha_2)}(f;x,y)$ *to the function* $f(x,y) = x + xy + 12y^2$ *for the increasing values of* $\alpha_1, \alpha_2 \in \{0.1, 1, 5\}$ *on the interval* $[0,1] \times [0,1]$.

EXAMPLE 26. Consider $f(x,y)=\begin{cases} x+xy+12y^2, & (x,y)\in[0,1]\times[0,1], \\ 14, & (x,y)\in(1,\infty)\times(1,\infty) \end{cases}$. Here we take the value of $\alpha_1,\alpha_1\in\{0.1,1,5\}$ and $n_1=n_2=10$ with $(x,y)\in[0,1]\times[0,1]$. The Figure 5 explains the convergence of the operators $K_{n_1,n_2}^{(\alpha_1,\alpha_2)}$ towards the function $f(x,y)$ for increasing values of $\alpha_1,\alpha_2\in\{0.1,1,5\}$ and fixed n_1 and n_2. Then, some numerical values of $E_{n_1,n_2}^{(\alpha_1\alpha_2)}(f;x,y)$ at certain points on the interval $[0,1]\times\ [0,1]$ for $n_1=n_2=10$ and $\alpha_1,\alpha_2\in\{0.1,1,5\}$ are given in Table 4.

±

x	y	$K_{10,10}^{0.1,0.1}(f;x,y)-f(x,y)$	$K_{10,10}^{1,1}(f;x,y)-f(x,y)$	$K_{10,10}^{5,5}(f;x,y)-f(x,y)$
1	0.1	0.641251476	0.3875	0.200992063
1	0.2	0.988524203	0.6325	0.36265873
1	0.3	1.33579693	0.8775	0.524325397
1	0.4	1.683069658	1.1225	0.685992063
1	0.5	2.030342385	1.3675	0.84765873
1	0.6	2.377615112	1.6125	1.009325397
1	0.7	2.724887839	1.8575	1.170992063
1	0.8	3.072160567	2.1025	1.33265873
1	0.9	3.419433294	2.3475	1.494325397
1	1	3.766706021	2.5925	1.655992063

Table 4: *Estimation of the absolute error function* $E_{10,10}^{(\alpha_1,\alpha_2)}$ *with* $f(x,y)=x+xy+12y^2$ *for some values of* (x,y) *in* $[0,1]\ \times[0,1]$ *and* $\alpha_1,\alpha_2\in\{0.1,1,5\}$.

As we increase the value of α_1 and α_2 and fixed n_1 and n_2, the approximation is good, i.e for the largest value of α_1 and α_2 and fixed n_1 and n_2, the error is minumum.

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Nazim Mahhmudov
Department of Mathematics
Eastern Mediterranean University
Gazimagusa 99628, T.R. Northen Cyprus, Mersin 10, Turkey

Mustafa Kara
Department of Mathematics
Eastern Mediterranean University
Gazimagusa 99628, T.R. Northen Cyprus, Mersin 10, Turkey