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SOME GENERALIZATIONS ON q–STEFFENSEN INEQUALITY

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Abstract. In this work, we study Steffensen inequality and obtain some generalizations on q-analogue of Steffensen inequality for infinite sums without restricted to the bounds. Since there are some differences between quantum and classical calculus, such as q-integral of a positive function from a to b ($a,b \in \mathbb{R}^+$, $0 < a < b$, $0 < q < 1$) does not have to be positive. The obtained results can not be exactly expressed as classical ones, nonetheless to say, they are q-extensions of the results in the classical ones.

1. Introduction

Quantum calculus also known as q-calculus can be described as a calculus without limits. Since it is a connection between mathematics and physics, many researchers interest in this type of calculus. Some of its applications in physics are in quantum field theory, the theory of relativity and mechanics. It also has many applications in mathematics such as basic hypergeometric series, number theory, combinatorics, time scales etc. Interested readers can find more information about quantum calculus in [7, 2, 3].

In this section, we give some definitions and facts on q-calculus in order to make this paper more understandable. Let us start with the q-derivative of a function f:

DEFINITION 1. [7] The q-derivative is defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$$

for an arbitrary function f. For a differentiable function f, $D_q(f) \to \frac{df(x)}{dx}$ as $q \to 1$.

Let f be a function defined on $[0,b]$. Then the q-integral of the function is defined as follows:

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DEFINITION 2. [7] Suppose $0 < a < b$ and $0 < q < 1$. The definite q-integral is defined as

$$\int_0^b f(x)d_qx = (1-q)b\sum_{j=0}^{\infty} q^j f(q^j b) \tag{1}$$

and

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{2}$$

Gauchman in [5] gives the definition of q-decreasing (or q-increasing) function and the special type of the definite q-integral as follows:

DEFINITION 3. [5] $f(x)$ is called q-increasing (respectively, q-decreasing) on $[a,b]$ if $f(qx) \leqslant f(x)$ (respectively, $f(qx) \geqslant f(x)$) whenever $x \in [a,b]$ and $qx \in [a,b]$. That is, $f(x)$ is q-increasing (respectively, q-decreasing) on $[a,b]$ if and only if $(D_q f(x) \geqslant 0)$ (respectively, $(D_q f(x) \leqslant 0)$) whenever $x \in [a,b]$ and $qx \in [a,b]$. Clearly, if f is increasing (decreasing), then it is also q-increasing (q-decreasing).

DEFINITION 4. [5] Let $0 < q < 1$, $b > 0$ and $n \in \mathbb{Z}^+$. The restricted q-integral is defined as $\int_{bq^n}^b f(x)d_qx$. In addition to $f(x)$, the restricted definite q-integral depends on a, b and n.

We clearly say that the q-restricted integral is convergent, since the series is finite sums.

One of the well-known classical integral inequality given by Steffensen [22] is as follows:

THEOREM 1. [22] *Suppose that the function f is decreasing and the function g is integrable on $[a,b]$ with $0 \leqslant g \leqslant 1$ and $\lambda = \int_a^b g(t)dt$. Then we have*

$$\int_{b-\lambda}^b f(t) \leqslant \int_a^b f(t)g(t)dt \leqslant \int_a^{a+\lambda} f(t)dt. \tag{3}$$

The inequalities are reversed if the function f is increasing.

Even nowadays many researchers interest in or inspire by the inequality and obtain this type of inequality on different subjects. For instances, some authors obtain the inequality for fractional integrals (see [19, 9, 20, 25]). Özkan and Yıldırım in [13] move the inequality to the time scales, Gauchman in [4], Jakšetić and Pečarić in [6] give the inequality on measure theory. Furthermore many researchers find generalizations, new proof or weaker conditions for the inequality (see [14, 12, 10, 26, 15]).

By the aid of Definition 3, Gauchman [5] gives the following q-analogue of Steffensen inequality for finite sum:

THEOREM 2. [5] *Suppose that* $0<q<1$, $b>0$, $n\in\mathbb{Z}^{+}$. *Let* $F,G:[a,b]\to\mathbb{R}$, *where* $a=bq^{n}$, *be two functions such that* F *is* q*-decreasing and* $0\leqslant G\leqslant 1$ *on* $[a,b]$. *Assume that* $k,l\in\{0,1,\ldots,n\}$ *are such that*

$$b-c_{l}\leqslant\int_{a}^{b}G(x)d_{q}x\leqslant c_{k}-a,\quad \text{if } F\geqslant 0,\quad \text{on } [a,b]$$

and

$$c_{k}-a\leqslant\int_{a}^{b}G(x)d_{q}x\leqslant b-c_{l},\quad \text{if } F\leqslant 0,\quad \text{on } [a,b].$$

Then

$$\int_{c_{l}}^{b}F(x)d_{q}x\leqslant\int_{a}^{b}F(x)G(x)d_{q}x\leqslant\int_{a}^{c_{k}}F(x)d_{q}x, \tag{4}$$

where $c_{j}=bq^{j}$ *for* $j\in\{0,1,\ldots,n\}$

Rajković et al. [18] firstly show the following results in order to obtain q-Steffensen inequality for infinite sums. In classical analysis, the integral of a positive function is also positive, but the q-integral of a positive function does not have to be positive. In the next lemma, authors give the condition for this fact:

LEMMA 1. [18] *If a function* $f(x)$ *is* q*-integrable, nonnegative and nondecreasing over* $[a,b]$, *then*

$$\int_{a}^{b}f(x)d_{q}x\geqslant 0;\quad (0\leqslant a\leqslant b;0<q<1).$$

LEMMA 2. [17, 18] *Let* $u(x)$ *be a continuous function on* $[a,b]$ *and* $v(x)$ *be a nonnegative and integrable function such that* $\int_{a}^{b}v(x)d_{q}x>0$ *for all* $q\in(0,1]$. *Then there exists* $\hat{q}\in(0,1)$ *such that for every* $q\in(\hat{q},1)$, *there exists* $\xi=\xi(q)\in(a,b)$ *so that*

$$\int_{a}^{b}u(x)v(x)d_{q}x=u(\xi)\int_{a}^{b}v(x)d_{q}x.$$

Thus Rajković et al. [18] point out that the inequality (3) can not be given with conditions in Theorem 1 for all $q\in(0,1)$ and they obtain q-Steffensen inequality as follows:

THEOREM 3. [18] *Let* $0<a<b$, $f(x)$ *and* $g(x)$ *are both continuous functions on* $[a,b]$, $f(x)$ *is decreasing and* $0<g(x)<1$ *on* $[a,b]$ *and* $\int_{a}^{d}g(x)d_{q}x>0$ *for every* $d\in(a,b)$. *If we denote* $\lambda=\int_{a}^{b}g(x)d_{q}x$, *then there is a* $\hat{q}\in(0,1)$ *such that*

$$\int_{b-\lambda}^{b}f(x)d_{q}x\leqslant\int_{a}^{b}f(x)g(x)d_{q}x\leqslant\int_{a}^{a+\lambda}f(x)d_{q}x \tag{5}$$

for all $q\in(\hat{q},1)$.

In the same paper, authors also improve Theorem 2 and find another version of q-Steffensen inequality on $(0,b)$.

THEOREM 4. [18] *Let* $0<q<1$, $b>0$, $f(x)$ *and* $g(x)$ *are both* q*-integrable functions on* $[0,b]$, $f(x)$ *is non-negative and decreasing and* $0\leqslant g(x)\leqslant 1$ *for each* $x\in[0,b]$ *and* $\lambda=\int_0^b g(x)d_qx$. *Let* $l,k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ *be such that*

$$l=\lfloor \log_q(1-\lambda/b)\rfloor,\quad k=\lfloor \log_q(\lambda/b)\rfloor.$$

Then

$$L_q(f;0,b)=\int_{bq^l}^b f(x)d_qx\leqslant\int_0^b f(x)g(x)d_qx\leqslant\int_0^{bq^k} f(x)d_qx=U_q(f;0,b). \tag{6}$$

Kalamir in [8] gives weaker conditions for the function g on $(0,b)$ as follows:

LEMMA 3. [8] *Let* $0<q<1$, $b>0$. *Let* f, g *be* q*-integrable functions on* $[0,b]$ *and* $\lambda=\int_0^b g(x)d_qx$. *Let* $l,k\in\mathbb{N}_0$ *be such that*

$$l=\lfloor \log_q(1-\lambda/b)\rfloor,\quad k=\lfloor \log_q(\lambda/b)\rfloor.$$

Then the following inequalities hold:

$$\begin{aligned}&\int_0^{bq^k} f(x)d_x-\int_0^b f(x)g(x)d_qx\\ &\geqslant\int_0^{bq^k}[f(x)-f(bq^k)][1-g(x)]d_qx+\int_{bq^k}^b[f(bq^k)-f(x)]g(x)d_qx\end{aligned} \tag{7}$$

and

$$\begin{aligned}&\int_0^b f(x)g(x)d_qx-\int_{bq^l} f(x)d_qx\\ &\geqslant\int_0^{bq^l}[f(x)-f(bq^l)]g(x)d_qx+\int_{bq^l}^b[f(bq^l]-f(x)][1-g(x)]d_qx.\end{aligned} \tag{8}$$

THEOREM 5. [8] *Let* $0<q<1$, $b>0$. *Let* f *and* g *be* q*-integrable functions on* $[0,b]$ *such that* f *is nonnegative and decreasing and* $\lambda=\int_0^b g(x)d_qx$. *Let* $k\in\mathbb{N}_0$ *be such that* $k=\lfloor \log_q(\lambda/b)\rfloor$. *If*

$$\int_0^{qx} g(t)d_qt\leqslant qx \text{ and } \int_{qx}^b g(t)d_qt\geqslant 0,\quad \textit{for every } x\in[0,b],$$

then

$$\int_0^b f(x)g(x)d_qx\leqslant\int_0^{bq^k} f(x)d_qx.$$

THEOREM 6. [8] *Let* $0 < q < 1$, $b > 0$. *Let* f *and* g *be* q*-integrable functions on* $[0,b]$ *such that* f *is nonnegative and decreasing and* $\lambda = \int_0^b g(x)d_qx$. *Let* $l \in \mathbb{N}_0$ *be such that* $l = \lfloor \log_q(1-\lambda/b) \rfloor$. *If*

$$\int_0^{qx} g(t)d_qt \geqslant 0 \text{ and } \int_{qx}^b g(t)d_qt \leqslant b - qx, \quad \text{for every } x \in [0,b],$$

then

$$\int_0^b f(x)g(x)d_qx \geqslant \int_{bq^l}^b f(x)d_qx.$$

In the same paper, author also gives the q-analogues of several generalizations of Steffen inequality, such as in the following theorem author obtains the q-analogue of the results given by Pečarić [14].

THEOREM 7. [8] *Let* $0 < q < 1$, $b > 0$. *Let* f, g *and* h *be* q*-integrable functions on* $[0,b]$ *such that* h *is positive,* f *is nonnegative,* f/h *is decreasing and* $0 \leqslant g(x) \leqslant 1$ *on* $[0,b]$. *Let* $k \in \mathbb{N}_0$ *be such that*

$$\int_0^{bq^k} h(x)d_qx \geqslant \int_0^b h(x)g(x)d_qx.$$

Then

$$\int_0^b f(x)g(x)d_qx \leqslant \int_0^{bq^k} f(x)d_qx.$$

Author in [8] points out that since $k, l \in \mathbb{N}_0$, the function f in Theorem 7 has to be positive. But the assumption for the function f in the generalization of classical Steffen inequality (see [14, Theorem 1]) does not have to satisfy such condition. Therefore, the author finds the following result by investigating the condition of the negativity of the function f in the next theorem.

THEOREM 8. [8] *Let* $0 < q < 1$, $b > 0$. *Let* f, g *and* h *be* q*-integrable functions on* $[0,b]$ *such that* h *is positive,* f *is negative,* f/h *is decreasing and* $0 \leqslant g(x) \leqslant 1$ *on* $[0,b]$. *Let* $k \in \mathbb{N}_0$ *be such that*

$$\int_0^{bq^k} h(x)d_qx = \int_0^b h(x)g(x)d_qx.$$

Then

$$\int_0^b f(x)g(x)d_qx \geqslant \int_0^{bq^k} f(x)d_qx.$$

More q-integral inequalities can be found in [1, 11, 16, 21, 23, 24] and references therein.

In this paper, our aim is to preserve the bounds of the integral on q-Steffensen inequality (3) and show which conditions should be added to the q-analogue's generalizations.

2. Generalizations of q-Steffensen inequality

As mentioned in Abstract and Introduction, the q-integral of a positive function f from a to b ($a,b \in \mathbb{R}^+$, $0<a<b$, $0<q<1$) does not have to be positive. Due to this fact, throughout the paper we call that the function f is positive q-integrable if the function f satisfies the conditions in Lemma 1, i.e. the function is q-integrable, continuous, positive and nondecreasing on $[a,b]$.

THEOREM 9. *Assume that the following conditions*

1. $0<a<b$,
2. h *is a positive* q*-integrable function on* $[a,b]$,
3. g *is a* q*-integrable continuous function such that* $\int_a^d g(x)d_qx > 0$ *for every* $d \in (a,b)$ *and* $0<g(x)<1$ *on* $[a,b]$

hold. If the function f *is a* q*-integrable continuous function such that* $\frac{f}{h}$ *is a decreasing function on* $[a,b]$, *then there exists a* $\hat{q} \in (0,1)$ *such that*

$$\int_a^{a+\lambda} f(x)d_qx \geqslant \int_a^b f(x)g(x)d_qx \tag{9}$$

for all $q \in (\hat{q},1)$, *where* λ *is the solution of the equation*

$$\int_a^{a+\lambda} h(x)d_qx = \int_a^b h(x)g(x)d_qx. \tag{10}$$

If $\frac{f}{h}$ *is an increasing function, then the inequality* (9) *is reversed.*

Proof. At first, we will show $0<\lambda<b-a$. By using Lemma 2 on the integral $\int_a^{a+\lambda} h(x)d_qx$, we can find a $q_1 \in (0,1)$ such that there exists $\xi_1 = \xi_1(q) \in (a,a+\lambda)$ so that

$$\int_a^{a+\lambda} h(x)d_qx = h(\xi_1)\int_a^{a+\lambda} d_qx = h(\xi_1)\lambda \tag{11}$$

for all $q \in (q_1,1)$ and also using the same lemma on the integral $\int_a^b h(x)g(x)d_qx$ yields that there exists a $q_2 \in (0,1)$ such that there is $\xi_2 = \xi_2(q) \in (a,b)$ so that

$$\int_a^b h(x)g(x)d_qx = g(\xi_2)\int_a^b h(x)d_qx \tag{12}$$

for all $q \in (q_2,1)$. Hence from the equations (10), (11) and (12), we get

$$h(\xi_1)\lambda = g(\xi_2)\int_a^b h(x)d_qx$$

for all $q \in (\bar{q}, 1)$, where $\bar{q} = \max\{q_1, q_2\}$. Since h is a positive q-integrable function and $0 < g(x) < 1$, we get $\lambda > 0$. Also using the equation (12) and $0 < g(x) < 1$ leads us to $\int_a^b g(x)h(x)d_qx < \int_a^b h(x)d_qx$. Then from the equation (10), we get

$$\int_a^{a+\lambda} h(x)d_qx < \int_a^b h(x)d_qx \Rightarrow \int_{a+\lambda}^b h(x)d_qx \geqslant 0 \tag{13}$$

for all $q \in (\bar{q}, 1)$. Again by using the Lemma 2 on the last integral, there exists a $q_3 \in (\bar{q}, 1)$ such that there can be found a $\xi_3 = \xi_3(q) \in (a+\lambda, b)$ such that $\int_{a+\lambda}^b h(x)d_qx = h(\xi_3)(b-a-\lambda)$. Since h is a nonnegative function, from the equation (13) we lastly obtain $b - a > \lambda$.

We just need to show that $I = \int_a^{a+\lambda} f(x)d_qx - \int_a^b f(x)g(x)d_qx \geqslant 0$. We have

$$\begin{aligned} I &= \int_a^{a+\lambda} f(x)d_qx - \int_a^{a+\lambda} f(x)g(x)d_qx - \int_{a+\lambda}^b f(x)g(x)d_qx \\ &= \int_a^{a+\lambda} (1-g(x))f(x)d_qx - \int_{a+\lambda}^b f(x)g(x)d_qx \end{aligned}$$

Since the function h is positive on $[a,b]$, we can rewrite that $\int_a^{a+\lambda} (1-g(x))f(x)d_qx = \int_a^{a+\lambda} [(1-g(x))h(x)]\frac{f(x)}{h(x)}d_qx$ and then letting $u = (1-g)h$ and $v = \frac{f}{h}$ in Lemma 2 yields that there exists a $\xi_4 \in (a, a+\lambda)$ so that

$$\int_a^{a+\lambda} (1-g(x))f(x)d_qx = \frac{f(\xi_4)}{h(\xi_4)} \int_a^{a+\lambda} (1-g(x))h(x)d_qx$$

for $q_4 \in (q_3, 1)$. By using Lemma 2 on the right side of the equation with $u = 1-g$ and $v = h$, one can show that $\int_a^{a+\lambda} (1-g(x))h(x)d_qx > 0$ for $q_5 \in (q_4, 1)$. By using the facts that $\frac{f}{h}$ also decreases on $[a, a+\lambda]$ and the equation (10), we obtain

$$\begin{aligned} I &> \frac{f(a+\lambda)}{h(a+\lambda)} \int_a^{a+\lambda} (1-g(x))h(x)d_qx - \int_{a+\lambda}^b f(x)g(x)d_qx \\ &= \frac{f(a+\lambda)}{h(a+\lambda)} \left[\int_a^b h(x)g(x)d_qx - \int_a^{a+\lambda} h(x)g(x)d_qx\right] - \int_{a+\lambda}^b f(x)g(x)d_qx \\ &= \int_{a+\lambda}^b g(x)h(x)\left[\frac{f(a+\lambda)}{h(a+\lambda)} - \frac{f(x)}{h(x)}\right] d_qx. \end{aligned}$$

Letting $u(x) = g(x)\left[\frac{f(a+\lambda)}{h(a+\lambda)} - \frac{f(x)}{h(x)}\right]$ and $v(x) = h(x)$ in Lemma 2 yields that there exists a $\hat{q} \in (q_5, 1)$ such that there exists $\xi_5 = \xi_5(q) \in (a+\lambda, b)$ so that

$$\int_{a+\lambda}^b g(x)h(x)\left[\frac{f(a+\lambda)}{h(a+\lambda)} - \frac{f(x)}{h(x)}\right] d_qx = g(\xi_5)\left[\frac{f(a+\lambda)}{h(a+\lambda)} - \frac{f(\xi_5)}{h(\xi_5)}\right] \int_{a+\lambda}^b h(x)d_qx.$$

The proof is completed by using the equation (13), $0 < g(x) < 1$ and monotonicity property of $\dfrac{f}{h}$.

For the reverse inequality, let us denote $I = \int_a^b f(x)g(x)d_qx - \int_a^{a+\lambda} f(x)d_qx$. Since $\frac{f}{h}$ is an increasing function, for $x \in (a, a+\lambda)$, by using similar technique above, one can obtain the following inequality

$$I \geqslant \int_{a+\lambda}^b f(x)g(x)d_qx - \frac{f(a+\lambda)}{h(a+\lambda)} \int_a^{a+\lambda} (1-g(x))h(x)d_qx$$
$$= \int_{a+\lambda}^b g(x)h(x)\left[\frac{f(x)}{h(x)} - \frac{f(a+\lambda)}{h(a+\lambda)}\right] d_qx.$$

Hence by letting $u(x) = g(x)\left[\dfrac{f(x)}{h(x)} - \dfrac{f(a+\lambda)}{h(a+\lambda)}\right]$ and $v(x) = h(x)$ in Lemma 2, we can find a $q' \in (\mathring{q}, 1)$ such that there exists $\varepsilon_5 = \varepsilon_5(q) \in (a+\lambda, b)$ for all $q \in (q', 1)$ so that

$$I = g(\varepsilon_5)\left[\frac{f(\varepsilon_5)}{h(\varepsilon_5)} - \frac{f(a+\lambda)}{h(a+\lambda)}\right] \int_{a+\lambda}^b h(x)d_qx.$$

Using the facts that $0 < g(x) < 1$ on $[a+\lambda, b]$, f/h increases on $[a+\lambda, b]$ and h is positive function on $[a+\lambda, b]$ leads us to the desired result. $\square$

REMARK 1. Letting $h(x) = 1$ leads us to the right hand side of the inequality (5) and taking the limit of the inequality (9) as $q \to 1^-$ tends to the reverse inequality in [14, Theorem 1].

THEOREM 10. *Assume that the following conditions*

1. $0 < a < b$,
2. h *is a positive* q*-integrable function on* $[a, b]$,
3. g *is a* q*-integrable continuous function such that* $\int_a^d g(t)d_qt > 0$ *for every* $d \in (a, b)$ *and* $0 < g(t) < 1$ *on* $[a, b]$

hold. If the function f *is a* q*-integrable continuous function such that* $\dfrac{f}{h}$ *is decreasing on* $[a, b]$, *then there exists a* $\hat{q} \in (0, 1)$ *such that*

$$\int_a^b f(x)g(x)d_qx \geqslant \int_{b-\lambda}^b f(x)d_qx \tag{14}$$

where λ *is given by*

$$\int_{b-\lambda}^b h(x)d_qx = \int_a^b h(x)g(x)d_qx. \tag{15}$$

If f/h *increases on* $[a, b]$, *then the inequality* (14) *reverses.*

Proof. From Lemma 2 and $0 < g(x) < 1$ on $[a,b]$, it is easy to see that $\int_a^d (1-g(x))d_qx > 0$ for $q \in (q_1, 1)$. Taking $1-g(x)$ and $b-a-\lambda$ instead of $g(x)$ and λ respectively in Theorem 9 yields

$$\int_a^{a+b-a-\lambda} f(x)d_qx \geqslant \int_a^b f(x)(1-g(x))d_qx$$
$$\int_a^{b-\lambda} f(x)d_qx \geqslant \int_a^b f(x)d_qx - \int_a^b f(x)g(x)d_qx$$
$$\int_a^b f(x)g(x)d_qx \geqslant \int_a^b f(x)d_qx + \int_{b-\lambda}^a f(x)d_qx$$
$$\int_a^b f(x)g(x)d_qx \geqslant \int_{b-\lambda}^b f(x)d_qx$$

and

$$\int_a^{a+b-a-\lambda} h(x)d_qx = \int_a^b h(x)(1-g(x))d_qx$$
$$\int_a^b h(x)g(x)d_qx = \int_{b-\lambda}^b h(x)d_qx$$

as desired. We want to note that this result can also be obtained by a similar technique in Theorem 9. □

REMARK 2. Taking $h(x) = 1$ leads to the left side of the inequality (5) and taking the limit of the inequality tends to the reverse inequality in [14, Theorem 2] as $q \to 1^-$.

COROLLARY 1. *Assume that the following conditions*

1. $0 < a < b$,
2. *h is a positive q-integrable function on $[a,b]$,*
3. *g is a q-integrable continuous function such that $\int_a^d \frac{g(x)}{h(x)}d_qx > 0$ for every $d \in (a,b)$ and $0 < g(x) < h(x)$ on $[a,b]$*

hold. If f is a decreasing function on $[a,b]$, then there exists a $\hat{q} \in (0,1)$ such that

$$\int_a^b f(x)g(x)d_qx \leqslant \int_a^{a+\lambda} f(x)h(x)d_qx \tag{16}$$

is valid for all $q \in (\hat{q}, 1)$, where λ is the solution of the equation

$$\int_a^{a+\lambda} h(x)d_qx = \int_a^b g(x)d_qx. \tag{17}$$

If f is an increasing function, then the reverse inequality in (16) is valid.

Proof. If we take $f(x)h(x)$ instead of $f(x)$ in Theorem 9, then $\frac{f(x)}{h(x)} \to f(x)$ is decreasing on $[a,b]$ which satisfy the fourth condition of the theorem. And also taking $\frac{g(x)}{h(x)}$ instead of $g(x)$ in Theorem 9 yields

$$\int_a^d \frac{g(x)}{h(x)} d_q x > 0 \quad \text{and} \quad 0 < \frac{g(x)}{h(x)} < 1$$

from the assumption. Hence by using Theorem 9, we obtain

$$\int_a^b f(x)g(x)d_q x \leqslant \int_a^{a+\lambda} f(x)h(x)d_q x$$

and

$$\int_a^{a+\lambda} h(x)d_q x = \int_a^b g(x)d_q x$$

as desired. □

REMARK 3. Taking $h(x) = 1$ in the Corollary 1 leads to the right side of the inequality (5). The inequality (16) is also a q-analogue of the result in [15].

COROLLARY 2. *Assume that the following conditions*

1. $0 < a < b$,
2. *h is a positive q-integrable function on $[a,b]$,*
3. *g is a q-integrable continuous function such that $\int_a^d \frac{g(x)}{h(x)} d_q x > 0$ for every $d \in (a,b)$ and $0 < g(x) < h(x)$ on $[a,b]$*

hold. If f is a decreasing function on $[a,b]$, then there exists a $\hat{q} \in (0,1)$ such that

$$\int_{b-\lambda}^b f(x)h(x)d_q x \leqslant \int_a^b f(x)g(x)d_q x \tag{18}$$

is valid for all $q \in (\hat{q},1)$, where λ is the solution of the equation

$$\int_{b-\lambda}^b h(x)d_q x = \int_a^b g(x)d_q x. \tag{19}$$

If f is an increasing function, then the reverse inequality in (18) is valid.

Proof. Taking $f(x) \to f(x)h(x)$ and $g(x) \to \frac{g(x)}{h(x)}$ in Theorem 10 satisfies the condition of the theorem. Hence we get the desired result. □

REMARK 4. Taking $h(x) = 1$ in Corollary 2 leads to the left side of the inequality (5). Taking the limit of the inequality (18) as $q \to 1^-$ tends to the inequality in [15].

THEOREM 11. *The following conditions*

1. $0 < a < b$,
2. *h and k are positive q-integrable functions on $[a,b]$,*
3. *g is a q-integrable continuous function such that $\int_a^d \frac{g(x)}{h(x)} d_qx > 0$ for every $d \in (a,b)$ and $0 < g(x) < h(x)$ on $[a,b]$*

hold. If $\frac{f}{k}$ is a decreasing function, then there exists a $\hat{q} \in (0,1)$ such that

$$\int_a^b f(x)g(x)d_qx \leqslant \int_a^{a+\lambda} f(x)h(x)d_qx \tag{20}$$

where λ is given by

$$\int_a^{a+\lambda} h(x)k(x)d_qx = \int_a^b g(x)k(x)d_qx$$

If $\frac{f}{k}$ is an increasing function, then the inequality (20) reverses.

Proof. Since the functions h and k are both positive q-integrable, from the assumption, the functions h and k are q-integrable, positive and nondecreasing. So we can easily see that the function $h(x)k(x)$ is also positive q-integrable. Thus by letting $h(x) \to h(x)k(x)$, $g(x) \to \frac{g(x)}{h(x)}$ and $f(x) \to f(x)h(x)$ in Theorem 9, we get

$$\int_a^b f(x)g(x)d_qx \leqslant \int_a^{a+\lambda} f(x)h(x)d_qx$$

for all $q \in (\hat{q},1)$ $(\hat{q} \in (0,1))$, where λ is given by

$$\int_a^{a+\lambda} h(x)k(x)d_qx = \int_a^b g(x)k(x)d_qx$$

as desired. $\square$

THEOREM 12. *The following conditions*

1. $0 < a < b$,
2. *h and k are positive q-integrable functions on $[a,b]$,*

3. g is a q-integrable continuous function such that $\int_a^d \frac{g(x)}{h(x)} d_q x > 0$ *for every* $d \in (a,b)$ *and* $0 < g(x) < h(x)$ *on* $[a,b]$

hold. If $\frac{f}{k}$ *is a decreasing function, then there exits a* $\hat{q} \in (0,1)$ *such that*

$$\int_a^b f(x)g(x) \geqslant \int_{b-\lambda}^b f(x)h(x)d_q x \tag{21}$$

where λ *is given by*

$$\int_{b-\lambda}^b h(x)k(x)d_q x = \int_a^b g(x)k(x)d_q x.$$

If $\frac{f}{k}$ *is an increasing function, then the inequality* (20) *reverses.*

Proof. The proof can be obtained by replacing

$$h(x) \to h(x)k(x), \quad g(x) \to \frac{g(x)}{h(x)} \quad \text{and} \quad f(x) \to f(x)h(x)$$

in Theorem 10. □

REMARK 5. By taking $k(x) = 1$ in Theorem 11 and Theorem 12, we can obtain Corollary 1 and Corollary 2, respectively. Also Theorem 11 and Theorem 12 are q-generalizations of the results obtained by Mercer [12, Theorem 3].

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