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CATER TYPE INEQUALITIES INVOLVING CATER PRODUCTS AND THEIR APPLICATIONS IN SPACE SCIENCE

JIA JIN WEN, TIAN YONG HAN AND JUN YUAN*

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Abstract. By means of the mathematical induction, stepwise adjustment method and the reorder method, under the proper hypotheses, we established the following Cater type inequalities involving Cater products:

$$X \otimes Y \geqslant KX \otimes Y \geqslant K^{+}X \otimes Y > e^{-1} \text{ and } f \otimes g \geqslant f(1-t) \otimes g > e^{-1}.$$

As applications, we solved the problem which proposed by M. Laub, Jerusalem and Israelin under the proper hypotheses, and an $\mathbf{l}$-isoperimetric inequality in the centered n-surround system $S^{(2)}\{P,\Gamma,\mathbf{l}\}$ is obtained as follows:

$$[\mu] \otimes [\mathbf{l}] \geqslant \left(\frac{|\Gamma|}{n}\right)^{\frac{2\pi}{n}}.$$

1. Introduction

In 1980, F. S. Cater established an interesting inequality involving the *Cater product* $K^0X \otimes X$ [1]. Since the Cater products $X \otimes Y$ and $f \otimes g$ are of extreme complexity and wide applicable so, it is of theoretical significance and application value that to establish Cater type inequalities involving Cater products.

The *surround system* has a wide application background in space science. In [32, 15, 34, 28, 26, 14, 33], the authors systematically established the surround system theory and some valuable *isoperimetric inequalities* [12] in space science are obtained.

In this paper, we will establish Cater type inequalities involving Cater products. As applications, we solved the problem which proposed by M. Laub, Jerusalem and Israelin [2] under the proper hypotheses, and an $\mathbf{l}$-*isoperimetric inequality* in the *centered n-surround system* $S^{(2)}\{P,\Gamma,\mathbf{l}\}$ is obtained.

The research methods of the paper are based on the *mathematical induction* [15, 13, 30, 22], *stepwise adjustment method* [29] and the *reorder method* [30, 22, 29]. The research tools of this paper include the theories of *functional analysis*, *discrete mathematics*, *optimization*, *convex geometry*, *inequality*, *mean value* and the *Mathematica* software, especially the mean value theory [32, 15, 34, 28, 26, 14, 33, 13, 30, 22, 29, 16, 17, 18, 24, 23, 10, 11].

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* Corresponding author.

2. Basic concepts and preliminary results

We will use the following hypotheses and notations throughout the paper:

$$\mathbb{R} \triangleq (-\infty, \infty),\ \mathbb{R}_+ \triangleq [0, \infty),\ \mathbb{R}_{++} \triangleq (0, \infty),\ X \triangleq (x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{R}^n,$$

$$Y \triangleq (y_1, \ldots, y_i, \ldots, y_n) \in \mathbb{R}^n,\ \overrightarrow{I}^n \triangleq \{X \in I^n : x_1 \leqslant \cdots \leqslant x_i \leqslant \cdots \leqslant x_n\},$$

$$\overleftarrow{I}^n \triangleq \{X \in I^n : x_1 \geqslant \cdots \geqslant x_i \geqslant \cdots \geqslant x_n\},\ \widehat{I^n} \triangleq \overrightarrow{I}^n \cup \overleftarrow{I}^n,$$

$C \triangleq \{\psi : [0,1] \to \mathbb{R} : \psi \text{ be continuous}\}$, $C_{++} \triangleq \{\psi : [0,1] \to \mathbb{R}_{++} : \psi \text{ be continuous}\}$, where $I \subseteq \mathbb{R}$ is an interval and $n \geqslant 2$.

DEFINITION 2.1. Let $K \triangleq [k_1 \cdots k_i \cdots k_n] \triangleq k_1 \cdots k_i \cdots k_n \in S_n$ be a *permutation* [30] of $1, 2, \ldots, n$. Then we define:

$$K^- \triangleq [k_1^- \cdots k_i^- \cdots k_n^-] :\ k_i^- = i,\ \forall i : 1 \leqslant i \leqslant n, \tag{1}$$

$$K^0 \triangleq [k_1^0 \cdots k_i^0 \cdots k_n^0] :\ k_i^0 = i+1,\ \forall i : 1 \leqslant i \leqslant n-1 \ \wedge\ k_n^0 = 1, \tag{2}$$

$$K^+ \triangleq [k_1^+ \cdots k_i^+ \cdots k_n^+] :\ k_i^+ = n+1-i,\ \forall i : 1 \leqslant i \leqslant n, \tag{3}$$

and

$$KX \triangleq \left(x_{k_1}, x_{k_2}, \ldots, x_{k_n}\right) \in \mathbb{R}^n, \tag{4}$$

where S_n is the symmetric group [30].

By Definition 2.1, we have

$$K^- X = X. \tag{5}$$

DEFINITION 2.2. [30,22] The points $X, Y \in \mathbb{R}^n$ are said to be *comonotone*, write as $X \backsim Y$, if

$$(x_i - x_j)(y_i - y_j) \geqslant 0,\ \forall i, j : 0 \leqslant i, j \leqslant n, \tag{6}$$

and X and Y are said to be *countermonotone*, write as $X \backsimeq Y$, if $-X \backsim Y$; The functions $f, g : [0,1] \to \mathbb{R}$ are said to be *comonotone*, write as $f \backsim g$, if

$$(f(x) - f(y))(g(x) - g(y)) \geqslant 0,\ \forall x, y \in [0,1], \tag{7}$$

and f and g are said to be *countermonotone*, write as $f \backsimeq g$, if $-f \backsim g$.

DEFINITION 2.3. We define the function

$$X \otimes Y : \mathbb{R}^n \times \mathbb{R}^n_{++} \to \mathbb{R}_{++},\ X \otimes Y \triangleq \frac{1}{n} \sum_{i=1}^{n} y_i^{x_i}, \tag{8}$$

as a *Cater product* of the points X and Y, and the functional

$$f \otimes g : C \times C_{++} \to \mathbb{R}_{++},\ f \otimes g \triangleq \int_0^1 [g(t)]^{f(t)}\, \mathrm{d}t, \tag{9}$$

as a *Cater product* of the functions f and g.

By Definition 2.3, we see that $X \otimes Y$ is the *mean value* [32, 15, 34, 28, 26, 14, 33, 13, 30, 22, 29, 16, 17, 18, 24, 23, 10, 11] of the positive real numbers $y_1^{x_1}, y_2^{x_2}, \dots, y_i^{x_i}, \dots, y_n^{x_n}$, and $f \otimes g$ is the mean value of the function $g^f \in C_{++}$.

In 1980, F. S. Cater established an interesting inequality involving the Cater product $K^0 X \otimes X$ as follows [1]:

$$K^0 X \otimes X > n^{-1}\left[1+(n-2)\min\left\{x_1^{x_2}, x_2^{x_3}, \cdots, x_{n-1}^{x_n}, x_n^{x_1}\right\}\right], \ \forall X \in \mathbb{R}_{++}^n, \tag{10}$$

which is called as the *Cater inequality*.

In 1985, M. Laub, Jerusalem and Israel proposed the following problem [2].

PROBLEM 2.1. Let $X \in \mathbb{R}_+^n$ and $K \in S_n$. Prove or disprove that

$$X \otimes X \geqslant K X \otimes X. \tag{11}$$

In 1990, Ishai Ilani proved that [2]

$$X \otimes X \geqslant K^0 X \otimes X. \tag{12}$$

In Section 3, we will establish the *discrete Cater type inequalities* which similar to the inequality (10). In Section 4, we will establish the *continuous Cater type inequalities*. In Section 5, we will study Problem 2.1. In Section 6, we will display the applications of our main results in space science.

3. Discrete Cater type inequalities

In this section, our main result is the following Theorem 3.1.

THEOREM 3.1. (Discrete Cater type inequalities) *Let $X \in \widehat{\mathbb{R}}^n$, $Y \in \widehat{\mathbb{R}}_{++}^n$ and $X \backsim Y$. If*

$$\min_{1 \leqslant i,j \leqslant n}\left\{y_j^{x_i}\right\} > e^{-1}, \tag{13}$$

then we have the following discrete Cater type inequalities:

$$X \otimes Y \geqslant K X \otimes Y \geqslant K^+ X \otimes Y > e^{-1}, \ \forall K \in S_n. \tag{14}$$

For any $K \in S_n$, the equalities in (14) hold if, and only if

$$x_1 = x_2 = \cdots = x_n \vee y_1 = y_2 = \cdots = y_n. \tag{15}$$

In order to prove Theorem 3.1, we need to establish the following Lemma 3.1.

LEMMA 3.1. *Under the hypotheses in Theorem 3.1, for any $i, j : 1 \leqslant i < j \leqslant n$ and any $k_i, k_j : 1 \leqslant k_i < k_j \leqslant n$, we have*

$$\chi\left(x_{k_i}, y_i\right) + \chi\left(x_{k_j}, y_j\right) \geqslant \chi\left(x_{k_j}, y_i\right) + \chi\left(x_{k_i}, y_j\right) \tag{16}$$

and, for any $i,j : 1 \leqslant i < j \leqslant n$ *and any* $k_i,k_j : 1 \leqslant k_i < k_j \leqslant n$,

$$\chi\left(x_{k_i},y_i\right)+\chi\left(x_{k_j},y_j\right)=\chi\left(x_{k_j},y_i\right)+\chi\left(x_{k_i},y_j\right), \tag{17}$$

if and and only if (15) *hold, where* $K \in S_n$ *and*

$$\chi : \mathbb{R}\times\mathbb{R}_{++}\to\mathbb{R}_{++},\ \chi(x,y)\triangleq x\otimes y=y^x. \tag{18}$$

Proof. Let $1 \leqslant i < j \leqslant n$ and $1 \leqslant k_i < k_j \leqslant n$. Now we prove the inequality (16). We define an auxiliary function as follows:

$$\varphi : \mathbb{R}_{++}\to\mathbb{R},\ \varphi(t)\triangleq\chi\left(x_{k_i},t\right)-\chi\left(x_{k_j},t\right). \tag{19}$$

Then the inequality (16) can be rewritten as:

$$\varphi(y_j)-\varphi(y_i)\leqslant 0,\ \forall y_i,y_j\in\mathbb{R}_{++}. \tag{20}$$

We first prove that

$$\left(x_{k_j}-x_{k_i}\right)(y_j-y_i)\geqslant 0. \tag{21}$$

Indeed, if $X,Y \in \widehat{\mathbb{R}}^n$, then, by Definition 2.2, we have

$$X \backsim Y \Leftrightarrow \left(X\in\overrightarrow{\mathbb{R}}^n\wedge Y\in\overrightarrow{\mathbb{R}}^n\right)\vee\left(X\in\overleftarrow{\mathbb{R}}^n\wedge Y\in\overleftarrow{\mathbb{R}}^n\right)\Leftrightarrow K^+X\simeq Y. \tag{22}$$

Since $Y \in \widehat{\mathbb{R}}^n_{++}$, we have

$$Y\in\overrightarrow{\mathbb{R}}^n_{++}\vee Y\in\overleftarrow{\mathbb{R}}^n_{++}. \tag{23}$$

Let $Y \in \overrightarrow{\mathbb{R}}^n_{++}$. Since $X \backsim Y$, by (22), we have $X \in \overrightarrow{\mathbb{R}}^n$. By $1 \leqslant i < j \leqslant n$ and $1 \leqslant k_i < k_j \leqslant n$, we get $y_j - y_i \geqslant 0$ and $x_{k_j} - x_{k_i} \geqslant 0$. Hence (21) holds. Similarly, if $Y \in \overleftarrow{\mathbb{R}}^n_{++}$, then $X \in \overleftarrow{\mathbb{R}}^n$, $y_j - y_i \leqslant 0$ and $x_{k_j} - x_{k_i} \leqslant 0$. Hence (21) also holds. Thus, the inequality (21) is proved.

For the convenience of narration, we might as well assume that

$$x_{k_j}\geqslant x_{k_i}\ \wedge\ y_j\geqslant y_i. \tag{24}$$

Since the function χ is continuous so, by the mathematical analysis theory, we have

$$\frac{\partial\chi(x,y)}{\partial x}\triangleq\chi_1(x,y)=y^x\log y, \tag{25}$$

$$\frac{\partial\chi(x,y)}{\partial y}\triangleq\chi_2(x,y)=xy^{x-1} \tag{26}$$

and

$$\frac{\partial^2\chi(x,y)}{\partial x\partial y}\triangleq\chi_{12}(x,y)=\chi_{21}(x,y)\triangleq\frac{\partial^2\chi(x,y)}{\partial y\partial x}=y^{x-1}\left(\log y^x+1\right). \tag{27}$$

According to (24)–(27) and the *Lagrange mean value theorem*, there exists a $\xi \in [y_i, y_j] \subseteq \mathbb{R}_{++}$ such that

$$\varphi(y_j) - \varphi(y_i) = (y_j - y_i)\left(\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}\right)_{t=\xi} = (y_j - y_i)\left(\chi_2(x_{k_i}, \xi) - \chi_2(x_{k_j}, \xi)\right), \tag{28}$$

and there exists an $\eta \in \left[x_{k_i}, x_{k_j}\right] \subseteq \mathbb{R}$ such that

$$\chi_2(x_{k_i}, \xi) - \chi_2(x_{k_j}, \xi) = \left(x_{k_i} - x_{k_j}\right)\chi_{21}(\eta, \xi). \tag{29}$$

By (13) and (27), we have

$$\begin{aligned}
\chi_{21}(\eta,\xi) &= \xi^{\eta-1}\left(\log \xi^{\eta} + 1\right) \\
&\geqslant \xi^{\eta-1}\left(\log \min\left\{\xi^{x_{k_i}}, \xi^{x_{k_j}}\right\} + 1\right) \\
&\geqslant \xi^{\eta-1}\left(\log \min_{1\leqslant m\leqslant n}\{\xi^{x_m}\} + 1\right) \\
&\geqslant \xi^{\eta-1}\left(\log \min_{1\leqslant m\leqslant n}\left\{\min\left\{y_i^{x_m}, y_j^{x_m}\right\}\right\} + 1\right) \\
&\geqslant \xi^{\eta-1}\left(\log \min_{1\leqslant m,j\leqslant n}\left\{y_j^{x_m}\right\} + 1\right) \\
&= \xi^{\eta-1}\left(\log \min_{1\leqslant i,j\leqslant n}\left\{y_j^{x_i}\right\} + 1\right) \\
&> \xi^{\eta-1}\left(\log e^{-1} + 1\right) \\
&= 0.
\end{aligned}$$

Hence

$$\chi_{21}(\eta,\xi) > 0. \tag{30}$$

Combining with (28)–(30) and (21), we get

$$\begin{aligned}
\varphi(y_j) - \varphi(y_i) &= (y_j - y_i)\left(\chi_2(x_{k_i}, \xi) - \chi_2(x_{k_j}, \xi)\right) \\
&= (y_j - y_i)\left(x_{k_i} - x_{k_j}\right)\chi_{21}(\eta,\xi) \\
&= -\left(x_{k_j} - x_{k_i}\right)(y_j - y_i)\,\chi_{21}(\eta,\xi) \\
&\leqslant 0 \\
&\Rightarrow (20) \\
&\Rightarrow (16).
\end{aligned}$$

Hence the inequality (16) is proved.

Based on the above proof, we can see that, for any $i, j : 1 \leqslant i < j \leqslant n$ and any $k_i, k_j : 1 \leqslant k_i < k_j \leqslant n$, the equalities (17) hold if and only if

$$\left(x_{k_j} - x_{k_i}\right)(y_j - y_i) = 0, \forall i, j : 1 \leqslant i < j \leqslant n \text{ and } \forall k_i, k_j : 1 \leqslant k_i < k_j \leqslant n. \tag{31}$$

Since $k_1 \cdots k_i \cdots k_n$ is an arbitrary permutation of $1,2,\ldots,n$, we see that (31) can be rewritten as (15). This ends the proof of Lemma 3.1. $\square$

Now we turn to the proof of Theorem 3.1.

Proof. We first prove that: There exists a permutation $k_{m+2}^{*}k_{m+3}^{*}\cdots k_n^{*}$ of $1,2,\ldots,$ $n-m-1$, such that

$$KX\otimes Y \geqslant \left[k_1^{+}k_2^{+}\cdots k_{m+1}^{+}k_{m+2}^{*}k_{m+3}^{*}\cdots k_n^{*}\right]X\otimes Y,\ \forall m: 0\leqslant m\leqslant n-1, \tag{32}$$

and there exists a permutation $k_{m+2}^{*}k_{m+3}^{*}\cdots k_n^{*}$ of $m+2,m+3,\ldots,n$, such that

$$KX\otimes Y \leqslant \left[k_1^{-}k_2^{-}\cdots k_{m+1}^{-}k_{m+2}^{*}k_{m+3}^{*}\cdots k_n^{*}\right]X\otimes Y,\ \forall m: 0\leqslant m\leqslant n-1. \tag{33}$$

We only prove the inequality (32) since the proof of the inequality (33) is similar.

Now we use the *mathematical induction* [15,13,30,22] for m and use the *stepwise adjustment method* [29] for K to prove the inequalities (32).

(A) Let $m=0$. If $k_1=k_1^{+}=n$, Then the inequality (32) is an equation. Assume that $k_1<k_1^{+}$. Then there exists an $r: 2\leqslant r\leqslant n$, such that $k_r=k_1^{+}$. Since $1\leqslant 2\leqslant r\leqslant$ n and $1\leqslant k_1<k_1^{+}=k_r\leqslant n$, by Lemma 3.1, we have

$$\chi\left(x_{k_1},y_1\right)+\chi\left(x_{k_r},y_r\right)\geqslant \chi\left(x_{k_r},y_1\right)+\chi\left(x_{k_1},y_r\right). \tag{34}$$

So, from the (34), we have

$$\begin{aligned}
KX\otimes Y &= \left[(k_1k_2\cdots k_r\cdots k_n)\right]X\otimes Y\\
&= \frac{1}{n}\sum_{i=1}^{n}\chi\left(x_{k_i},y_i\right)\\
&= \frac{1}{n}\left(\chi\left(x_{k_1},y_1\right)+\chi\left(x_{k_r},y_r\right)+\sum_{1\leqslant i\leqslant n,\ i\neq 1,r}\chi\left(x_{k_i},y_i\right)\right)\\
&\geqslant \frac{1}{n}\left(\chi\left(x_{k_r},y_1\right)+\chi\left(x_{k_1},y_r\right)+\sum_{1\leqslant i\leqslant n,\ i\neq 1,r}\chi\left(x_{k_i},y_i\right)\right)\\
&= \left[(k_rk_2\cdots k_1\cdots k_n)\right]X\otimes Y\\
&= \left[k_1^{+}k_2^{*}k_3^{*}\cdots k_n^{*}\right]X\otimes Y\\
&\Rightarrow (32),
\end{aligned}$$

where $k_2^{*}k_3^{*}\cdots k_n^{*}\triangleq k_2\cdots k_1\cdots k_n$ ia a permutation of $1,2,\ldots,n-1$. Thus, the inequalities (32) holds when $m=0$.

(B) Suppose that there exists a permutation $k_{m+2}^{*}k_{m+3}^{*}\cdots k_n^{*}$ of $1,2,\ldots,n-m-1$, such that (32) holds, where $0\leqslant m\leqslant n-2$. Now we prove that there exists a permutation $k_{m+3}^{**}k_{m+4}^{**}\cdots k_n^{**}$ of $1,2,\ldots,n-m-2$, such that

$$KX\otimes Y\geqslant \left[k_1^{+}k_2^{+}\cdots k_{m+2}^{+}k_{m+3}^{**}k_{m+4}^{**}\cdots k_n^{**}\right]X\otimes Y. \tag{35}$$

To this end, we prove that there exists a permutation $k_{m+3}^{**}k_{m+4}^{**}\cdots k_n^{**}$ of $1,2,\ldots,$ $n-m-2$, such that

$$\left[k_1^{+}k_2^{+}\cdots k_{m+1}^{+}k_{m+2}^{*}k_{m+3}^{*}\cdots k_n^{*}\right]X\otimes Y\geqslant \left[k_1^{+}k_2^{+}\cdots k_{m+2}^{+}k_{m+3}^{**}k_{m+4}^{**}\cdots k_n^{**}\right]X\otimes Y. \tag{36}$$

Indeed, if $k^*_{m+2}=k^+_{m+2}=n-m-1$, then (36) is an equation. Assume that $k^*_{m+2}<k^+_{m+2}$. Then, there exists an $r: m+3\leqslant r\leqslant n$ such that $k^*_r=k^+_{m+2}$. Since $1\leqslant m+3\leqslant r\leqslant n$ and $1\leqslant k^*_{m+2}<k^+_{m+2}=k^*_r\leqslant n$, by Lemma 3.1, we have

$$\chi\left(x_{k^*_{m+2}},y_{m+2}\right)+\chi\left(x_{k^*_r},y_r\right)\geqslant\chi\left(x_{k^*_r},y_{m+2}\right)+\chi\left(x_{k^*_{m+2}},y_r\right). \tag{37}$$

Let

$$k^*_i\triangleq k^+_i=n+1-i,\ i=1,2,\ldots,m+1. \tag{38}$$

Then, by (37) and (38), we have

$$\begin{aligned}
&\left[k^+_1k^+_2\cdots k^+_{m+1}k^*_{m+2}k^*_{m+3}\cdots k^*_n\right]X\otimes Y\\
=&\left[k^*_1k^*_2\cdots k^*_{m+1}k^*_{m+2}k^*_{m+3}\cdots k^*_r\cdots k^*_n\right]X\otimes Y\\
=&\frac{1}{n}\sum_{i=1}^{n}\chi\left(x_{k^*_i},y_i\right)\\
=&\frac{1}{n}\left(\chi\left(x_{k^*_{m+2}},y_{m+2}\right)+\chi\left(x_{k^*_r},y_r\right)+\sum_{1\leqslant i\leqslant n,\ i\neq m+2,r}\chi\left(x_{k^*_i},y_i\right)\right)\\
\geqslant&\frac{1}{n}\left(\chi\left(x_{k^*_r},y_{m+2}\right)+\chi\left(x_{k^*_{m+2}},y_r\right)+\sum_{1\leqslant i\leqslant n,\ i\neq m+2,r}\chi\left(x_{k^*_i},y_i\right)\right)\\
=&\left[k^*_1k^*_2\cdots k^*_{m+1}k^*_rk^*_{m+3}\cdots k^*_{m+2}\cdots k^*_n\right]X\otimes Y\\
=&\left[k^+_1k^+_2\cdots k^+_{m+2}k^*_{m+3}\cdots k^*_{m+2}\cdots k^*_n\right]X\otimes Y\\
=&\left[k^+_1k^+_2\cdots k^+_{m+2}k^{**}_{m+3}k^{**}_{m+4}\cdots k^{**}_n\right]X\otimes Y\\
\Rightarrow&\ (36),
\end{aligned}$$

where $k^{**}_{m+3}k^{**}_{m+4}\cdots k^{**}_n\triangleq k^*_{m+3}\cdots k^*_{m+2}\cdots k^*_n$ is a permutation of $1,2,\ldots,n-m-2$. Thus, inequality (36) is proved.

Combining with (32) and (36), we get

$$\begin{aligned}
KX\otimes Y&\geqslant\left[k^+_1k^+_2\cdots k^+_{m+1}k^*_{m+2}k^*_{m+3}\cdots k^*_n\right]X\otimes Y\\
&\geqslant\left[k^+_1k^+_2\cdots k^+_{m+2}k^{**}_{m+3}k^{**}_{m+4}\cdots k^{**}_n\right]X\otimes Y\\
&\Rightarrow(35).
\end{aligned}$$

Hence the inequality (35) is proved.

According to the principle of the mathematical induction, the inequalities (32) are proved.

Finally, we prove the inequalities (14). In (32) and (33), set $m=n-1$. Then, by Definition 2.3, (5) and (13), we get

$$\begin{aligned}
X\otimes Y&=K^-X\otimes Y=[k^-_1k^-_2\cdots k^-_n]X\otimes Y\\
&\geqslant KX\otimes Y\geqslant[k^+_1k^+_2\cdots k^+_n]X\otimes Y\\
&=K^+X\otimes Y>e^{-1}\Rightarrow(14).
\end{aligned}$$

Thus, the inequalities (14) are proved.

Based on the above proof and Lemma 3.1, for any $K \in S_n$, the equalities in (14) hold if and only if (15) hold. The proof of Theorem 3.1 is completed. □

REMARK 3.1. By (5) and Theorem 3.1, we see that the mean value of the positive real numbers $y_1^{x_{k_1}}, y_2^{x_{k_2}}, \ldots, y_i^{x_{k_i}}, \ldots, y_n^{x_{k_n}}$ reaches the maximum when $K = K^-$, while reaches the minimum when $K = K^+$, where $K^-X \backsimeq K^+X$. This is the significance of Theorem 3.1 in the *optimization theory*.

REMARK 3.2. An important hypothesis of Theorem 3.1 is $X \backsim Y$. Using this method to deal with the inequality problems is called as the *reorder method* [30,22,29].

REMARK 3.3. The equalities conditions of the inequalities (14) are similar to that of the *Chebyshev inequality* [30,29,3,21,20,9,4,35,25,5].

REMARK 3.4. We remark here that,

$$x_1 = x_2 = \cdots = x_n = -1 \ \wedge \ y_1 = y_2 = \cdots = y_n = e \Rightarrow K^+X \otimes Y = e^{-1}. \tag{39}$$

4. Continuous Cater type inequalities

In this section, our main result is the following Theorem 4.1.

THEOREM 4.1. (Continuous Cater type inequalities) *Let $f \in C$, $g \in C_{++}$ be monotone functions, and let $f \backsim g$. If*

$$\min_{(x,y)\in[0,1]^2} \left\{ [g(y)]^{f(x)} \right\} > e^{-1}, \tag{40}$$

then we have the following continuous Cater type inequalities:

$$f \times g \geqslant f(1-t) \times g > e^{-1}, \tag{41}$$

where $f \triangleq f(t) \ \wedge \ g \triangleq g(t)$.

Proof. According to the theory of functional analysis, for any continuous function $\phi : [0,1] \to \mathbb{R}$, we have

$$\int_0^1 \phi(t)\mathrm{d}t = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \phi\left(\frac{i-1}{n-1}\right). \tag{42}$$

Let $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^n_{++}$, and let

$$x_i = f\left(\frac{i-1}{n-1}\right) \ \wedge \ y_i = g\left(\frac{i-1}{n-1}\right), \ i = 1,2,\ldots,n. \tag{43}$$

By Definition 2.1 and (43), we have

$$K^{+}X=\left(x_{k_1^+},\ldots,x_{k_i^+},\ldots,x_{k_n^+}\right)\ \wedge x_{k_i^+}=x_{n+1-i}=f\left(1-\frac{i-1}{n-1}\right),\ i=1,2,\ldots,n. \quad (44)$$

By Definition 2.3 and (42)–(44), we have

$$\lim_{n\to\infty}X\otimes Y=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\chi\left(f\left(\frac{i-1}{n-1}\right),g\left(\frac{i-1}{n-1}\right)\right)=\int_0^1 g^f\,\mathrm{d}t \qquad (45)$$

and

$$\lim_{n\to\infty}K^{+}X\otimes Y=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\chi\left(f\left(1-\frac{i-1}{n-1}\right),g\left(\frac{i-1}{n-1}\right)\right)=\int_0^1 g^{f(1-t)}\mathrm{d}t. \qquad (46)$$

By (43) and the assumptions of Theorem 4.1, we have

$$X\in\widehat{\mathbb{R}}^n,\ Y\in\widehat{\mathbb{R}}^n_{++},\ X\backsim Y \text{ and } \min_{1\leqslant i,j\leqslant n}\left\{y_j^{x_i}\right\}>e^{-1}. \qquad (47)$$

By (47) and Theorem 3.1, we have

$$X\otimes Y\geqslant K^{+}X\otimes Y. \qquad (48)$$

Combining with (45), (46) and (48), we get

$$\begin{aligned} f\otimes g &= \int_0^1 g^f\,\mathrm{d}t=\lim_{n\to\infty}X\otimes Y\\ &\geqslant \lim_{n\to\infty}K^{+}X\otimes Y=\int_0^1 g^{f(1-t)}\mathrm{d}t\\ &= f(1-t)\otimes g>e^{-1}\\ &\Rightarrow (41). \end{aligned}$$

Hence the inequality (41) is proved. This completes the proof of Theorem 4.1. □

5. Research on Problem 2.1

In this section, our main result is the following Theorem 5.1.

THEOREM 5.1. (Discrete Cater type inequalities) *Let* $X\in\widehat{[e^{-1},1]}^n\cup\widehat{[1,\infty)}^n$. *Then we have the following discrete Cater type inequalities:*

$$X\otimes X\geqslant KX\otimes X\geqslant K^{+}X\otimes X\geqslant 2^{-1}\left(e^{-1}+1\right)=0.6839397205857212\cdots,\ \forall K\in S_n. \qquad (49)$$

In order to prove Theorem 5.1, we need to establish the following Lemma 5.1.

LEMMA 5.1. *If* $(x,y) \in [e^{-1},1]^2$, *then we have*

$$1 \geqslant (x,y) \otimes (y,x) \geqslant 2^{-1}\left(e^{-1}+1\right) = 0.6839397205857212\cdots \tag{50}$$

and, if $(x,y) \in [1,\infty)^2$, *then we have*

$$(x,y) \otimes (y,x) \geqslant 1. \tag{51}$$

Proof. We just need to prove the inequalities (50) since the inequality (51) is clear. Define the function

$$F : [e^{-1},1]^2 \to \mathbb{R}_{++},\ F(x,y) \triangleq (x,y) \otimes (y,x) = 2^{-1}\left(x^y + y^x\right). \tag{52}$$

The graph of the function $F(x,y)$ is depicted in Figure 1.

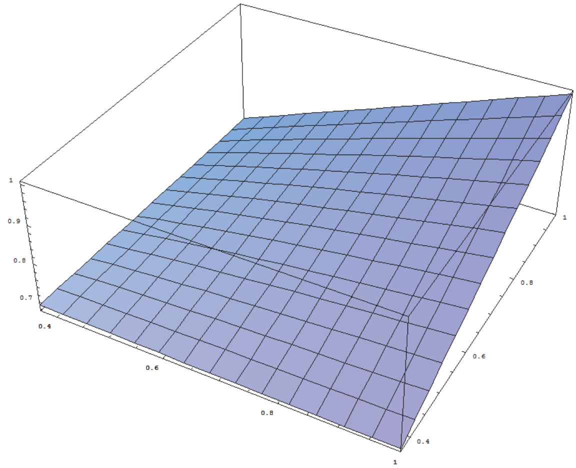

Figure 1: *The graph of the function* $F(x,y)$.

We first prove that: In $(e^{-1},1)^2$, the function F has no any stagnation points. Indeed, assume that there exists an $(x,y) \in (e^{-1},1)^2$ such that

$$\frac{\partial F}{\partial x} = 2^{-1}\left(yx^{y-1} + y^x \log y\right) = 0 \text{ and } \frac{\partial F}{\partial y} = 2^{-1}\left(x^y \log x + xy^{x-1}\right) = 0. \tag{53}$$

By (53), we have

$$\log x \log y = \left(-x^{1-y}y^{x-1}\right)\left(-y^{1-x}x^{y-1}\right) = 1. \tag{54}$$

Since $(x,y) \in (e^{-1},1)^2$, we have $0 < -\log x < 1$ and $0 < -\log y < 1$. Hence

$$0 < \log x \log y = (-\log x)(-\log y) < 1. \tag{55}$$

The equality (54) and the inequalities (55) are contradictory. This proves our assertion.

Let (x,y) be the boundary points of the closed region $[e^{-1},1]^2$.

Case 1: $x=e^{-1},\ y\in[e^{-1},1]$. Now we prove that the function

$$F(e^{-1},y)=2^{-1}\left(e^{-y}+y^{e^{-1}}\right)$$

in $(e^{-1},1)$ has no any stagnation points.

Indeed, assume that there exists an $y\in(e^{-1},1)$ such that

$$\frac{\partial F(e^{-1},y)}{\partial y}=2^{-1}\left(-e^{-y}+e^{-1}y^{e^{-1}-1}\right)=0\Leftrightarrow\psi(y)\triangleq(e^{-1}-1)\log y+y-1=0. \tag{56}$$

Since

$$\psi(e^{-1})=\psi(y)=\psi(1)=0, \tag{57}$$

according to the *Rolle theorem*, there exist $\xi_1,\xi_2,\xi:\xi_1\in(e^{-1},y),\ \xi_2\in(y,1),\ \xi\in(\xi_1,\xi_2)\subset(e^{-1},1)$, such that

$$\psi'(\xi_1)=\psi'(\xi_2)=0 \text{ and } \psi''(\xi)=(1-e^{-1})\xi^{-2}=0. \tag{58}$$

The equlities (58) and the inequality $(1-e^{-1})\xi^{-2}>0$ are contradictory. This proves our assertion.

Case 1.1: $y=e^{-1}$. Then

$$1>F(x,y)=e^{-e^{-1}}=0.6922006275553464\cdots>2^{-1}\left(e^{-1}+1\right).$$

Case 1.2: $y=1$. Then

$$1>F(x,y)=2^{-1}\left(e^{-1}+1\right).$$

Case 2: $y=e^{-1},\ x\in[e^{-1},1]$. By the proof of the Case 1, we have

$$1>F(x,y)=F(y,x)\geqslant 2^{-1}\left(e^{-1}+1\right).$$

Case 3: $x=1,\ y\in[e^{-1},1]$. Then

$$1\geqslant F(x,y)=2^{-1}(1+y)\geqslant 2^{-1}\left(e^{-1}+1\right).$$

Case 4: $y=1,\ x\in[e^{-1},1]$. Then

$$1\geqslant F(x,y)=2^{-1}(x+1)\geqslant 2^{-1}\left(e^{-1}+1\right).$$

According to the theory of mathematical analysis, the inequalities (50) are proved. This proves Lemma 5.1. □

Now we turn to the proof of Theorem 5.1.

Proof. Set $Y=X$ and let $X\in\widehat{[e^{-1},1)}^n\cup\widehat{[1,\infty)}^n$. By Definition 2.2, we have $X,Y\in\widehat{\mathbb{R}}^n_{++}$ and $X\backsim Y$. If $X\in\widehat{[e^{-1},1)}^n$, then

$$\min_{1\leqslant i,j\leqslant n}\left\{y_j^{x_i}\right\}=\min_{1\leqslant i,j\leqslant n}\left\{x_j^{x_i}\right\}\geqslant\min_{1\leqslant i\leqslant n}\left\{(e^{-1})^{x_i}\right\}>(e^{-1})^1=e^{-1},$$

that is, (13) holds. If $X \in \widehat{[1,\infty)}^n$, then

$$\min_{1\leqslant i,j\leqslant n}\left\{y_j^{x_i}\right\} = \min_{1\leqslant i,j\leqslant n}\left\{x_j^{x_i}\right\} \geqslant \min_{1\leqslant i\leqslant n}\left\{1^{x_i}\right\} = 1 > e^{-1},$$

hence (13) also holds.

According to Theorem 3.1 and $Y = X$, for any permutation K of $1,2,\ldots,n$, we have

$$X\otimes X \geqslant KX\otimes X \geqslant K^{+}X\otimes X. \tag{59}$$

Now we prove that

$$K^{+}X\otimes X \geqslant 2^{-1}\left(e^{-1}+1\right). \tag{60}$$

Indeed, if $X \in \widehat{[e^{-1},1)}^n$, then, by Definition 2.3 and Lemma 5.1, we have

$$\begin{aligned}
K^{+}X\otimes X &= \frac{1}{n}\sum_{i=1}^{n}\chi\left(x_{k_i^+},x_i\right)\\
&= \frac{1}{n}\sum_{i=1}^{n}\chi\left(x_{n+1-i},x_i\right)\\
&\equiv \frac{1}{n}\sum_{i=1}^{n}\chi\left(x_i,x_{n+1-i}\right)\\
&= \frac{1}{2}\left(\frac{1}{n}\sum_{i=1}^{n}\chi\left(x_{n+1-i},x_i\right)+\frac{1}{n}\sum_{i=1}^{n}\chi\left(x_i,x_{n+1-i}\right)\right)\\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}\left(\chi\left(x_{n+1-i},x_i\right)+\chi\left(x_i,x_{n+1-i}\right)\right)\\
&= \frac{1}{n}\sum_{i=1}^{n}\left(x_{n+1-i},x_i\right)\otimes\left(x_i,x_{n+1-i}\right)\\
&\geqslant \frac{1}{n}\sum_{i=1}^{n}2^{-1}\left(e^{-1}+1\right)\\
&= 2^{-1}\left(e^{-1}+1\right)\\
&\Rightarrow (60).
\end{aligned}$$

Similarly, if $X \in \widehat{[1,\infty)}^n$, then

$$K^{+}X\otimes X = \frac{1}{n}\sum_{i=1}^{n}\left(x_{n+1-i},x_i\right)\otimes\left(x_i,x_{n+1-i}\right) \geqslant \frac{1}{n}\sum_{i=1}^{n}1 = 1 > 2^{-1}\left(e^{-1}+1\right) \Rightarrow (60).$$

Thus, the inequality (60) is proved.

Combining with (59) and (60), we get the inequalities (49). That is, the inequalities (49) hold when $X \in \widehat{[e^{-1},1)}^n \cup \widehat{[1,\infty)}^n$. Since the functions $X\otimes X$, $KX\otimes X$ and $K^{+}X\otimes X$ are continuous so, the inequalities (49) also hold when $X \in \widehat{[e^{-1},1]}^n \cup \widehat{[1,\infty)}^n$. The proof of Theorem 5.1 is completed. □

REMARK 5.1. By Theorem 5.1, we see that, under the hypothesis $X \in \widehat{[e^{-1},1]}^n \cup \widehat{[1,\infty)}^n$, we solved Problem 2.1. In other word, if $X \in \widehat{[e^{-1},1]}^n \cup \widehat{[1,\infty)}^n$, then the inequality (11) holds.

REMARK 5.2. If $X \in \widehat{[0,1]}^n$, then the inequalities (59) do not hold. Indeed, when $n = 3$ and $0 = x_1 < x_2 < x_3 < 1$, we have

$$K^0X \otimes X = x_1^{x_2} + x_2^{x_3} + x_3^{x_1} = x_2^{x_3} + 1 < x_2^{x_2} + 1 = x_1^{x_3} + x_2^{x_2} + x_3^{x_1} = K^+X \otimes X.$$

REMARK 5.3. If $n = 2m$ and $X = \left(e^{-1}\mathbf{1}^m, \mathbf{1}^m\right)$, where $\mathbf{1}^m \triangleq (1,1,\ldots,1) \in \mathbb{R}^m$ and m is a positive integer, then the equality in (60) holds.

REMARK 5.4. Figure 1 and the relevant calculations of this paper are based on the *Mathematica* software. The references on using mathematical software to deal with inequality problems can be see [32, 15, 28, 13, 31].

6. Applications in space science

Let $S^{(2)}\{P,\Gamma,l\}$ be a centered 2-*surround system* [32, 15, 34, 28, 26, 14, 33] and A, A_+ be two *satellites* of the surround system, and let the curve $\widetilde{AA_+} \subset \Gamma$. Then

$$0 < \left|\widetilde{AA_+}\right| = l < \frac{|\Gamma|}{2} \tag{61}$$

and

$$\mu \triangleq \angle APA_+ \in (0,\pi),\ \forall A, A_+ \in \Gamma, \tag{62}$$

where $|\Gamma| \triangleq \oint_\Gamma \mathrm{d}s$ is the length of the smooth and convex *Jordan closed curve* [8] Γ, l is a constant and $\left|\widetilde{AA_+}\right|$ is the length of the curve $\widetilde{AA_+}$. We say that the μ is l-*observation angle* of the surround system $S^{(2)}\{P,\Gamma,l\}$, and the function

$$\mu \otimes l : (0,\pi) \times \left(0, \frac{|\Gamma|}{2}\right),\ \mu \otimes l \triangleq l^\mu, \tag{63}$$

is the l-*feature function* of the surround system.

According to the above definition, we know that the l-observation angle μ can be represented by the l-feature function $\mu \otimes l$ as follows:

$$\mu = \log_l(\mu \otimes l). \tag{64}$$

In [15] (see Lemma 2.12 in [15]), the authors proved that

$$\overline{\mu} \triangleq \frac{1}{|\Gamma|}\oint_\Gamma \mu\,\mathrm{d}s = \frac{1}{|\Gamma|}\oint_\Gamma \angle APA_+\mathrm{d}s = \frac{2l\pi}{|\Gamma|}, \tag{65}$$

where $\overline{\mu}$ is the *mean value* [32,15,34,28,26,14,33,13,30,22,29,16,17,18,24,23,10,11] of the l-observation angle μ.

Let $S^{(2)}\{P,\Gamma,\mathbf{l}\}$ be a centered n-*surround system* [32] and $A_1,A_2,\dots,A_n$ be n *satellites* of the surround system. Then

$$\mathbf{l}=(l_1,l_2,\dots,l_n),\ 0<\left|\widetilde{A_jA_{j+1}}\right|=l_j<\frac{|\Gamma|}{2},\ j=1,2,\dots,n,\ \sum_{j=1}^{n}l_j=|\Gamma|,\ A_{n+1}\triangleq A_1. \tag{66}$$

Set

$$\mu_j\triangleq\angle A_jPA_{j+1}\in(0,\pi),\ j=1,2,\dots,n,\ A_{n+1}\triangleq A_1,\ \mu\triangleq(\mu_1,\mu_2,\dots,\mu_n). \tag{67}$$

Then

$$\sum_{j=1}^{n}\mu_j=2\pi. \tag{68}$$

Let

$$[\mathbf{l}]=([l_1],[l_2],\dots,[l_n])\ \wedge\ [\mu]\triangleq([\mu_1],[\mu_2],\dots,[\mu_n]), \tag{69}$$

where $[l_1][l_2]\cdots[l_n]$ is a permutation of $l_1,l_2,\dots,l_n$, and $[\mu_1][\mu_2]\cdots[\mu_n]$ is a permutation of $\mu_1,\mu_2,\dots,\mu_n$, which such that

$$[l_1]\leqslant[l_2]\leqslant\cdots\leqslant[l_n]\ \wedge\ [\mu_1]\leqslant[\mu_2]\leqslant\cdots\leqslant[\mu_n]. \tag{70}$$

Then, by Definition 2.2, we have

$$[\mu]\backsim[\mathbf{l}]. \tag{71}$$

We say that the function

$$[\mu]\otimes[\mathbf{l}]:(0,\pi)^n\times\left(0,\frac{|\Gamma|}{2}\right)^n\to\mathbb{R}_{++} \tag{72}$$

is the $\mathbf{l}$-*feature function* of the centered n-surround system $S^{(2)}\{P,\Gamma,\mathbf{l}\}$.

In this section, our main results are the following Theorems 6.1 and 6.2.

THEOREM 6.1. ($\mathbf{l}$-isoperimetric inequality) *Let $S^{(2)}\{P,\Gamma,\mathbf{l}\}$ be a centered n-surround system, where $3\leqslant n\leqslant 6$. If*

$$[l_1]\geqslant e^{-\frac{1}{\pi}}=0.7273773492952165\cdots, \tag{73}$$

then we have the following $\mathbf{l}$-isoperimetric inequality [12]:

$$[\mu]\otimes[\mathbf{l}]\geqslant\left(\frac{|\Gamma|}{n}\right)^{\frac{2\pi}{n}}. \tag{74}$$

Proof. By (67) and (73), we have

$$\min_{1\leqslant i,j\leqslant n}\left\{[l_j]^{[\mu_i]}\right\}=\min_{1\leqslant i\leqslant n}\left\{[l_1]^{[\mu_i]}\right\}\geqslant\min_{1\leqslant i\leqslant n}\left\{\left(e^{-\frac{1}{\pi}}\right)^{[\mu_i]}\right\}>\left(e^{-\frac{1}{\pi}}\right)^{\pi}=e^{-1}. \tag{75}$$

By (71), (75) and Theorem 3.1, we have

$$[\mu]\otimes[\mathbf{l}] \geqslant [\mu]\otimes K[\mathbf{l}],\ \forall K\in S_n. \tag{76}$$

By (76) and Definition 2.3, we have

$$\begin{aligned}
[\mu]\otimes[\mathbf{l}] &= \frac{1}{n!}\sum_{K\in S_n}[\mu]\otimes[\mathbf{l}]\\
&\geqslant \frac{1}{n!}\sum_{K\in S_n}[\mu]\otimes K[\mathbf{l}]\\
&= \frac{1}{n!}\sum_{K\in S_n}\frac{1}{n}\sum_{i=1}^{n}[l_i]^{[\mu_{k_i}]}\\
&= \frac{1}{n\cdot n!}\sum_{i=1}^{n}\sum_{K\in S_n}[l_i]^{[\mu_{k_i}]}\\
&= \frac{1}{n\cdot n!}\sum_{i=1}^{n}(n-1)!\sum_{k_i=1}^{n}[l_i]^{[\mu_{k_i}]}\\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{n}\sum_{j=1}^{n}[l_i]^{[\mu_j]},
\end{aligned}$$

that is,

$$[\mu]\otimes[\mathbf{l}] \geqslant \frac{1}{n}\sum_{i=1}^{n}\frac{1}{n}\sum_{j=1}^{n}[l_i]^{[\mu_j]}. \tag{77}$$

Since for a fixed l_i, the function $[l_i]^t : (0,\pi)\to\mathbb{R}$ for the variable t is a strictly *convex function* [32, 14, 16, 27, 19, 6, 7] so, by the classical *Jensen inequality* [32, 27] and (68), we have

$$\frac{1}{n}\sum_{j=1}^{n}[l_i]^{[\mu_j]} \geqslant [l_i]^{\frac{1}{n}\sum_{j=1}^{n}[\mu_j]} = [l_i]^{\frac{2\pi}{n}}. \tag{78}$$

Since $3\leqslant n\leqslant 6$, we have $\frac{2\pi}{n}>1$. By the classical *power mean inequality* [26, 13, 31] and (66), we have

$$\frac{1}{n}\sum_{i=1}^{n}[l_i]^{\frac{2\pi}{n}} \geqslant \left(\frac{1}{n}\sum_{i=1}^{n}[l_i]\right)^{\frac{2\pi}{n}} = \left(\frac{|\Gamma|}{n}\right)^{\frac{2\pi}{n}}. \tag{79}$$

Combining with (77), (78) and (79), we get

$$[\mu]\otimes[\mathbf{l}] \geqslant \frac{1}{n}\sum_{i=1}^{n}\frac{1}{n}\sum_{j=1}^{n}[l_i]^{[\mu_j]} \geqslant \frac{1}{n}\sum_{i=1}^{n}[l_i]^{\frac{2\pi}{n}} \geqslant \left(\frac{|\Gamma|}{n}\right)^{\frac{2\pi}{n}} \Rightarrow (74).$$

The proof of Theorem 6.1 is completed. □

THEOREM 6.2. ($\mathbf{l}$-isoperimetric inequality) *Let* $S^{(2)}\{P,\Gamma,\mathbf{l}\}$ *be a centered* n*-surround system, where* $3\leqslant n\leqslant 6$. *If*

$$[l_1]\times[l_2]\geqslant 1, \tag{80}$$

then we have the **1***-isoperimetric inequality (74).*

Proof. Let us first recall the classical Chebyshev inequality [30, 29, 3, 21, 20, 9, 4, 35, 25, 5] as follows.

Let $X, Y \in \mathbb{R}^n$. If $X \backsim Y$, then we have

$$\frac{1}{n}\sum_{j=1}^{n} x_j y_j \geqslant \left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) \times \left(\frac{1}{n}\sum_{j=1}^{n} y_j\right). \tag{81}$$

The equalities in (81) holds if, and only if (15) hold.

Next, we proof that

$$\left(\log^m[l_1], \log^m[l_2], \ldots, \log^m[l_n]\right) \backsim \left([\mu_1]^m, [\mu_2]^m, \ldots, [\mu_n]^m\right),\ m = 0, 1, 2, \ldots. \tag{82}$$

Indeed, if $m = 2k+1$, $k = 0, 1, 2, \ldots$, then, by (70), we have

$$\left(\log^m[l_i] - \log^m[l_j]\right)\left([\mu_i]^m - [\mu_j]^m\right) \geqslant 0,\ \forall i, j : 1 \leqslant i, j \leqslant n \Rightarrow (82).$$

Let $m = 2k$, $k = 0, 1, 2, \ldots$. If $k = 0$ or $i = j$, then (82) holds. Assume that $k \geqslant 1$ and $i \neq j$. Without losing of generality, we may assume that $1 \leqslant j < i \leqslant n$. By (70) and (80), we have

$$[\mu_i]^m - [\mu_j]^m \geqslant 0, \tag{83}$$

and

$$\begin{aligned} |\log[l_i]|^2 - |\log[l_j]|^2 &= \log^2[l_i] - \log^2[l_j] \\ &= \log([l_i][l_j])\left(\log[l_i] - \log[l_j]\right) \\ &\geqslant \log([l_1][l_2])\left(\log[l_i] - \log[l_j]\right) \geqslant 0 \\ &\Rightarrow |\log[l_i]| \geqslant |\log[l_j]| \\ &\Rightarrow \log^m[l_i] - \log^m[l_j] \\ &= |\log[l_i]|^m - |\log[l_j]|^m \geqslant 0. \end{aligned}$$

Hence

$$\log^m[l_i] - \log^m[l_j] \geqslant 0 \Leftrightarrow \log[l_i] + \log[l_i] \geqslant 0. \tag{84}$$

Combining with (83) and (84), we get

$$\left(\log^m[l_i] - \log^m[l_j]\right)\left([\mu_i]^m - [\mu_j]^m\right) \geqslant 0,\ \forall i, j : 1 \leqslant i, j \leqslant n \Rightarrow (82).$$

Next, we proof that

$$\frac{1}{n}\sum_{i=1}^{n} \log^m[l_i] \geqslant 0,\ m = 0, 1, 2, \ldots. \tag{85}$$

Indeed, if $m=2k$, $k=0,1,2,\ldots$, then (85) hold. Let $m=2k+1$, $k=0,1,2,\ldots$. By $1\leqslant j<i\leqslant n$, (70) and (84), we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\log^m[l_i] &= \frac{1}{n}\left(\sum_{i=1}^{2}\log^m[l_i]+\sum_{j=3}^{n}\log^m[l_j]\right)\\ &\geqslant \frac{1}{n}\left(\sum_{i=1}^{2}\log^m[l_i]+\sum_{j=3}^{n}\frac{\sum_{i=1}^{2}\log^m[l_i]}{2}\right)\\ &= \frac{1}{2}(\log^m[l_1]+\log^m[l_2])\\ &\geqslant \frac{1}{2}\{(-\log[l_2])^m+\log^m[l_2]\}\\ &=0\Rightarrow (85).\end{aligned}$$

Next, we proof that

$$\frac{1}{n}\sum_{i=1}^{n}[\mu_i]^m\geqslant\left(\frac{2\pi}{n}\right)^m,\ m=0,1,2,\ldots. \tag{86}$$

Indeed, if $m=0$, then, then (86) hold. Assume that $m\geqslant 1$. By the classical power mean inequality [26, 13, 31] and (68), we have

$$\frac{1}{n}\sum_{i=1}^{n}[\mu_i]^m\geqslant\left(\frac{1}{n}\sum_{i=1}^{n}[\mu_i]\right)^m=\left(\frac{2\pi}{n}\right)^m\Rightarrow(86).$$

Finally, we prove the inequality (74). Combining with (81), (82), (85), (86), $\frac{2\pi}{n}>1$ and the power mean inequality , we get

$$\begin{aligned}[\mu]\otimes[\mathbf{l}] &= \frac{1}{n}\sum_{i=1}^{n}[l_i]^{[\mu_i]}\\ &= \frac{1}{n}\sum_{i=1}^{n}e^{[\mu_i]\log[l_i]}\\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{([\mu_i]\log[l_i])^m}{m!}\\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\frac{1}{n}\sum_{i=1}^{n}([\mu_i])^m(\log[l_i])^m\\ &\geqslant \sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{1}{n}\sum_{i=1}^{n}[\mu_i]^m\right)\left(\frac{1}{n}\sum_{i=1}^{n}\log^m[l_i]\right)\\ &\geqslant \sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{2\pi}{n}\right)^m\left(\frac{1}{n}\sum_{i=1}^{n}\log^m[l_i]\right)\\ &= \frac{1}{n}\sum_{i=1}^{n}e^{\frac{2\pi}{n}\log[l_i]}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n}[l_i]^{\frac{2\pi}{n}}\\
&\geqslant \left(\frac{1}{n}\sum_{i=1}^{n}[l_i]\right)^{\frac{2\pi}{n}}\\
&= \left(\frac{|\Gamma|}{n}\right)^{\frac{2\pi}{n}}\\
&\Rightarrow (74).
\end{aligned}
$$

This completes the proof of Theorem 6.2. □

Based on the Theorems 6.1 and 6.2, we see that Theorem 3.1 is of the application value in space science.

REMARK 6.1. Let $S^{(2)}\{P,\Gamma,\mathbf{l}\}$ be a centered *n-surround system* [32] and $A_1, A_2, \dots, A_n$ be n *satellites* of the surround system. Then we may think that the particle P is the Earth, and the particles $A_1, A_2, \dots, A_n$ are the *synchronous satellites* of the Earth, and Γ is their orbit. In general, we have $3 \leqslant n \leqslant 6$. Therefore, Theorems 6.1 and 6.2 are of the practical in space science.

REMARK 6.2. In space science, by Theorem 6.1, we need the lower bound $\left(\frac{|\Gamma|}{n}\right)^{\frac{2\pi}{n}}$ to be maximal. Since the function $t^t : (e^{-1}, \infty) \to \mathbb{R}$ is strictly incremental, and

$$
\frac{|\Gamma|}{n} \geqslant [l_1] > e^{-\frac{1}{\pi}} > e^{-1} \wedge \left(\frac{|\Gamma|}{n}\right)^{\frac{2\pi}{n}} = \left[\left(\frac{|\Gamma|}{n}\right)^{\frac{|\Gamma|}{n}}\right]^{\frac{2\pi}{|\Gamma|}},
$$

we see that the sequence $\left\{\left(\frac{|\Gamma|}{n}\right)^{\frac{2\pi}{n}}\right\}_{n=3}^{6}$ is strictly decreasing. Hence, we may choose the parameter $n = 3$.

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REFERENCES

[1] *Problems and Solutions*, Amer. Math. Monthly, **87** (4) (1980), 302–303.

[2] PROBLEMS AND SOLUTIONS, Amer. Math. Monthly, **97** (1) (1990), 65–67.

[3] A. M. ACU AND M. D. RUSU, *New results concerning Chebyshev-Grüss-type inequalities via discrete oscillations*, Appl. Math. Comput., **243** (2014), 585–593.

[4] H. AGAHI, R. MESIAR AND Y. OUYANG, *Chebyshev type inequalities for pseudo-integrals*, Nonlinear Anal., Theory Methods and Applications, **72** (6) (2010), 2737–2743.

[5] G. ANASTASSIOU, *Chebyshev-Grüss type inequalities on $\mathbb{R}^N$ over spherical shells and balls*, Appl. Math. Lett., **21** (2008), 119–127.

[6] A. COLESANTI, M. LUDWIG AND F. MUSSNIG, *A homogeneous decomposition theorem for valuations on convex functions*, J. Funct. Anal., **279** (5) (2020), Paper No. 108573, 25 pp.

[7] R. CORREA, A. HANTOUTE AND P. PĚREZ-AROS, *Characterizations of the subdifferential of convex integral functions under qualification conditions*, J. Funct. Anal., **277** (1) (2019), 227–254.

[8] N. DAFNIS, A. GIANNOPOULOS AND A. TSOLOMITIS, *Asymptotic shape of a random polytope in a convex body*, J. Funct. Anal. 257 (2009), no. 9, 2820–2839.

[9] B. DARABY, *Investigation of a Stolarsky type inequality for integrals in pseudo-analysis*, Fract. Calc. Appl. Anal., **13** (5) (2010), 467–473.

[10] J. DASCĂL AND J. JARCZYK, *Computer assisted solution of an equality problem of mean values*, Appl. Math. Comput., **219** (2) (2012), 475–481.

[11] K. E. DIETHELM, *The mean value theorems and a Nagumo-type uniqueness theorem for Caputo's fractional calculus*, Fract. Calc. Appl. Anal., **20** (6) (2017), 1567–1570.

[12] J. DOU AND M. ZHU, *L^1 isoperimetric inequality on the unit disk*, Adv. Math., **404** (2022), Paper No. 108383, 25 pp.

[13] C. B. GAO AND J. J. WEN, *A dimensionality reduction principle on the optimization of function*, J. Math. Inequal., **7** (3) (2013), 357–375.

[14] C. B. GAO AND J. J. WEN, *Theories and inequalities on the satellite system*, ISRN Math. Anal., **2011**, Article ID 909261, 22 pages.

[15] C. B. GAO AND J. J. WEN, *Theory of surround system and associated inequalities*, Comput. Math. Appl., **63** (2012), 1621–1640.

[16] D. Y. HWANG AND S. S. DRAGOMIR, *Comparing two integral means for absolutely continuous functions whose absolute value of the derivative are convex and applications*, Appl. Math. Comput., **230** (2014), 259–266.

[17] K. JOTSAROOP AND S. SHRIVASTAVA, *Maximal estimates for bilinear Bochner-Riesz means*, Adv. Math., **395** (2022), Paper No. 108100, 38 pp.

[18] Y. M. JUNG, J. J. WHANG AND S. YUN, *Sparse probabilistic K-means*, Appl. Math. Comput., **382** (2020), 125328, 12 pp.

[19] J. KNOERR, *The support of dually epi-translation invariant valuations on convex functions*, J. Funct. Anal., **281** (5) (2021), Paper No. 109059, 52 pp.

[20] L. M. MEAUX, J. W. J. SEAMAN AND T. L. BOULLION, *Calculation of multivariate Chebyshev-type inequalities*, Comput. Math. Appl., **20** (12) (1990), 55–60.

[21] Y. OUYANG, R. MESIAR AND J. LI, *On the comonotonic-$\star$-property for Sugeno integral*, Appl. Math. Comput., **211** (2) (2009), 450–458.

[22] J. E. PEČARIĆ, J. J. WEN, W. L. WANG AND T. LU, *A generalization of Maclaurin's inequalities and its applications*, Math. Inequal. Appl., **8** (4) (2005), 583–598.

[23] S. PONNUSAMY, J. QIAO AND X. WANG, *Schwarzian derivative, integral means, and the affine and linear invariant families of biharmonic mappings*, Appl. Math. Comput., **216** (12) (2010), 3468–3479.

[24] G. UR RAHMAN, R. P. AGARWAL AND Q. DIN, *Mathematical analysis of giving up smoking model via harmonic mean type incidence rate*, Appl. Math. Comput., **354** (2019), 128–148.

[25] A. WAGENE, *Chebyshevs algebraic inequality and comparative statics under uncertainty*, Math. Soc. Sci., **52** (2006), 217–221.

[26] J. J. WEN AND C. B. GAO, *Geometric inequalities involving the central distance of the centered 2-surround system*, Acta. Math. Sinica. **51** (4) (2008), 815–832, (in Chinese).

[27] J. J. WEN, C. B. GAO AND W. L. WANG, *Inequalities of J-P-S-F type*, J. Math. Inequal., **7** (2) (2013), 213–225.

[28] J. J. WEN, T. Y. HAN AND J. YUAN, *Stability inequalities involving gravity norm and temperature*, J. Math. Inequal., **14** (4) (2020), 1007–1037.

[29] J. J. WEN, J. E. PEČARIĆ AND T. Y. HAN, *Weak monotonicity and Chebyshev type inequality*, Math. Inequal. Appl., **18** (1) (2015), 217–231.

[30] J. J. WEN AND W. L. WANG, *Chebyshev type inequalities involving permanents and their applications*, Linear Algebra Appl., **422** (1) (2007), 295–303.

[31] J. J. WEN AND W. L. WANG, *The optimization for the inequalities of power means*, J. Inequal. Appl., **2006**, Article ID 46782, 25 pages.

[32] J. J. WEN, S. H. WU, J. YUAN AND T. Y. HAN, *Mean central distance–central distance inequalities*, J. Math. Inequal., **11** (4) (2017), 1131–1149.

[33] J. J. WEN, J. YUAN AND S. H. WU, *Isoperimetric inequalities in surround system and space science*, J. Inequal. Appl., **2016**: 74 (2016), 28 pp.

[34] J. J. WEN, J. YUAN, S. H. WU AND T. Y. HAN, *Gravity inequalities and the mean temperature on a planet*, J. Inequal. Appl., **2016**: 264, 18 pp.

[35] YEWALE, R. BHAGWAT, PACHPATTE AND B. DEEPAK, *Some new Chebyshev type inequalities via extended generalized fractional integral operator*, J. Fract. Calc. Appl., **12** (2) (2021), 11–19.

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Jia Jin Wen
College of Mathematics and Computer Science
Chengdu University
Chengdu, Sichuan, 610106, P. R. China
e-mail: wenjiajin623@163.com

Tian Yong Han
College of Mathematics and Computer Science
Chengdu University
Chengdu, Sichuan, 610106, P. R. China
e-mail: hantian123_123@163.com

Jun Yuan
School of Information Engineering
Nanjing Xiaozhuang University
Nanjing, Jiangsu, 211171, P. R. China
e-mail: yuanjun_math@126.com